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Master Stability Equations of Complex Dynamical Networks with General Topology ⋆

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Abstract: The master stability equations for a complex dynamical network with general topology are obtained. Compared to prior work, we remove almost all the restrictions on the graph of the network. The coupling configuration matrix is not necessarily diagonalizable, the coupling coefficients are not necessarily nonnegative, and the graph of the network can be directed. These new master stability equations as for those in the previous studies are still very effective in analyzing the stability of complex dynamical networks in terms of synchronization to a manifold. We present some new observations on stability. A new concept "heavily connected", which can be regarded as the generalization of both "connected" for an undirected graph and "strong connected" for a directed graph, is proposed. The proofs of the two main theorems are very short but can substitute many of those in the literature.

1. INTRODUCTION

Recently, complex dynamical networks have attracted increasing attention among researchers (see Barabási (1999), Song et al. (2005), Watts et al. (1998), Wang et al. (2002)). Many characteristics and complicated behaviors such as small-world (Watts et al. (1998)) scale-free(Barabási (1999), Wang et al. (2002)), self-similarity (Song et al. (2005)), robustness, fragility, and synchronization have been studied widely. Among these, synchronization is one of the most important issues.

The master stability function (MSF) provides an effective method to study the synchronization problem for complex dynamical networks (Wang et al. (2002, 2003), Li et al. (2006), Lü et al. (2004, 2005), Pecora et al. (1998), Zhou et al. (2006)). The MSF is in essence the largest Lyapunov exponent and can be calculated from the so-called master stability equation (MSE) (Pecora et al. (1998)). The MSF facilitates analyzing the stability of the synchronized manifold by specifying the region of parameters, in which the largest Lyapunov exponent can be ensured to be negative. The MSEs are obtained by linearizing the dynamical network at the synchronized manifold firstly and then using a special transformation to convert the network into a set of linear time-variant systems. Each has the same structure and thus is called the MSE. Because of the simplicity and usefulness, MSEs have been employed by numerous researchers. Some results only are now discussed. Wang et al. (2002, 2003) investigated the synchronization in small-world and scale-free dynamical networks. A threshold of the coupling strength, which is a lower bound to determine the synchronization, was obtained. A further result to precisely determine this threshold was gained in Li et al. (2003, 2004). Li et al. (2006) studied the asymptotic stability of complex dynamical networks whose dynamical equations can be regarded as the MSEs with extra disturbances from the point of view of Lyapunov theory. Both the chaos and periodic orbit synchronization for the time-variant complex dynamical networks were studied in Lü et al. (2004, 2005). Some stability criteria for pinning control for a kind of asymmetric and heterogeneous connected networks are derived in Xiang et al. (2007). These guarantee that the whole network can be pinned to its equilibrium by placing feedback control only on a small fraction of nodes. Similar ideas were used to study networks with time delay in Liu et al. (2007). Zhou et al. (2006) studied weighted networks and introduced a simple but generic scheme of weight adaptation according to a local synchronization property, which leads to global synchronization of the whole network.

In the previous results, some deficiencies should be noted that require further work. Firstly, the previous studies have a common restriction that the coupling configuration matrix must be diagonalizable (we just confine our field of view within the MSE methods). Secondly, most of the work requires that the coupling coefficients between nodes are nonnegative (Xiang et al. (1998), Li et al. (2003, 2004), Wang et al. (2002, 2003), Li et al. (2006), Xiang et al. (2007), Liu et al. (2007), Zhou et al. (2006)). Finally, much research focused on undirected networks (i.e., the graph of a network is undirected) whose coupling configuration matrices are symmetric. In practice, the coupling of a real network is often asymmetric and heterogeneous (Xiang

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et al. (2007)). For practical situations, it is too restrictive to restrain all the coupling coefficients of a network to be nonnegative. On the contrary, we often encounter negative feedback loops in the networks of process control, in which these loops feed some negative errors to their forward components. Further, in most cases the coupling configuration matrix of a complex dynamical network should be asymmetric, neither nonnegative nor nonpositive, and therefore not necessarily diagonalizable.

In this paper, we deduce the MSEs for the complex dynamical networks with general topology, which means their coupling configuration matrices can be arbitrary without any restrictions except only the fundamental one that every row sum of them is zero. Positive and negative coupling coefficients can be combined with each other in this kind of networks. We will also apply these new MSEs to study the stability and the manifold synchronization for the complex dynamical networks with general topology. Our study shows that the MSEs for the complex dynamical networks with general topology are similar in form to those with diagonalizable coupling configuration matrices. Moreover, these new MSEs hold the same usefulness and flexibility as the previous ones for analyzing the stability of the isolated equilibrium as well as the synchronized manifold for a complex dynamical network. Therefore, our work can be regarded as an extension of the past results on MSEs for complex dynamical networks.

The rest of this paper is organized as follows. The model of a complex dynamical network is described based on graph theory in Section 2. Section 3 gives the main results and their applications in the two aspects: stability of the equilibrium and, realizability of the manifold synchronization. Some comparison between our results and the previous ones is also discussed in this section. Finally, Section 4 contains the concluding remarks.

Notation:

I_n: $n \times n$ unit matrix. $\mathbf{R}^{n\times n}$ ($\mathbf{C}^{n\times n}$): the set of all the $n \times n$ Real (Complex) matrices. j: the imaginary unit satisfying $j^2 = -1$. $Re()$: the real part of a complex number, vector or matrix. $Im()$: the imaginary part of a complex number, vector or matrix. $\otimes$: the Kronecker product for two matrices. $\|\bullet\|$: the Euclidian vector norm.

2. MODEL DESCRIPTIONS AND SOME LEMMAS

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a directed graph of order N with the set of nodes $\mathcal{V} = \{v_1, \ldots, v_N\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacency matrix $A = (a_{ij}) \in \mathbf{R}^{N\times N}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_j, v_i)$. The adjacency elements associated with the edges of the graph are nonzero, i.e., $e_{ij} \in \mathcal{E} \Longleftrightarrow a_{ij} \neq 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I} =: \{1, 2, \ldots, N\}$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} \ : \ (v_j, v_i) \in \mathcal{E}\}$. We call a graph *undirected*, if its adjacency matrix A is symmetric, i.e., $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{I}$. For a directed (undirected) graph, we call it *strong connected* (*connected*), if for any two nodes, there exists a path between them.

Suppose each node of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is a dynamical system described by

$$\dot{x}_i(t) = f(x_i(t)) + u_i, \quad i \in \mathcal{I} \tag{1a}$$

where $x_i(t) \in R^n$ is the state variable of node i, u_i takes the form as

$$u_i = \sum_{v_j \in N_i} a_{ij}\Gamma(x_j - x_i) \tag{1b}$$

where $\Gamma \in \mathbf{R}^{n\times n}$.

If we redefine the diagonal elements of matrix A [1] as

$$a_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij} \tag{2}$$

then system (1) can be rewritten as

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} a_{ij}\Gamma x_j(t), \ i \in \mathcal{I} \tag{3}$$

System (3) is called a complex dynamical network, which can be regarded as a dynamical network consisting of N linearly and diffusively coupled nodes, with each node being a $n-$dimensional nonlinear system. The graph $\mathcal{G}$ represents the topology of the network. Matrix $A = (a_{ij}) \in \mathbf{R}^{N\times N}$ is called the coupling configuration matrix, whose elements a_{ij} represent the coupling strength from node j to node i. When $a_{ij} > 0$ (< 0, ≥ 0, ≤ 0), we call the coupling from node j to node i to be positive (negative, nonnegative, nonpositive). Matrix $\Gamma \in \mathbf{R}^{n\times n}$ is called the inner-coupling matrix, which describes the way of linking the components in each pair of connected node vector $x_j - x_i$.

Remark 1 There are no extra restrictions on the coupling configuration matrix A and the inner-coupling matrix Γ except that every row sum of A is zero. So the model studied in this paper represents a wider and more general class of networks.

When the graph $\mathcal{G}$ is undirected and Γ is a nonnegative diagonal matrix, system (3) will degenerate into the models in Xiang et al. (1998) and Li et al. (2004). Moreover, if $a_{ij} = cb_{ij}$ ($i \neq j$), where $b_{ij} = 0$ or 1 and $c > 0$, then system (3) becomes the models in Wang et al. (2002, 2003), Li et al. (2003), Li et al. (2006).

When the graph $\mathcal{G}$ is directed, few studies can be found in the literature. The model in Zhou et al. (2006) is a directed network, but all the coupling is set to be nonnegative. The models in Lü et al. (2004, 2005) can be regarded as networks under the case that the graph $\mathcal{G}$ is both directed and time-variant. However, it is required that the coupling configuration matrix is diagonalizable. In this paper, we will remove all these strict conditions.

In order to reach our main results, we need to introduce some lemmas stated below.

Lemma 1 The equilibrium $\xi \equiv 0$ of a complex-valued linear time-invariant system

$$\dot{\xi}(t) = A\xi(t) \tag{4}$$

[1] Strictly speaking, we should introduce another terminology, i.e., the Laplacian of a graph (Godsil et al. (2001)). However, there will be no misunderstanding if we call A whose diagonal elements are as (2) both the adjacency matrix of a graph and the coupling configuration matrix of the corresponding complex dynamical network.

where $\xi \in \mathbf{C}^n$, $A \in \mathbf{C}^{n\times n}$, is asymptotically stable if and only if the following $2n \times 2n$ real matrix

$$\overline{A} = \begin{bmatrix} Re(A) & -Im(A) \\ Im(A) & Re(A) \end{bmatrix} \quad (5)$$

is Hurwitz stable.

Proof In system (4), let $\xi(t) = \xi_1(t)+j\xi_2(t)$, $A = Re(A)+jIm(A)$. Then system (4) is transformed into

$$\dot{\xi}_1(t) + j\dot{\xi}_2(t) = [Re(A) + jIm(A)][\xi_1(t) + j\xi_2(t)] \quad (6)$$

or equivalently

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} Re(A) & -Im(A) \\ Im(A) & Re(A) \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \quad (7)$$

This leads Lemma 1 directly. ■

Lemma 2 For system (4), if $A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$ or $A = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}$, then its asymptotic stability is equivalent to the asymptotic stability of the following two complex-valued systems.

$$\dot{\xi}_1(t) = A_1\xi_1(t) \quad (8)$$

and

$$\dot{\xi}_2(t) = A_2\xi_2(t) \quad (9)$$

Lemma 3 (Schur's Decomposition) If $A \in \mathbf{C}^{N\times N}$, then there exists a unitary matrix $T \in \mathbf{C}^{N\times N}$ such that $TAT^{-1} = U$ is an upper triangular matrix, and the diagonal elements of U are the N eigenvalues of A. Matrix T is called a Schur transformation of A.

3. MAIN RESULTS

Our aim is to obtain the MSEs for the complex dynamical network (3). It is helpful to suppose firstly that the function $f(x_i)$ in (3) is a linear function, i.e., $f(x_i) = Fx_i$, where $F \in \mathbf{R}^{n\times n}$. Then the equations of system (3) are

$$\dot{x}_i(t) = Fx_i(t) + \sum_{j=1}^{N} a_{ij}\Gamma x_j(t),\ i \in \mathcal{I} \quad (10)$$

Let $x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T \in R^{Nn}$. By using the Kronecker product, system (10) can be rewritten in the following concise form

$$\dot{x}(t) = [(I_N \otimes F) + (A \otimes \Gamma)]x(t) \quad (11)$$

Theorem 1 Zero is an asymptotically stable equilibrium of dynamical network (11), if and only if zero is the asymptotically stable equilibrium of the following N linear systems

$$\dot{\xi}(t) = [F + \lambda_i\Gamma]\xi(t), \qquad i \in \mathcal{I} \quad (12)$$

where $\xi(t) \in C^n$, λ_i are the eigenvalues of A.

Proof Let $T \in C^{N\times N}$ be a Schur transformation of A, meaning the unitary matrix, such that $U = TAT^{-1}$ is upper triangular with the eigenvalues of A along the diagonal. Obviously $T \otimes I_n$ is invertible and $(T \otimes I_n)^{-1} = (T^{-1} \otimes I_n)$. Let $\tilde{x}(t) = (T \otimes I_n)x(t)$, then system (11) can be transformed into

$$\dot{\tilde{x}}(t) = (T \otimes I_n)[(I_N \otimes F) + (A \otimes \Gamma)](T \otimes I_n)^{-1}\tilde{x}(t) \quad (13)$$

We have

$$\begin{aligned}&(T \otimes I_n)[(I_N \otimes F) + (A \otimes \Gamma)](T \otimes I_n)^{-1} \\ &= [I_N \otimes F] + [(TAT^{-1}) \otimes \Gamma] \\ &= [I_N \otimes F] + [U \otimes \Gamma]\end{aligned}$$

Note that $I_N \otimes F$ is a block diagonal matrix with each identical diagonal block F and that $U \otimes \Gamma$ is block upper-triangular matrix with the diagonal block $\lambda_i\Gamma$. Therefore, by Lemma 2 the stability of system (13) is equivalent to the stability of system (12). This completes the proof. ■

In accordance to the research in the literature, we call the one-parameter linear system $\dot{\xi}(t) = [F + \alpha\Gamma]\xi(t)$ the MSE of complex dynamical network (10). It is the same in form with those in Li et al. (2003, 2004), Wang et al. (2002, 2003), Li et al. (2006), Lü et al. (2004, 2005), Xiang et al. (2007), Liu et al. (2007), Pecora et al. (1998). So we can deal with all the similar problems described in the literature for a larger variety of complex dynamical networks. The largest Lyapunov exponent L_{max} of network (10), which can be calculated from the MSE and is a function of α, is referred to as the MSF. In addition, the region S of complex number α where L_{max} is negative is called the *synchronized region* of network (10). Based on Theorem 1, network (10) is asymptotically stable if, and only if all the eigenvalues of the coupling configuration matrix belong to S. For simplicity, we directly call N linear systems (12) the MSEs of complex dynamical network (10) in this paper.

For a complex dynamical network consisting of N coupled identical linear time-invariant nodes, its stability is equivalent to that of the MSEs, which are low-dimension linear systems. Since some of the eigenvalues of A may be complex numbers, we can make use of Lemma 1 to judge the stability of systems (12).

Corollary 1 Zero is an asymptotically stable equilibrium of dynamical network (10), if and only if the following N matrices $\overline{F}_i$ are Hurwitz stable.

$$\overline{F}_i = \begin{bmatrix} F + Re(\lambda_i)\Gamma & -Im(\lambda_i)\Gamma \\ Im(\lambda_i)\Gamma & F + Re(\lambda_i)\Gamma \end{bmatrix}, i \in \mathcal{I}. \quad (14)$$

3.1 Application 1: Stability

Suppose $s \equiv 0$ is an isolated equilibrium of $\dot{s} = f(s)$. Let $\frac{\partial f(s(t))}{\partial s} |_{s=0} = F$.

Then, linearizing (3) yields

$$\dot{x}(t) = [(I_N \otimes F) + (A \otimes \Gamma)]x(t) \quad (15)$$

which is the same as (11) in form.

Therefore the previous results can be used directly. We will omit the repetitive statement of a corresponding theorem but some observations on special cases are worth noting:

Case 1. When graph $\mathcal{G}$ is undirected and connected, and suppose $A = cB$, where the off-diagonal elements of B are 0 or 1, then the eigenvalues of B are all real numbers and satisfy

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_N$$

The Hurwitz stability of matrices (14) is equivalent to that of matrices $F + c\lambda_i\Gamma$, which is consistent with the MSEs conditions in Wang et al. (2002, 2003), Li et al. (2003), Li et al. (2006).

Case 2. When graph $\mathcal{G}$ is directed, basically the eigenvalues of A are at least partly purely imaginary numbers.

The Hurwitz stability of matrices (14) can be tested by solving the following Lyapunov inequalities

$$\overline{F}_i^T P_i + P_i \overline{F}_i < -I_{2n}, \quad i \in \mathcal{I} \tag{16}$$

where $P_i \in \mathbf{R}^{2n\times 2n}$ are the symmetric positive definite solutions of (16) to be sought.

Case 3. Whatever the graph $\mathcal{G}$ is, we can get simpler but sufficient conditions to guarantee the Hurwitz stability of matrices (14), whenever the inner-coupling matrix Γ is set to be the unit matrix I_n.

Proposition 1 If the inner-coupling matrix Γ is set to be the unit matrix I_n, then the Hurwitz stability of matrices (14) can be guaranteed by the existence of symmetric positive definite solutions to the following n-dimensional Lyapunov inequalities

$$[F+Re(\lambda_i)I_n]^T P_i + P_i[F+Re(\lambda_i)I_n] < -I_n, \quad i \in \mathcal{I} \tag{17}$$

where $P_i \in \mathbf{R}^{n\times n}$ are the symmetric positive definite solutions of (17) to be sought.

Proof Let $\overline{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & P_i \end{bmatrix}$, then we have

$$\begin{aligned} &\overline{F}_i^T \overline{P}_i + \overline{P}_i \overline{F}_i \\ &= \begin{bmatrix} (F+Re(\lambda_i)I_n)^T P_i + P_i(F+Re(\lambda_i)I_n) & 0 \\ 0 & (F+Re(\lambda_i)I_n)^T P_i + P_i(F+Re(\lambda_i)I_n) \end{bmatrix} \\ &< -\begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \\ &= -I_{2n} \end{aligned}$$

therefore, $\overline{F}_i$, $i \in \mathcal{I}$, are Hurwitz stable. ■

Condition (2) guarantees that zero is an eigenvalue of the coupling configuration matrix A. So, Theorem 1 implies a fact that the isolated system

$$\dot{s}(t) = Fs(t) \tag{18}$$

must be stable if the dynamical network (10) or (11) is stable. In the following, it is shown that we can always set up a stable dynamical network only if the isolated system (18) is stable.

Proposition 2 Suppose the isolated system (18) is stable, the inner-coupling matrix Γ is the unit matrix I_n and all the coupling is nonnegative, then for any graph, directed or not, the corresponding dynamical network will be stable.

To proof proposition 2, we need a lemma about the spectral location of the adjacency matrix of a directed graph.

Lemma 4 (Murray et al. (2004)) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a directed graph. Suppose that all the coupling is nonnegative, then all the eigenvalues of A are located in the following disk:

$$D(\mathcal{G}) = \{z \in \mathbf{C}: \ |z + d_{max}| \le d_{max}\} \tag{19}$$

centered at $z = -d_{max} + 0j$ in the complex plane, where $d_{max} = \max_{i\in\mathcal{I}} |a_{ii}|$.

Remark 2 From Lemma 4, we know that all the eigenvalues of the adjacency matrix of a directed graph with nonnegative coupling should have nonpositive real parts. Obviously, if $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is an undirected graph with nonnegative coupling, then all the eigenvalues of its adjacency matrix should be nonpositive real numbers.

Proof of Proposition 2 Let λ_i, $i \in \mathcal{I}$ be the N eigenvalues of the adjacency matrix A. By Lemma 4, we have that $Re(\lambda_i) \le 0$, $i \in \mathcal{I}$.

The stability of the isolated system (18) means the matrix F is Hurwitz stable, i.e., all the eigenvalues of F have negative real parts. Therefore $F + Re(\lambda_i)I_n$, $i \in \mathcal{I}$ are all Hurwitz stable. By Proposition 1, Proposition 2 follows. ■

Proposition 2 tells us a simple fact that it is very easy to construct a stable complex dynamical network from N identical and asymptotically stable nodes. We can set the inner-coupling matrix Γ to be I_n, and randomly design a network topology so long as all its coupling is nonnegative. This also shows that the nonnegative coupling is a somewhat strong and conservative condition for a dynamical network.

3.2 Application 2: Manifold Synchronization

For a nonlinear dynamical system, basically its behavior might be very complicated. This can arise from the facts that 1) the number of its equilibrium points is usually more than one or even infinite, 2) sometimes there is no equilibrium, 3) there usually exist some invariant sets such as limit cycles, periodic orbits and 4) chaos occurs sometimes (see Kuznetsov (1995), Hale et al. (1991), Kaye (1993)). Therefore, for a complex dynamical network, it is more meaningful and practical to study the consensus behavior such as periodic orbit synchronization and chaos synchronization.

Definition 1(Synchronization)

The complex dynamical network (3) is said to achieve (asymptotic) synchronization if

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \qquad \forall i,\ j \in \mathcal{I}$$

Remark 3 Synchronization is in fact a local concept, which is only valid within a so-called *region of synchrony*. For a precise definition, see for instance Lü et al. (2004).

It is noticed that condition (2) ensures that the synchronous solution of dynamical network (3), $x_1(t) = x_2(t) = \ldots = x_N(t)$, denoted by $s(t)$, is a solution of each individual node, namely,

$$\dot{s}(t) = f(s(t)) \tag{20}$$

Therefore, the synchronization dynamics $s(t)$ in dynamical network (3) corresponds to the motion in the invariant manifold: $x_1(t) = x_2(t) = \ldots = x_N(t)$. It can be an isolated equilibrium, a periodic orbit, or even a chaotic orbit. In the following, we will adjust our focus to the periodic orbit and chaos synchronization and derive a common sufficient condition for these two kinds of manifold synchronization. To attain this target, we need to introduce a new concept for the directed graphs.

Definition 2 We call a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ *heavily connected* if the adjacency matrix A (the diagonal elements are defined as (2)) has rank $N-1$ (N is the number of the nodes).

Remark 4 The concept "heavily connected" defined here is a generalization of both the "strong connected" concept for a directed graph with nonnegative coupling and the "connected" concept for an undirected graph with non-negative coupling because the adjacency matrices of these two kinds of graphs all have rank $N-1$ (see Godsil et al. (2001)).

Example 1 Consider the following graph $\mathcal{G}_1$ (see Fig. 1), it is not strong connected because there is not a path from node 5 to node 1, but it can be heavily connected if the adjacency matrix is set to be

$$A_1 = \begin{bmatrix} -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 4 & -5 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 \end{bmatrix}$$

However, if the adjacency matrix is set to be

$$A_2 = \begin{bmatrix} -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 4 & -8 & 4 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 \end{bmatrix}$$

this graph will fail to be heavily connected. Therefore, the "heavily connected" property is determined not only by the connected structure of the graph but also by the coupling strength between each pair of nodes. This is partly why we call it "heavily connected".

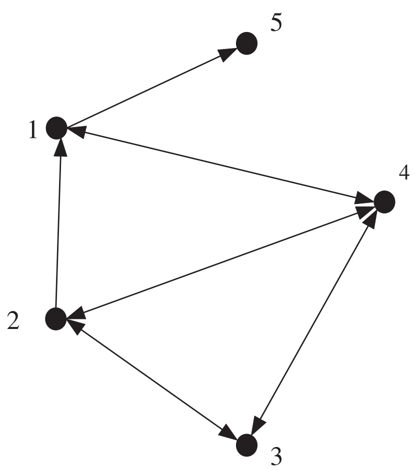

Fig. 1. The graph $\mathcal{G}_1$ in Example 1

Obviously, for a heavily connected graph, zero is an eigenvalue of the adjacency matrix with multiplicity 1 , associated with eigenvector $k(1,1,\ldots,1)^T$, $k \in \mathbf{R}, k \neq 0$.

In the following, we suppose the topology graph of dynamical network (3) is heavily connected and for simplicity, the eigenvalues of the coupling configuration matrix A are assumed to be

$$\lambda_1 = 0, \quad \lambda_i \neq 0, \quad i = 2,3,\ldots,N$$

Theorem 2 Consider the dynamical network (3), we assume the manifold $s(t)$ is a chaotic attractor of (20). If the topology graph of dynamical network (3) is heavily connected and the following $N-1$ linear time-variant systems are asymptotically stable

$$\dot{\omega}(t) = [Jf(s(t)) + \lambda_i \Gamma]\omega(t), \quad i = 2,3,\ldots,N \tag{21}$$

where $Jf(s(t))$ is the Jacobian matrix of $f(s(t))$, then asymptotic synchronization is achieved.

Proof Without loss of generality, select $x_1(t) = s(t)$ to be the reference direction of the synchronous manifold: $x_1(t) = x_2(t) = \ldots = x_N(t)$. Define the transverse errors as

$$\eta_i(t) = x_i(t) - s(t) \tag{22}$$

We have $\eta_1(t) \equiv 0$, and

$$\dot{\eta}_i(t) = f(s(t) + \eta_i(t)) - f(s(t)) + \sum_{j=1}^{N} a_{ij}\Gamma\eta_j(t), \quad i \in \mathcal{I} \tag{23}$$

According to Lü et al. (2004), the asymptotic chaos synchronization is achievable if and only if all the transverse errors $\eta_i(t)$ $(2 \leq i \leq N)$ tend to zero.

Let $\eta(t) = (\eta_1^T(t), \eta_2^T(t), \ldots, \eta_N^T(t))^T \in \mathbf{R}^{nN}$, $F(\eta(t)) = (f^T(s(t)+\eta_1(t)) - f^T(s(t)), \ldots, f^T(s(t)+\eta_N(t)) - f^T(s(t)))^T \in R^{nN}$. Systems (23) are rewritten as

$$\dot{\eta}(t) = F(\eta(t)) + (A \otimes \Gamma)\eta(t) \tag{24}$$

The corresponding linearized system of (24) at $\eta(t) = 0$ is

$$\dot{\eta}(t) = [(I_N \otimes Jf(s(t))) + (A \otimes \Gamma)]\eta(t) \tag{25}$$

Let $T \in \mathbf{C}^{N \times N}$ be a Schur transformation of A, which satisfies that $U = (u_{ij})_{N \times N} = TAT^{-1}$ is an upper triangular matrix and $u_{ii} = \lambda_i$.

Let $\xi(t) = (\xi_1^T(t), \ldots, \xi_N^T(t))^T = (T \otimes I_n)\eta(t)$. Then (25) becomes

$$\dot{\xi}(t) = [(I_N \otimes Jf(s(t))) + (U \otimes \Gamma)]\xi(t) \tag{26}$$

On one hand, we point out that in (26), $\xi_1(t) \to 0$ can be guaranteed by $\xi_j(t) \to 0$, $j = 2,3,\ldots,N$.

Let $T^{-1} = (l_{ij})_{N \times N}$. The facts $U = TAT^{-1}$, $\lambda_1 = 0$ and $Rank(A) = N-1$ imply $l_{11} \neq 0$. In fact if we let $L_1 \in R^N$ be the first column of T^{-1}, we have

$$AL_1 = 0 \tag{27}$$

Then L_1 is a eigenvector of A corresponding the zero eigenvalue. So $L_1 = k(1,\ldots,1)^T$, $k \neq 0$.

From the transformation $\xi(t) = (T \otimes I_n)\eta(t)$, we know that $\eta(t) = (T^{-1} \otimes I_n)\xi(t)$. It follows that

$$0 \equiv \eta_1(t) = \sum_{k=1}^{N} l_{1k}\xi_k(t) \tag{28}$$

Then we have

$$\xi_1(t) = -\frac{1}{l_{11}}\sum_{k=2}^{N} l_{1k}\xi_k(t) \tag{29}$$

On the other hand, we know from Lemma 2 that $\xi_j(t) \to 0$, $(j = 2,3,\ldots,N)$ is equivalent to the asymptotic stability of (21). ∎

Remark 5 Theorem 2 is applicable to periodic orbit synchronization under the weaker condition of removing the "heavily connected" restriction on the topology graph (in this case, condition (29) is not necessarily satisfied). This can be proofed by setting the transformations $\eta_i(t) = x_i(t) - s(t)$, $i \in \mathcal{I}$. It can be easily checked that (26) is also valid.

However, some differences between chaos and periodic orbit synchronization should be noted here. When $i = 1$ in (21), the linear system is $\dot{\omega}(t) = Jf(s(t))\omega(t)$, which is the corresponding linearized system of an individual node $\dot{x}_i = f(x_i)$ at $x_i = s(t)$. It is definitely unstable because $s(t)$ is a chaotic attractor of (20) (see Lü et al. (2004, 2005)). However, if $s(t)$ is an exponentially stable periodic orbit, the linear system $\dot{\omega}(t) = Jf(s(t))\omega(t)$ is spontaneously asymptotically stable.

The MSEs (21) are n-dimensional linear time-variant systems. There are not universal methods to judge their stability. Even the Lyapunov function approach may sometimes be inconvenient because of the difficulties of finding the Lyapunov functions. For most engineers, the maximal Lyapunov exponent is more welcomed since there have been so many numerical algorithms to calculate it (see Christiansen et al. (1997), Zeng et al. (1991)). The maximal Lyapunov exponent is relatively convenient to use to measure the asymptotic behaviors of a dynamical system. If the maximal Lyapunov exponent is less than zero then the system converges to a fixed point or stable periodic orbit. If the maximal Lyapunov exponent is zero then the system is neutrally stable; such systems are conservative and in a steady state mode. If the maximal Lyapunov exponent is positive then the system is unstable or even chaotic.

Numerical examples are omitted here due to the space limitation.

4. CONCLUSIONS

The master stability equations (MSEs) are preferred by many researchers to study the stability and manifold synchronization of complex dynamical networks. However, the constraints on the topology of the network in the prior work are overstrict. We have obtained the MSEs for the complex dynamical networks with general topologies. These new MSEs can be effectively used to study the stability and manifold synchronization of complex dynamical networks.

It is worthwhile pointing out that the approaches proposed in this paper can be generalized to the discrete-time case without too much endeavor. They should also be suitable to study the other problems of complex dynamical networks, such as pinning control, robustness and fragility analysis.

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