Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A Neural Network Observer-Based Approach for Synchronization of Discrete-Time Chaotic Systems

S. V. Naghavi*, A.A. Safavi**

* Department of Electrical Engineering, Shiraz University, Shiraz, Iran
Vahid_166@yahoo.com

** Department of Electrical Engineering, Shiraz University, Shiraz, Iran

(Tel/Fax: +98 711 2303081; safavi@shirazu.ac.ir,)

Abstract: This paper presents a new approach to solve synchronization problem of a large class of discrete chaotic systems. The chaotic systems can be reformulated as an appropriate class of linear parameter varying (LPV) systems. Then, based on the LPV representation, a neural network observer-based approach is proposed to solve the synchronization problem. The simulation results show the advantages of combining the LPV techniques and the neural networks to determine the appropriate observer gain within the context of chaotic system synchronization.

1. INTRODUCTION

The possibility of synchronization of two coupled chaotic systems was first shown by Pecora and Carrol (1990) . The importance of this discovery was quickly appreciated by He (1992), and soon this topic aroused great interest as a potential mean for communications (kolumban et al, 1997; Tse et al,2003). In recent years a great deal of effort has been devoted to extend the chaotic communication applications to the field of secure communications. A detailed survey of chaotic secure communication systems is presented by Yang (2004). As the chaos synchronization problem can be reformulated as an observer design problem, the observer-based approach becomes one of the most attractive techniques for chaotic systems. This kind of approach has extensively been investigated in the recent research works by Grassi and Mascolo (2002); Morgul (1996) and Solak (1997) ; Nijmeijer and Mareels (1997); Ushio (1999); Celikovsky and Chen (2002).

Neural networks (NNs) have been recognized as valuable tools that offer simple solutions to difficult problems in various science and engineering fields due to their inherent adaptability and universal approximation properties (Suykens, 1996; Luo et al, 1997; Cherkassky et al, 1998). Especially in the area of control, neural networks have experienced an increased interest in the last decade. The later works on this field have mostly focused on the application of recurrent neural networks for system identification and observer design for general nonlinear systems. Zhu et al (1997) focus on the application of dynamic recurrent neural networks (DRNN) as observers for nonlinear systems. They consider a class of single-input-single-output (SISO) nonlinear time-varying systems in their work, where they prove the bounded ness of the observer error and the DRNN weights during adaptation using Lyapunov stability theory and the well-known universal approximation theorem for neural networks (Zhu et al ,1997; Hornik et al ,1989). With an alternative approach, Wang and Wu (1994) exploit the multiplayer recurrent neural networks to synthesize linear state observers in real-time application. There are also examples of static feed forward neural network applications to observer and controller design. Ahmed and Riyaz (2000) consider an off-line training scheme for an MLP based observer design for nonlinear systems. They note that although the NN observer requires more computation in the training phase, it is more computational efficient compared to the EKF in the implementation phase. An interesting approach is presented by Vargas and Hemerly (2000), where they employ linearly parametrized neural networks (LPNN) for the design of an adaptive observer for general nonlinear systems. LPNN include a wide class of networks including radial-basis -function (RBF) networks, adaptive fuzzy systems, and wavelet networks. Fretheim et al (2000) utilize the feed forward MLP in the observer design problem with a little twist. They formulate the problem as a multi-step prediction, and exploit the extrapolation capabilities of the MLP to obtain the state estimates. Erdogmus et al (2002) investigate the use of adaptive extended Luenberger state estimators for general nonlinear and possibly time-varying systems. The association between dynamic neural networks and the Luenberger observer leaded to an obvious modification on the proposed observer scheme that would allow handling state estimation for those systems. On the other side, some researchers followed a more conservative approach to nonlinear observer design for the class of nonlinear discrete-time chaotic systems. An extended Kalman filtering was proposed by Cruz and Nijmeijer (1999); while extended observers are used by Huijberts (2001); Lilge (1999). Linear parameter-varying (LPV) techniques were also used successfully in the context of chaotic systems synchronization (Bara et al ,2005). Based on their results, a sufficient condition was given in order to design the observer gain with guaranteed stability of the synchronization error. This condition was expressed as an LMI (linear matrix inequalities) solvability problem. In such a case, one needs to solve the (LMI) in order to find the observer gain if the LMIs are feasible. Therefore, NN based techniques as described above seems more general and easier to implement. In this paper, a new approach is proposed for the synchronization of discrete-time chaotic systems. This approach is based on the fact that many chaotic systems can be transformed into LPV systems when the output signal is chosen appropriately. Then the synchronization problem is reformulated as an observer design. In order to find the observer gain, one seeks the connection between the designed observer and Grossberg's additive model (J.C. Principe ,1999) for dynamic neural networks (DNN). The proposed DNN uses the on-line training algorithm to train observer gain and finds the weights of the network such that the synchronization of this class of chaotic systems is achieved.

This paper is organized as follows. In Section 2, the class of systems under study and the chaos synchronization problem are described. Formulation of the observer-LPV form is also presented in this section. Section 3 presents the main contribution of this paper which consists of a new LPV neural network observer-based approach for the problem. In order to demonstrate the validity of the approach two numerical examples including discrete-time Henon and Lorenz systems are presented in Section 4. This paper concludes in Section 5.

2. PROBLEM FORMULATIOM

Consider the class of chaotic systems described by the following nonlinear state equations:

$$\begin{cases} x(k+1) = Ax(k) + f(x(k), y(k), k) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

Where $x(k) \in \Re^n$ is the state vector and $y(k) \in \Re$ is the scalar output signal. A and C are constant matrices of appropriate dimensions and $f : \Re^n \times \Re \times \Re_+ \to \Re^n$ is a nonlinear function. The pair (A, C) is assumed to be detectable (Polderman et al, 1998).

Given the chaotic drive-system (1), the chaos synchronization problem consists of finding a response system (also called a slave-system) whose state $\hat{x}(k)$ converges towards the drive-system state $x(k)$ using the transmitted signal $y(k)$.

The synchronization problem for the discrete-time LPV chaotic systems using neural network observers is investigated. As it is showed by Bara et al (2005), the following assumptions can lead to the reformulation of the chaotic system as an LPV one. Note that these assumptions are not restrictive. In fact, the class of systems satisfying these conditions includes an extensive variety of chaotic systems such as the discrete-time version of the Henon's or the Rossler's and Lorenz's systems (Liao et al , 1999).

A1: For a particular choice of the output matrix C, the nonlinear part can be rewritten as:

$$f(x(k), y(k), k) = g_1(y(k), k)Hx(k) + g_2(y(k), k) \tag{2}$$

Where $H \in \Re^{n\times n}$, $g_1 : \Re\times\Re_+ \to \Re$ and $g_2 : \Re\times\Re_+ \to \Re^n$.

A2: It is assumed that the function $g_1(y(k)), k)$ is bounded when y(k) is bounded. Note that this assumption is not restrictive because the state vector x(k) and the transmitted signal y(k) of a chaotic system are always bounded. Now, the following notations are introduced:

$$\rho(k) = g_1(y(k), k) \tag{3a}$$

$$\overline{A}(\rho(k)) = A + \rho(k)H \tag{3b}$$

Using Equation (2) and the notations in (3), system (1) can be rewritten as:

$$\begin{cases} x(k+1) = \overline{A}(\rho(k))x(k) + g_2(y(k), k) \\ y(k) = Cx(k) \end{cases} \tag{4}$$

Using the output measurement, one can compute $\rho(k)$ at any instant k. Hence, $\rho(k)$ is considered as a known time-varying parameter and system (4) can be seen as a linear parameter varying (LPV) system with a nonlinear term. A state observer corresponding to (4) is given by:

$$\begin{cases} \hat{x}(k+1) = \overline{A}(\rho(k))\hat{x}(k) + g_2(y(k), k) \\ \qquad\qquad + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C\hat{x}(k) \end{cases} \tag{5}$$

Where $\hat{x}(k)$ denotes the estimate of the state $x(k)$.

3. MAIN RESULTS

The most widely used dynamic neural network is the so called additive model by Grossberg. The state dynamics of the additive model is described by

$$x(k+1) = -T.x(k) + \sigma(W_s.x(k)) + W_I.I(k) \tag{6}$$

Usually, it is desired to identify the weights matrix W_I multiplying the input vector $I(k)$. The passive decay matrix T is commonly a diagonal positive definite matrix and the interactions between states are provided through W_s and the nonlinearity of the neurons, $\sigma(.)$. But these may not be always the case. A special case of interest is when the nonlinearity of the neurons in (6) is chosen to be a linear function. For the choice $\sigma(a) = a$, (6) is reduced to a linear dynamic neural network whose dynamics are of the form

$$x(k+1) = (W_s - T).x(k)) + W_I.I(k) \tag{7}$$

One may express the designed observer (5) in the form of the additive model. Then it is possible to generalize this equation as (8), with a sufficient number of neurons and a proper choice of the weight matrix :

$$\begin{aligned}&\hat{x}(k+1) = (W_s - T).\hat{x}(k) + W_I.y(k) + g_2(y(k),k)\\ &Where \quad (W_s - T) = (A + \rho(k).H - L.C)\\ &W_I = L \quad and \quad I(k) = y(k)\end{aligned} \tag{8}$$

Then, with a proper adaptation of parameters in (8), one may design a stable observer for this discrete class of chaotic systems as it is described below.
Suppose a network is to be trained to find the gain matrix L such that the synchronization error,

$$e(k) = x(k) - \hat{x}(k) \tag{9}$$

converges asymptotically toward zero. If the system under consideration is observable and LTI, off-line training of the network is readily led to a globally asymptotically stable observer. However, for nonlinear and time-varying systems, off-line training is not much useful. Instead, invoking an on-line training algorithm called Widrow's stochastic gradient adaptation algorithm is very promising (Widrow et al, 1985).
Suppose one is to train the network to find an optimal L based on the minimization of instantaneous squared error (ISE). The ISE may be considered as a stochastic approximation to the mean square error (MSE) which was used in the off-line training. It should be noted that one may use other performance criteria instead of ISE. The learning algorithm used in this paper is the steepest descent algorithm. The gradient the stochastic cost function (i.e. ISE) with respect to the weights was computed to get Widrow's stochastic gradient for MSE. Though the cost function depends only on the instantaneous value of the error, the error recursion still exhibits a back propagation property and it is easy to oversee this. One may use the actual gradient expression computed with full consideration of the recursive structure of the topology or use an approximate version of the gradient, which is very accurate if the learning rate is chosen to be a small value. The stochastic cost function and its approximate gradient are simply computed using (10) and the current value of the observer gains:

$$J = (y(k) - \hat{y}(k))^T (y(k) - \hat{y}(k))$$

$$\begin{aligned}&\frac{\partial J}{\partial L} = -2(y(k) - \hat{y}(k))^T C \frac{\partial \hat{x}(k)}{\partial L}\\ &\frac{\partial \hat{x}(k)}{\partial L} = ((A + \rho(k-1)H + \frac{\partial g_2(y(k-1),k-1)}{\partial y(k-1)}.C) - L.C)...\\ &(y(k-2) - \hat{y}(k-2)) + (y(k-1) - \hat{y}(k-1)).ones(n)\\ &L(k+1) = L(k) - \eta(\frac{\partial J}{\partial L})^T\end{aligned} \tag{10}$$

Here J is the cost function, η is the learning rate and *ones(n)* is an all ones matrix of the size of the state vector. With this algorithm, the weights converge toward optimal solution and the synchronization of the LPV chaotic system is possible.

3.1 Approximation Improvement

As it was mentioned the algorithm (10) is close to the actual gradient when the step size in steepest descent is small. The advantage of using the approximate gradient is that it is computationally much simple. However, it requires the use of smaller step size values for stability of the weights. This is investigated in below.
Suppose we assign a time index to each gain vector during the adaptation process in the following manner (we will drop the input and time indexes from the expressions for simplicity):

$$J_k = (y(k) - \hat{y}(k))^T (y(k) - \hat{y}(k)) \tag{11}$$

$$\begin{aligned}&\frac{\partial J_k}{\partial L_k} = -2(y(k) - \hat{y}(k))^T C \frac{\partial \hat{x}(k)}{\partial L_k}\\ &\frac{\partial \hat{x}(k)}{\partial L_k} = ((A + \rho(k-1)H + \frac{\partial g_2(y(k-1),k-1)}{\partial y(k-1)}.C) - L_k.C)...\\ &\frac{\partial \hat{x}(k-1)}{\partial L_k} + (y(k-1) - \hat{y}(k-1)).ones(n)\end{aligned}$$

Now we may use the chain rule to express

$$\frac{\partial \hat{x}(k-1)}{\partial L_k} = \frac{\partial \hat{x}(k-1)}{\partial L_{k-1}} . \frac{\partial L_{k-1}}{\partial L_k} \tag{12}$$

Notice that when the step size is small, the second term on the right hand side of (12) will be approximately one(or identity matrix). This is the approximation that links this actual gradient expression to the approximated one given in (10). To avoid this approximation, one may use the steepest descent update rule to determine this second derivative. The matrix we seek is the inverse of

$$\frac{\partial L_k}{\partial L_{k-1}} = I - (2\eta(y(k-1) - \hat{y}(k-1))CC^T(y(k-3) - \hat{y}(k-3))\frac{\partial L_{k-2}}{\partial L_{k-1}}) \tag{13}$$

In summary, the actual gradient expression can be computed by iterating the equations presented in (11-13). This modification provides faster convergence rates at the cost of increased computational requirements. For example if one supposes that $\frac{\partial L_{k-2}}{\partial L_{k-1}} = 1$ and substitutes this into the algorithm (11) with consideration of (12) and (13), it leads to the following algorithm:

$$\begin{aligned}&J_k = (y(k) - \hat{y}(k))^T (y(k) - \hat{y}(k))\\ &\frac{\partial J_k}{\partial L_k} = -2(y(k) - \hat{y}(k))^T C \frac{\partial \hat{x}(k)}{\partial L_k}\\ &\frac{\partial \hat{x}(k)}{\partial L_k} = ((A + \rho(k-1)H + \frac{\partial g_2(y(k-1),k-1)}{\partial y(k-1)}.C) - L.C)(y(k-2) - \hat{y}(k-2)).\\ &(I - (2\eta(y(k-1) - \hat{y}(k-1))CC^T(y(k-3) - \hat{y}(k-3)))^{-1} + (y(k-1) - \hat{y}(k-1)).ones(n)\\ &L(k+1) = L(k) - \eta(\frac{\partial J_k}{\partial L_k})^T\end{aligned} \tag{14}$$

3.2 Remarks on stability

From the assumption A2 and the definition of $\rho(k)$, we deduced that the parameter $\rho(k)$ is bounded. Then, assume the notations

$$\underline{\rho} = \min_k(\rho(k)) \qquad \overline{\rho} = \max_k(\rho(k)) \tag{15}$$

Now, consider the quadratic Lyapunov function:

$$V(k) = e^T(k)\,P\,e(k) \tag{16}$$

According to the Lyapunov stability theory, the error (9) converges exponentially towards zero if and only if:

1. The function V (k) is positive definite.
2. $\Delta V(k) = V(k+1) - V(k)$ (17)
is negative definite
for all $e(k) \neq 0$ and all possible trajectories $\rho(k)$.

The achievement of the first condition can be done by choosing P as the positive definite matrix. The second condition is satisfied if and only if :

$$[(A+\rho(k)H - LC)^T P(A+\rho(k)H - LC)\_P] \leq 0 \tag{18}$$

This can be proved as bellow. The variation of this Lyapunov function is $\Delta J = J(k+1) - J(k)$
$= e^T(k)[(A+\rho(k)H - LC)^T P(A+\rho(k)H - LC)\_P]e(k)$

The parameter dependence of the inequality (18) implies an infinite number of inequalities to satisfy. In order to reduce this infinite number to a finite one, we apply the convexity principle. Then, since (18) is affine with respect to the parameter $\rho(k)$, the in $\rho(k) \in [\underline{\rho}, \overline{\rho}]$ equality (18) is satisfied for all possible trajectories if it is satisfied on the vertices of $[\underline{\rho}, \overline{\rho}]$. This condition yields the inequality conditions (19) and (20):

$$[(A+\underline{\rho}H - LC)^T P(A+\underline{\rho}H - LC)\_P] \leq 0 \tag{19}$$

$$[(A+\overline{\rho}H - LC)^T P(A+\overline{\rho}H - LC)\_P] \leq 0 \tag{20}$$

These conditions determine a bounded range for L which guaranties the convergence of the synchronization error to zero and in this range we have:

$$e(k) \to 0$$

Then considering the system described by (1) we deduce:

$$e_o(k) = y(k) - \hat{y}(k) = C\,e(k) \;\to 0$$

Therefore it is obvious that the cost function J_k converges asymptotically toward zero for this range of the weight matrix L. Now one can invoke the proposed algorithms to find the optimal L. This is done by restricting the search for optimum weights of the network to this bounded region. The following case studies show the applicability of the proposed approaches.

4. THE CASE STUDIES

Example1. Consider the discrete-time Henon chaotic system:

$$A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}, c = [1 \quad 0]$$

$$f(x(k), y(k), k) = \begin{bmatrix} 1 - 1.4x_1^2 \\ 0 \end{bmatrix}$$

The chaotic behaviour of this system is depicted in the phase plot of Fig. 1(a), with the initial conditions $x(0) = [1\ 1]^T$. One may rewrite this system as in (4) with $\overline{A}(k)$ defined by (3b) with

$$H = T\begin{bmatrix} -1.4 & 0 \\ 0 & 0 \end{bmatrix},\ \rho(k) = y(k),\ g_2(y(k),k) = [1\ 0]^T$$

The Fig. 1(b) shows the evolution of parameter $\rho(k)$ which is identical to the system output. From this figure one can see that $\rho(k)$ is bounded: $\underline{\rho} = -1.2846 \qquad \overline{\rho} = 1.2728$

Using the proposed algorithms (10) and (14), one can readily compute the results as shown in Table1. This table contains the initial value of L (L_0), the learning rate (η), the training algorithm which was used (Alga.), number of epochs to find the optimal observer gain (L) and its value.
Fig. 3 shows simulation results for Henon system (Table 1, case 3). As it is seen from the figures, the synchronization achieved quite fast and with almost zero error. As it was stated in Section 3, for small learning rates the algorithms given by (10) and (14) give the same results (see Table 1,case 1 and 2) but if η increases then the algorithm (14) provides faster convergence rates (see Table1,case 4 and 5) .
In this example, it is impossible to find L such that conditions (18) and (19) are satisfied. This means that although one cannot guarantee the stability of the system (see Table1, case 6) but the synchronization can be achieved for some values of initial conditions (Table 1, case 1-5) . Such values of initial conditions can be worked out by trial and error or judicious selection based on satisfaction of at least some of the conditions in (18) and (19).

Table1.Simulation results for Henon map (Minimum cost equal is to 1e-9)

Case	L_0	η	Alg.	Epochs	L
1	$[1.5\ 1]^T$	.02	10	72	$[.3314\ .3338]^T$
2	$[1.5\ 1]^T$	.02	14	72	$[.3156\ .3335]^T$
3	$[1.5\ 1]^T$	.03	10	28	$[.3222\ .3140]^T$
4	$[1.5\ 1]^T$	.035	10	24	$[.5419\ .1277]^T$
5	$[1.5\ 1]^T$	.035	14	19	$[.5934\ .1125]^T$
6	$[-2.78\ 1]^T$	.03	10, 14	max	No result

Example 2. Consider the discrete-time version of the Lorenz chaotic system [30]. This discrete-time version is obtained by using the Euler discretization method with a sampling period of T = 0:01. The system is described by:

$$A = \begin{bmatrix} 1-10T & 10T & 0 \\ 28T & 1-T & 0 \\ 0 & 0 & 1-8/3T \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$f(x(k), y(k), k) = T\begin{bmatrix} 0 \\ -x_3(k)x_1(k) \\ x_2(k)x_1(k) \end{bmatrix}$$

The chaotic behaviour of this system is depicted in the phase plot of Fig.2, with the initial conditions $x(0) = \begin{bmatrix} -2 & 2 & 0.1 \end{bmatrix}^T$. One may rewrite this system as in (4) with $\overline{A}(k)$ defined by (3b) with

$$H = T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \rho(k) = y(k), \; g_2(y(k), k) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

Using the proposed algorithms (10) and (14) one can readily compute the results as given in Table 2. Fig. 4 shows simulation results for Lorenz attractor (Table 2, case 1). As it is seen from the figures, the synchronization achieved quite fast and with almost zero error. As it was stated in Section 3, for small learning rates the algorithms given by (10) and (14) give the same results (see Table 2, case 1 and 2) but if η increases then the algorithm (14) provides faster convergence rates (see Table 2,case 3 and 4).

Table2.Simulation results for Lorenz system (Minimum cost is equal to 1e-9)

Case	L_0	η	Alg.	Epochs	L
1	$[1.5\ 1\ .8]^T$	.5	10	149	$[1.0794\ .8418\ .5988]^T$
2	$[1.5\ 1\ .8]^T$	.5	14	149	$[1.3597\ .9589\ .7468]^T$
3	$[1.5\ 1\ .8]^T$	2	10	78	$[.67\ .8047\ .4191]^T$
4	$[1.5\ 1\ .8]^T$	2	14	54	$[1.4649\ .7976\ .5427]^T$

5. CONCLUSION

This paper presented a new approach to solve synchronization problem of a large class of discrete chaotic systems. The idea was to reformulate the chaotic system as an appropriate class of linear parameter varying (LPV) systems. Then, based on the LPV representation, a neural network observer-based approach was proposed to solve the synchronization problem. The simulation results for two typical chaotic system examples showed effectiveness of the proposed approach.

Acknowledgments: The authors would like to thanks Iran's Telecommunication Research Centre for their financial support on this research.

REFERENCES

Ahmed M.S, S.H. Riyaz (2000). Design of Dynamic Neural Observers. IEE Proc. Cont. Theory Appl., **vol.147**, n.3, pp.257-266.

Bara G.Iulia ,Ali Zemouche and Mohamad boutayeb (2005). A new approach for the observer-based synchronization of chaotic systems. IFAC.

Carrol T.L and L.M Pecora (1991). Synchronizing chaotic circuits.IEEE. Trans. Of circuits and systems.**vol.38**,453-456.

Cherkassky V.S., F.M. Mulier (1998). Learning from Data: Concepts, Theory and Methods. Wiley, NY.

Celikovsky S. and G. Chen (2002). Synchronization of a class of chaotic systems via a nonlinear observer approach. In Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, Nevada, USA.

Cruz C. and H. Nijmeijer (1999). Synchronization through extended Kalman filtering. Lecture notes in control and information sciences,**vol. 244**,Springer Verlag.

Erdogmus D., A.U. Genc and Jose C. Principe (2002).A neural network perspective to extended Luenberger observers. Measurement & control. Special feature on recent advances in neural networks, **vol .35** , no.1.

Fretheim T., R. Shoureshi, T. Vincent (2000). A General Approach to Non-Linear Output Observer Design Using Neural Network Models. Proc. of Amer. Cont. Conf., pp. 924-928.

Grassi G. and D. A. Miller (2002). Theory and experimental realization of observer-based discrete-time hyper chaos synchronization. IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, **vol.49** (3),373-378.

Huijberts H.J.C., T. Lilge, and H. Nijmeijer (2001). Nonlinear discrete-time synchronization via extended observers. International Journal of Bifurcation and Chaos,**vol.11**(7),1997-2006.

He R. and P.G.Raidyn (1992). Analysis and synthesis of synchronous periodic and chaotic systems.A.46,7387-7392 .

Hornik K., M. Stinchcombe, H. White (1989).MLP's are universal approximators. Neural Networks, **vol. 2**,359-366.

kolumban G. and M.P.kenedy and L.O.chua (1997).The role of synchronization in digital communication using chaotic part I : Fundamental of digital communications.IEEE Trans. Circuits sys., **I.44**,927-93.

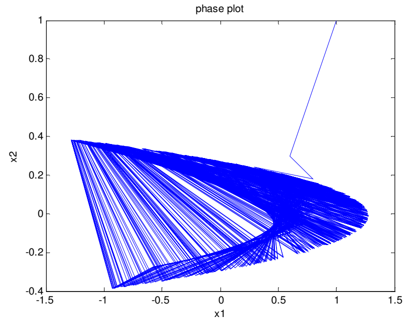
Liao T. L. and N. S. Huang (1999) . An observer-based approach for chaotic synchronization with applications to secure communications. IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, **vol.46** (9),1144-1150.

Lilge T (1999). Nonlinear discrete-time observers for synchronization problems. Lecture notes in control and information sciences, **vol. 244**, Springer Verlag.

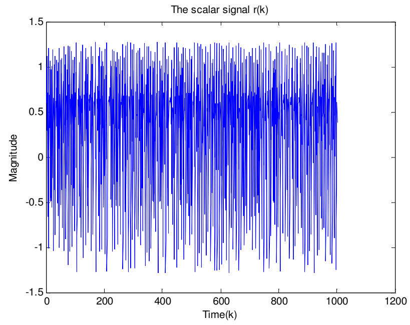
Luo F.L., U. Rolf (1997). Applied Neural Networks for Signal Processing. Cambridge University Press, Cambridge.

Morgu O. l and E. Solak (1997). On the synchronization of chaotic systems by using state observations. International Journal of Bifurcation and Chaos, **vol.7**(6),1307-1322.

Morgul O. and E. Solak (1996). Observer based synchronization of chaotic systems. Physical Review ,**E.54**,4802-4811.
Nijmeijer H. and I.M.Y. Mareels (1997). An observer looks at Synchronization. IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications,**vol.44** (1),882-890.
Pecora L.M. and T.L. Carroll (1990). Synchronization in chaotic systems. Physical Review Letters, **vol.64**, 821-824.
Polderman J.W and J.C. Williams (1998). Introduction to Mathematical System Theory–A Behaviour approach. Springer.
Principe J.C., N. Euliano, C. Lefebvre (1999). Neural and Adaptive Systems. John Wiley and Sons Inc, NY.
Puskorius G.V., L.A. Feldkamp (1994). Neurocontrol of Nonlinear Dynamical Systems with Kalman Filter Trained Recurrent Networks. IEEE Trans. Neural Networks, **vol. 5**, no. 2,279-297.
Ruiz J.A. Vargas, E.M. Hemerly (2000). Neural adaptive observer for general nonlinear systems. Proc. Amer. Cont. Conf., pp. 708-712, Chicago, IL.
Suykens J.A.K., J.P.L. Vandewalle, B.L.R. Moor (1996). Artificial Neural Networks for Modeling and Control of Non-Linear Systems. Kluwer Academic Publishers, Netherlands.
Tse C.K. and F.C.M.lau (2003). Chaos based digital communication systems. Operating principles , analysis methods and performance evaluation. Springer velag.
Ushio T (1999). Synthesis of synchronized chaotic systems based on observers. International Journal of Bifurcation and Chaos, **vol.9**(3),541-546.
Wang J., G. Wu (1994).Real-Time Synthesis of Linear State Observers Using a Multilayer Recurrent Neural Network. Proc. IEEE Int. Conf. **Industrial Tech.**, 278-282.
Widrow B., S.D.Stearns (1985). Adaptive Signal Processing. Prentice Hall, NJ.
Yang Tao (2004). A survey of chaotic secure communication systems. International Journal of Computational Cognition, **vol.2**(2),81-130.
Zhu R., T. Chai, C. Shao (1997). Robust Nonlinear Adaptive Observer Design Using Dynamic Recurrent Neural Networks. Proc. Amer. Cont. Conf., pp. 1096-1100, New Mexico.



(a)Phase plot



(b) The scalar signal $\rho(k)$

Fig.1. The Henon chaotic system.

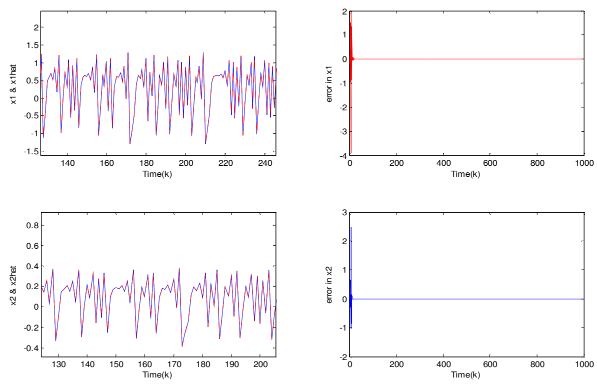


Fig.3. Henon states, estimates and synchronization errors

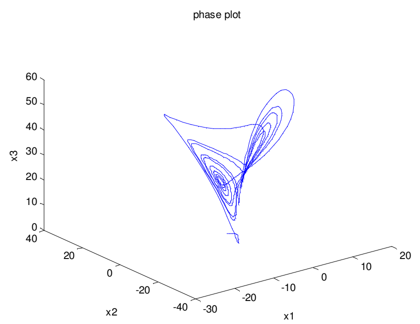


Fig.2. The Lorenz chaotic system..

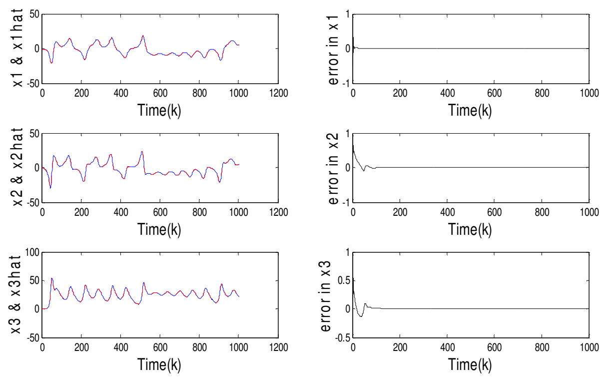


Fig.4. Lorenz states, estimates and synchronization errors