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Extensions of LaSalle's Invariance Principle for Switched Nonlinear Systems ⋆

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Abstract: In this paper the extension of LaSalle's Invariance Principle for switched nonlinear systems is studied. Unlike most existing results in which each switching mode in the system needs to be asymptotically stable, in this paper we allow the switching modes to be only stable. Under certain ergodicity assumptions of the switching signals, two extensions of LaSalle's Invariance Principle for global asymptotic stability of switched nonlinear systems are obtained, using the method of common joint Lyapunov function.

1. INTRODUCTION

In recent years, the problem of stability and stabilization of switched systems has attracted a considerable attention from control community (refer to Liberzon (1999); Agrachev (2001); Zhao (2004), et.al). They arise from many engineer problems, such as in robot manipulators (Tan (2004)), power systems (Sira-Ranirez (1991)), multi-agent models (Jadbabaie (2003); Cheng (2007); Moreau (2005)), etc. The stability of a switched system can be assured by a common Lyapunov function (CLF) of all switching modes under arbitrary switching law (Dayawansa (1999); Mancilla-Aguilar (2000)). Finding a common Lyapunov function is still an interesting and challenging problem. There is a large amount of literatures concerning it. We refer to Agrachev (2001), Cheng (2003), Shorten (2003), Hespanha (1999) and the references therein for detailed discussions.

The method of multiple Lyapunov functions is also a useful tool for stability analysis of switched systems. In comparison with common Lyapunov function, it allows each switching mode to have its own Lyapunov function (Branicky (1998)). However, as a compensation, some additional conditions are necessary to assure the value of each Lyapunov function on its corresponding mode will decrease.

In practical applications, many switched systems don't share a common Lyapunov function, yet they still may be asymptotically stable under some properly chosen switching laws. Searching certain admissible classes of switching laws is necessary for this kind of problems (Hespanha (2004)). Roughly speaking, stability can be assured if the switching is sufficiently slow. Hespanha (2004) introduced several admissible switching signals.

When the derivative of a candidate Lyapunov function with respect to each mode is only non-positive, the function is called a weak Lyapunov function (Bacciotti (2005)). In order to solve the asymptotic stability problem in such case, various extensions of LaSalle's invariance principle for switched systems have been investigated. By imposing some restrictions on the admissible trajectories, global asympotic stability results using multiple weak Lyapunov functions are obtained for switched linear systems (Hespanha (2004)). Then it is extended to switched nonlinear systems (Hespanha (2005)). A more traditional style extension of LaSalle's invariance principle is proposed in Bacciotti (2005). Its statement is closer in spirit to the classical one. But it only shows that the solution is attracted to a weakly invariant set M, and the asymptotical stability can't be obtained unless $M = \{0\}$. Under certain restrictions, another extension of LaSalle's invariance principle for switched nonlinear systems and criteria for asymptotic stability are obtained in Mancilla-Aguilar (2006).

To the best of our knowledge, all these extensions of LaSalle's invariance principle require each switching mode to be asymptotically stable. Naturally, if we do not impose certain restrictions on the switching signals, each switching mode must be asymptotically stable. Otherwise, when the system stays on a non-asymptotically-stable mode for ever, the overall system will not be asymptotically stable.

In this paper we consider the following nonlinear switched system

$$\dot{x} = f_{\sigma(t)}(x), \quad x \in \mathbf{R}^n, \tag{1}$$

where $\sigma : [0, +\infty) \to \Lambda = \{1, 2, \cdots, N\}$ is a piece-wise constant function and continuous from the right, called a switching signal (or switching law). Each $f_i(x)$ is a smooth vector field of $\mathbf{R}^n$ such that $f_i(0) = 0$, $i \in \Lambda$. Lyapunov function approach is a fundamental and powerful tool for stability analysis. It is well known that if there exists a common Lyapunov function, i.e., a positive definite C^1 function $V(x) > 0$, radially unbounded, such that

$$\dot{V}|_i = \nabla V(x) f_i(x) < 0, \quad x \neq 0, \quad i = 1, \cdots, N,$$

then the switched system is globally asymptotically stable. If we ask for globally uniformly asymptotical stability (GUAS), then the existence of a common Lyapunov function becomes necessary and sufficient (Dayawansa (1999); Mancilla-Aguilar (2000)).

Different from other results, in this paper, each mode does not need to be asymptotically stable. Under certain ergodicity assumption on the switching signals, we investigate

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two extensions of LaSalle's invariance principle, which are easily verifiable. As we have done in Cheng (2007), if the switched system is linear, the results are useful for the consensus of multi-agent systems.

The rest of this paper is organized as follows: Section 2 contains some preliminary knowledge and an introduction for a new kind of weak Lyapunov functions, called common joint Lyapunov function (CJLF). Certain properties are also investigated. Then in Sections 3 and 4, two extensions of LaSalle's invariance principle are proposed respectively. In Section 3, disjoint $Z\backslash\{0\}$ is assumed. Section 4 considers a class of $\{f_i\}$, which have a special relationship with the largest weakly invariant set contained in Z_i. Both assure the global asymptotical stability of the switched system under certain ergodicity assumptions. Section 5 is a short conclusion.

2. PRELIMINARIES

To begin with, we recall some basic concepts used in this paper.

Definition 1. The equilibrium point $x = 0$ of (1) is

- stable if for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that
$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \quad \forall\ t \geq 0;$$
- asymptotically stable if it is stable and given an $\eta > 0$, and for each $\epsilon > 0$ there exists $T > 0$ such that
$$||x(0)|| < \eta \Rightarrow ||x(t)|| < \epsilon, \quad \forall\ t > T; \tag{2}$$
- globally asymptotically stable if (2) holds for all $\eta > 0$.

It is said that the above stabilities hold "uniformly" if they hold for all switching law σ.

Definition 2. A function $V(x)$ is said to be

- positive definite, if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$;
- positive semi-definite, if $V(x) \geq 0$ for all $x \neq 0$;
- negative definite or negative semi-definite, if $-V(x)$ is positive definite or positive semi-definite.

Consider a nonlinear system
$$\dot{x} = f(x), \quad x \in \mathbf{R}^n. \tag{3}$$
By the well-known LaSalle's invariance principle (Khalil (2002)), if there exists a continuously differential, positive definite, radially unbounded function $V(x) : \mathbf{R}^n \to \mathbf{R}$ such that $\dot{V}(x) \leq 0$ for all $x \in \mathbf{R}^n$, then every solution of (3) converges to the largest invariant set M contained in $Z = \{x \in \mathbf{R}^n \mid \dot{V}(x) = 0\}$. Moreover, if $M = \{0\}$, the origin of (3) is globally asymptotically stable.

Unfortunately, the classical LaSalle's invariance principle can't be applied to switched systems directly. For switched systems, there are also some extended results of LaSalle's invariance principle as we have mentioned in Section 1. Among them, certain restrictions on the switching signals are necessary. A switched system is said to have a non-vanishing dwell time, if there exists a positive time period $\tau_0 > 0$, such that the switching instances $\{\tau_k \mid k = 1, 2, \cdots\}$ satisfy
$$\inf_k(\tau_{k+1} - \tau_k) \geq \tau_0. \tag{4}$$

Through this paper we assume

A1. Admissible switching signals have a dwell time $\tau_0 > 0$.

We need to recall another concept: weakly invariant set.

Definition 3. (Bacciotti (2005)) A compact set M is weakly invariant with respect to (1), if for each point $x \in M$, there exist a $\lambda \in \Lambda$, a solution $\varphi(t)$ of the vector field $f_\lambda(x)$ and a real number $b > 0$ such that $\varphi(0) = x$ and $\varphi(t) \in M$ for either $t \in [-b, 0]$ or $t \in [0, b]$.

Now for system (1) assume $V(t)$ is the candidate Lyapunov function concerned, we denote by $Z_i = \{x \mid \dot{V}(x)|_{f_i} = 0\}, \ \forall\ i \in \Lambda$.

With some mild modification, we state Theorem 1 of Bacciotti (2005) as

Proposition 4. (Bacciotti (2005)) Assume system (1) has a CWLF,
$$Z = \bigcup_{i\in\Lambda} Z_i,$$
and M is the largest weakly invariant set contained in Z. Then every solution $\varphi(t, x_0)$ of system (1) is attracted to M.

This result is the starting point of our following discussion.

Since we only require each mode to be stable, in addition to **A1**, we need to pose certain ergodicity property for switching signals.

A2. For any $T > 0$, and any $\lambda \in \Lambda$, there exists $t > T$ such that
$$\sigma(t) = \lambda. \tag{5}$$

Or a stronger assumption is

A2$'$. There exists a $T > 0$, such that for any $t_0 \geq 0$,
$$\{t \mid \sigma(t) = \lambda\} \bigcap [t_0, t_0 + T] \neq \emptyset, \forall\ \lambda \in \Lambda. \tag{6}$$

Remark.

- *Assumptions A1 and A2 imply that each mode will be active infinite times and the total time length for each mode i being active is infinity, i.e.,*
$$|\{t \mid \sigma(t) = \lambda\}| = \infty, \quad \forall\ \lambda \in \Lambda,$$
where $|\cdot|$ denotes the Lebesgue measure. We call such a switching "ergodic switching".
- *A2$'$ may be called "finite time ergodic switching". It is easy to see that A2$'$ implies A2.*
- *If both A1 and A2$'$ hold, then there exists $T > 0$ (replacing original T of A2$'$ by $T + \tau_0$) such that*
$$|\{t \mid \sigma(t) = \lambda\} \bigcap [t_0, t_0 + T]| \geq \tau_0, \ \forall\ \lambda \in \Lambda, \ t_0 \geq 0. \tag{7}$$

Next, we recall a new Lyapunov-type function, called the joint Lyapunov function. The following definition is mimic to the linear case in Cheng (2007).

Definition 5. Consider system (1).

- If there exists a positive definite C^1 function $V(x) > 0$, radially unbounded, such that
$$\dot{V}(x)|_{f_i} = \nabla V(x) f_i(x) := Q_i(x) \leq 0, \quad x \neq 0, \tag{8}$$
$$Q_i(0) = 0, \quad i \in \Lambda,$$
then $V(x)$ is called a common weak Lyapunov function (CWLF) of system (1).

- A common weak Lyapunov function of system (1) is called a common joint Lyapunov function (CJLF) if

$$\sum_{i=1}^{N} Q_i(x) < 0, \quad x \neq 0. \tag{9}$$

Remark. *For a switched linear system*

$$\dot{x} = A_{\sigma(t)}x, \quad x \in \mathbf{R}^n, \tag{10}$$

where $\sigma : [0, +\infty) \to \Lambda = \{1, 2, \cdots, N\}$ *is the switching signal. If there exists a quadratic function* $V(x) = x^T Px$ *with* $P > 0$ *satisfying*

- $PA_i + A_i^T P = Q_i \leq 0, \quad i \in \Lambda;$
- $Q := \sum_{i=1}^{N} Q_i < 0.$

Then $V(x)$ *(or briefly, P) is called a common joint quadratic Lyapunov function (CJQLF) of system (10).*

According to the definition, we get the following property at once.

Proposition 6. For system (1), assume there exists a CWLF $V(x) > 0$, then V is a CJLF if and only if

$$\bigcap_{i\in\Lambda} Z_i = \{0\}, \tag{11}$$

where $Z_i = \{x \mid Q_i(x) = 0\}$ is the kernel of $Q_i, \ i \in \Lambda$.

Proof. ($\Rightarrow$) Obviously, $0 \in Z_i, \ i \in \Lambda$. If there exists $0 \neq \eta \in \bigcap_{i\in\Lambda} Z_i$, then $Q_i(\eta) = 0, \ \forall \ i \in \Lambda$ which implies $\sum_{i\in\Lambda} Q_i(\eta) = 0$, a contradiction.

($\Leftarrow$) If $V(x)$ is not a CJLF, then there exists $\xi \neq 0$ such that $\sum_{i\in\Lambda} Q_i(\xi) = 0$. Since every $Q_i(x)$ is negative semi-definite, then $Q_i(\xi) = 0, \ \forall \ i \in \Lambda$, that is, $\xi \in Z_i, \ \forall \ i \in \Lambda$, which is a contradiction to (11). □

Unfortunately, under the assumptions of $A1$ and $A2$ (or $A2'$), even for a switched linear system, a CJLF is not enough to assure the global asymptotical stability. Cheng (2007) gave a counter example.

Therefore, in addition to $A1, A2$ ($A2'$) and the existence of CJLF, in the next two sections we will give some additional conditions to assure the system being globally asymptotically stable.

3. LASALLE'S INVARIANCE PRINCIPLE FOR DISCONNECTED $Z \setminus \{0\}$

Now we present our first LaSalle type of stability result.

Theorem 7. Consider system (1). Assume

- A1, A2 hold;
- there exists a CJLF;
- $Z \setminus \{0\}$ is disconnected, where $Z = \bigcup_{i\in\Lambda} Z_i$ and Z_i is the kernel of $Q_i, \ i \in \Lambda$.

Then system (1) is globally asymptotically stable.

Proof. By the common weak Lyapunov function, system (1) is stable. Then we only need to prove the convergence.

For any x_0, construct a nonempty compact set

$$W = \{x \in \mathbf{R}^n \mid V(x) \leq V(x_0)\}.$$

Since $\bigcup_{i\in\Lambda} Z_i \backslash \{0\}$ is disconnected, without loss of generality, we assume it is composed of two connected components, denoted by

$$Z_I = \bigcup_{i\in I} Z_i \setminus \{0\}, \quad Z_J = \bigcup_{j\in J} Z_j \setminus \{0\},$$

where $I \bigcup J = \Lambda$ and $I \bigcap J = \emptyset$.

Define $N_I = \{x \in W \mid d(x, Z_I) < \epsilon_0\}$, $N_J = \{x \in W \mid d(x, Z_J) < \epsilon_0\}$, and $N_I^c = W \setminus N_I$, $N_J^c = W \setminus N_J$, where $\epsilon_0 > 0$ can be chosen properly. Then under subspace topology $N_I, \ N_J$ are open sets containing 0 and $N_I^c, \ N_J^c$ are compact sets.

For any $\epsilon > 0$, let $W_\epsilon = \{x \in W \mid ||x|| < \epsilon\}$. We can choose $\epsilon_0 > 0$ small enough such that $N_I \bigcap N_J \subset W_\epsilon$ and $\bar{N}_I \backslash W_\epsilon$ and $\bar{N}_J \backslash W_\epsilon$ are disjoint. Let $d = d(\bar{N}_I \backslash W_\epsilon, \ \bar{N}_J \backslash W_\epsilon) > 0$.

Note that when $i \in I$ mode is active, $\dot{V}(x)|_{f_i} < 0, \ \ \forall \ x \in N_I^c$, then there exists a $\delta_I > 0$ such that $\max_{x\in N_I^c, \ i\in I} \dot{V}(x)|_{f_i} = -\delta_I < 0$. Similarly, there exists a $\delta_J > 0$ such that $\max_{x\in N_J^c, \ i\in J} \dot{V}(x)|_{f_j} = -\delta_J < 0$ and $\max_{x\in N_I^c \cap N_J^c, \ i\in\Lambda} \dot{V}(x)|_{f_i} = -\delta < 0$ with $\delta = \max\{\delta_I, \ \delta_J\}$.

We claim that there exists $T > 0$ such that

$$x(t) \in N_I \bigcap N_J \subset W_\epsilon, \ \forall \ t > T, \tag{12}$$

where $x(t)$ is any solution of system (1).

We prove it case by case as follows:

(i) If $x(t) \in (N_I \bigcup N_J)^c$, then no matter which mode is active, $V(x)$ decreases strictly, because $\dot{V}(x)|_{f_i} \leq -\delta, \ \ \forall \ i \in \Lambda$. Then we have

$$V(x(t + \triangle t)) \leq V(x(t)) - \delta\triangle t. \tag{13}$$

(13) remains true as long as $x(t)$ stays in $(N_I \cup N_J)^c$. Then $V(x(t + \triangle t)) \to -\infty$ as $\triangle t \to \infty$. Therefore, we assume $x(t)$ will not stay in $(N_I \bigcup N_J)^c$ for ever.

(ii) If $x \in N_I \bigcup N_J$, $V(x)$ remains non-increasing. Since the switching set is ergodic, system (1) can not dwell on any one mode for ever.

If $x(t)$ enters N_I (same for N_J) only finite times, then after a $T_0 > 0$, the trajectory will stay in N_I^c for ever. Then

$$V(x(t)) < V(x(T_0)) - \delta_I \tau, \tag{14}$$

where

$$\tau = |\{T_0 < s < t \mid \sigma(s) \in I\}|.$$

Since as $t \to \infty$, $\tau \to \infty$, we have $V(x(t)) \to -\infty, \ t \to \infty$, a contradiction.

(iii) Assume $x(t)$ travels between $N_I \setminus W_\epsilon$ and $N_J \setminus W_\epsilon$ infinite times. Since $f_i(x)$ is continuous, there exists $b_i > 0$ such that as mode i is active, $\|\dot{x}(t)\| = \|f_i(x)\| \leq b_i, \ x \in (N_I \bigcup N_J)^c$. Taking $0 < b = \max_{i\in\Lambda} b_i$, then the time that $x(t)$ travels between $N_I \backslash W_\epsilon$ and $N_J \backslash W_\epsilon$ satisfies $|\triangle t| \geq \frac{d}{b}$. Denote $W_0 = W_\epsilon^c \bigcap N_I^c \bigcap N_J^c$. Then there exists an infinite time sequence $t_1, t_2, \cdots$ at which $x(t)$ goes through the following regions: $N_I \xrightarrow{t_1} W_0 \xrightarrow{t_2} N_J \xrightarrow{t_3} W_0 \xrightarrow{t_4} N_I \xrightarrow{t_5} W_0 \xrightarrow{t_6} \cdots$, with $x(t) \in W_0$ for $t \in [t_{2k-1}, t_{2k}]$ and $t_{2k} - t_{2k-1} \geq \frac{d}{b}$. By (13)

$$V(x(t_{2k})) \leq V(x(t_{2k-1})) - \delta\frac{d}{b} \leq V(x(t_{2k-3})) - 2\delta\frac{d}{b}$$
$$\leq \cdots \leq V(x(t_1)) - k\delta\frac{d}{b} \to -\infty,\ k \to \infty,$$

a contradiction.

Therefore, after a finite time, the trajectory of $x(t)$ will stay in $N_I \bigcap N_J$ for ever, which means (12) holds. The conclusion follows. □

Taking Proposition 6 into consideration, the second condition in Theorem 7 can be replaced by CWLF, because CWLF plus the third condition implies CJLF.

Also note that when $N = 2$, we have $Z_1 \bigcap Z_2 = \{0\}$, so condition 3 is automatically satisfied. This observation leads to

Corollary 8. Theorem 7 remains true if the last condition is replaced by $N = 2$.

Taking Proposition 4 into consideration, we have the following stronger result.

Corollary 9. Let M be the largest weakly invariant set contained in Z. Then Theorem 7 remains true if in the last condition $Z \setminus \{0\}$ is replaced by $M \setminus \{0\}$.

Remark. *Obviously, Theorem 7 is also true for switched linear system (10). To assure the global asymptotical stability, we can find a CJQLF.*

Before ending this section, we give two simple examples to illustrate the effectiveness of our theorem. The first example is for the linear case.

Example 10. Consider the following switched system

$$\dot{x} = A_{\sigma(t)}x, \quad x \in \mathbf{R}^3, \tag{15}$$

where $\sigma(t) \in \Lambda = \{1, 2, 3\}$,

$$A_1 = \begin{bmatrix} 0 & -3 & -2 \\ 3 & -9 & -5 \\ -3 & 9 & 5 \end{bmatrix}, A_2 = \begin{bmatrix} -4 & 3 & 1 \\ -6 & 5 & 2 \\ 8 & -7 & -3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 4 & -6 & -2 \\ 8 & -12 & -4 \\ -9 & 12 & 3 \end{bmatrix}.$$

Choosing

$$P = \begin{bmatrix} 5 & -4 & -1 \\ -4 & 6 & 3 \\ -1 & 3 & 2 \end{bmatrix} > 0,$$

Then $Q_i = PA_i + A_i^T P$, which are

$$Q_1 = \begin{bmatrix} -18 & 21 & 8 \\ 21 & -30 & -13 \\ 8 & -13 & -6 \end{bmatrix} \leq 0, \quad Q_2 = \begin{bmatrix} -8 & 6 & 2 \\ 6 & -6 & -3 \\ 2 & -3 & -2 \end{bmatrix} \leq 0,$$

$$Q_3 = \begin{bmatrix} -6 & 11 & 5 \\ 11 & -24 & -13 \\ 5 & -13 & -8 \end{bmatrix} \leq 0.$$

And

$$Q = Q_1 + Q_2 + Q_3 = \begin{bmatrix} -32 & 38 & 15 \\ 38 & -60 & -29 \\ 15 & -29 & -16 \end{bmatrix} < 0.$$

Obviously,

$$Z_1 = \{x \in \mathbf{R}^3 \mid x_1 = x_2 = 0\},$$
$$Z_2 = \{x \in \mathbf{R}^3 \mid x_1 = x_3 = 0\},$$
$$Z_3 = \{x \in \mathbf{R}^3 \mid x_2 = x_3 = 0\},$$

and $\bigcup_{i=1}^{3} Z_i \setminus \{0\}$ is not connected.

We conclude by Theorem 7 that system (15) is globally asymptotically stable if the switching signal satisfies $A1$ and $A2$. Choose the initial values $[6,\ 1,\ -5]^{\mathrm{T}}$. Fig.1- Fig.3 show the convergence of each component of system (15) with $T = 2$ and different dwell time τ_0. □

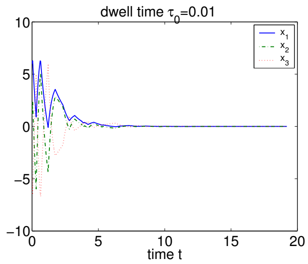

Fig. 1. the convergence of system (15) with $\tau_0 = 0.01$

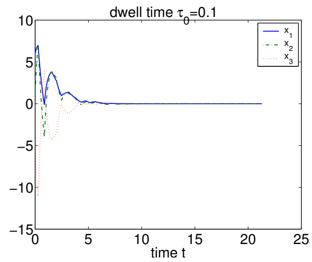

Fig. 2. the convergence of system (15) with $\tau_0 = 0.1$

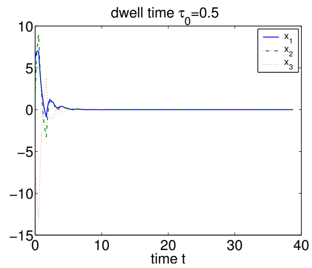

Fig. 3. the convergence of system (15) with $\tau_0 = 0.5$

Example 11. Consider the following switched system

$$\dot{x} = f_{\sigma(t)}(x), \quad x \in \mathbf{R}^2, \tag{16}$$

where $\sigma(t) \in \Lambda = \{1, 2\}$ and

$$f_1(x) = \begin{pmatrix} -(2x_2)^k \\ -2^{k-1}x_2^k \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} -(x_1 - 2x_2)^k \\ 0 \end{pmatrix},$$

$k \geq 1$ is an odd integer. Obviously, every switching mode is stable, but not asymptotically stable. Choose $V(x) = (x_1 - 2x_2)^2 + 4x_2^2$, then

$$Q_1(x) := \dot{V}(x)|_{f_1} = -2(2x_2)^{k+1} \leq 0,$$
$$Q_2(x) := \dot{V}(x)|_{f_2} = -2(x_1 - 2x_2)^{k+1} \leq 0,$$
$$Q_1(x) + Q_2(x) = -2[(2x_2)^{k+1} + (x_1 - 2x_2)^{k+1}] < 0,$$
$$\forall\ (x_1, x_2) \neq (0, 0).$$

Therefore, $V(x)$ is a CJLF. We get by Corollary 8 that system (16) is globally asymptotically stable if the switching signal satisfies $A1$ and $A2$. □

4. LASALLE'S INVARIANCE PRINCIPLE FOR A CLASS OF F_I

In this section, we impose certain constrains on system (1). We need some preparations first.

Lemma 12. Consider system (1). Assume every switching mode is stable. Denote $K_i = ker(f_i) = \{x \mid f_i(x) = 0\}$, $K = \bigcap_{i\in\Lambda} K_i$, and let $y \in K$. Assume the switching signal satisfies $A1$ and $A2'$, then for any $R > 0$, there exists $r > 0$, such that if $x_0 \in B_r(y)$ then

$$\varphi(t, x_0) \in B_R(y), \quad 0 \le t \le T, \tag{17}$$

where $\varphi(t, x_0)$ is the solution of system (1) with $\varphi(0, x_0) = x_0$ and T is the same as in $A2'$.

Proof. Since every switching mode is stable, $y \in K$ is a stable equilibrium for every switching mode. Then for any $R > 0$, we can find $r_i > 0$ ($i \in \Lambda$), associated with every subsystem of (1), such that as long as $\|x_0 - y\| < r_i$, $\|\varphi(t, x_0) - y\| < R$, $t \ge 0$.

Now suppose the switching moments over $[0, T]$ are t_i, $i = 1, 2, \cdots, s$. Denote $x_i = \varphi(t_i, x_0)$, $i = 1, 2, \cdots, s$. Since every switching mode is stable, for any $R > 0$, there exists $0 < R_s < R$ such that $\|x_s - y\| < R_s$ implies $\|\varphi(t, x_s) - y\| < R$, $t_s \le t \le T$. For $R_s > 0$, there exists $0 < R_{s-1} < R_s$ such that $\|x_{s-1} - y\| < R_{s-1}$ implies $\|\varphi(t, x_{s-1}) - y\| < R_s$, $t_{s-1} \le t \le t_s$. Continuing this argument, then for $R_1 > 0$, there exists $0 < r < R_1$ such that $\|x_0 - y\| < r$ implies $\|\varphi(t, x_0) - y\| < R_1$, $0 \le t \le t_1$. From the above procedure, it follows that as long as $x_0 \in B_r(y)$, (17) holds. □

Lemma 13. $\ker(f_i) \subset \ker(Q_i)$, $\forall\, i \in \Lambda$.

Proof. For any $x_0 \in \ker(f_i)$, we have $f_i(x_0) = 0$. Then $Q_i(x_0) = \dot V(x_0)|_{f_i} = \nabla V(x_0) f_i(x_0) = 0$. The conclusion follows. □

Denote by M the largest weakly invariant set contained in $Z = \bigcup_{i\in\Lambda} Z_i$, and let

$$V_i = M \bigcap Z_i, \quad i \in \Lambda.$$

It is easy to see that $\ker(f_i)$ itself is a weakly invariant set contained in $Z_i \subset Z$, hence $\ker(f_i) \subset V_i$. Next, we give one more assumption.

A3. $\ker(f_i) = V_i$, $i \in \Lambda$.

The next proposition was obtained in Bacciotti (2005), which gives a property of the ω-limit set.

Proposition 14. (Bacciotti (2005)) Let $\varphi(t, x_0)$ be a solution of system (1) with dwell time τ_0. $\Omega(x_0)$ is its ω-limit set. Then $\Omega(x_0)$ is a weakly invariant set contained in Z.

Now we are ready to state our second main result.

Theorem 15. Consider system (1). Assume $A1$, $A2'$ and $A3$ hold and there exists a CJLF, then system (1) is globally asymptotically stable.

Proof. Let $x(t) = \varphi(t, x_0)$ be any solution of system (1) with $\varphi(0, x_0) = x_0$. Since $V(x)$ is monotonically not increasing and bounded, we have

$$\lim_{t\to\infty} V(x(t)) = V_0.$$

If $V_0 = 0$, we are done. So we assume $V_0 > 0$ and will draw a contradiction.

Since $x(t)$ is bounded, then there exists an infinite sequence $\{t_k\}$ such that

$$x_k := x(t_k) \to y, \quad t \to \infty,$$

and $\lim_{k\to\infty} V(x(t_k)) = V(y) = V_0$. Now since y is an ω-limit point, by Proposition 14, we have $y \in M \subset Z$ and by the assumption $V_0 > 0$, $y \ne 0$.

Split Λ into two disjoint subsets, $I \subset \Lambda$ and $J = \Lambda \setminus I$, satisfying

$$y \in Z_i, \forall\, i \in I, \quad y \notin Z_j, \forall\, j \in J.$$

Since $y \in M$, thus $I \ne \emptyset$ and $y \in V_i$, $\forall\, i \in I$. According to Proposition 6, $J \ne \emptyset$.

Denote

$$d = \min_{j\in J} d(y, Z_j) > 0, \tag{18}$$

we can choose $0 < R < d/2$ and define a ball $B_R(y) = \{x \mid ||x - y|| < R\}$. Then we have

$$d(x, Z_j) > R, \quad \forall\, x \in B_R(y), \quad j \in J. \tag{19}$$

For any $x \in \bar B_R(y)$, the closure of $B_R(y)$, when mode $j \in J$ is active, we have

$$\dot V(x(t))|_{f_j} < 0.$$

Since $\bar B_R(y)$ is compact, there exists an $\alpha > 0$ such that $\max_{x\in \bar B_R(y), j\in J} \dot V(x(t))|_{f_j} = -\alpha < 0$.

Now assume $0 < R_1 < R$ is small enough such that as $x_0 \in B_{R_1}(y)$, $x(t) \in B_R(y)$, $\forall\, t \in [t_0, t_0 + \tau_0]$. Then when $x_0 \in B_{R_1}(y)$ and t_0 is the moment when mode $j \in J$ becomes active, we have

$$V(x(t_0 + \tau_0)) < V(x_0) - \alpha\tau_0. \tag{20}$$

On the other hand, using Lemma 12 associated with assumption $A3$, we can find $0 < r < R_1$ such that when $x_0 \in B_r(y)$ and only modes $i \in I$ are active, we have

$$\varphi(t, x_0) \in B_{R_1}(y), \quad 0 \le t \le T. \tag{21}$$

Since y belongs to the ω-limit set, there exists $N > 0$ such that $x_k \in B_r(y)$ for all $k > N$. Recalling assumption $A2'$, the finite time ergodic property, on every interval $[t_k, t_k + T]$, all the modes will be active at least once. Let $t'_k \in [t_k, t_k + T]$ be the moment when a $j \in J$ mode is triggered, then by (21), $\varphi(t'_k, x_k) \in B_{R_1}(y)$. According to (20), we get

$$V(x(t'_k + \tau_0)) < V(x(t'_k)) - \alpha\tau_0, \quad \forall\, k > N.$$

Then

$$\begin{aligned} V(x(t'_{N+l} + \tau_0)) \le & V(x(t'_{N+l})) - \alpha\tau_0 \\ \le & V(x(t'_{N+l-1})) - 2\alpha\tau_0 \le \cdots \\ \le & V(x(t'_{N+1}) - l\alpha\tau_0 \to -\infty, \quad l \to \infty, \end{aligned}$$

which is a contradiction. □

In general, it is not straightforward to verify A3. We thus give a sufficient condition here.

Proposition 16. If $\ker(f_i) = Z_i$, $\quad i \in \Lambda$, then A3 is satisfied.

Proof. If $\ker(f_i) = Z_i$, then $V_i \subset \ker(f_i)$. The conclusion follows. □

Remark. *In general, for nonlinear switched systems, it is not easy to get the global asymptotic stability result. Sometimes, we only need the local stability. If the Lyapunov function is defined on a neighborhood of the origin which is a compact set, then the conclusions of Theorem 7 and 15 hold locally.*

Example 17. Consider the following switched system

$$\dot{x} = f_{\sigma(t)}(x), \quad x \in \mathbf{R}^4, \tag{22}$$

where $\sigma(t) \in \Lambda = \{1, 2, 3\}$,

$$f_1(x) = \begin{pmatrix} -x_1^5 \\ x_1^3 x_2 - x_2^3 \\ 0 \\ -2x_4^3 - x_3^2 x_4 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} 0 \\ 0 \\ -x_3^3 \\ 2x_3^2 - 3x_4 \end{pmatrix},$$

$$f_3(x) = \begin{pmatrix} 0 \\ -2x_2^3 + x_2 x_3^2 \\ -x_3^3 \\ 0 \end{pmatrix}.$$

Choosing $V(x) = \frac{1}{2}\sum_{i=1}^{4} x_i^2$, then

$$Q_1(x) := \dot{V}(x)|_{f_1} = -(x_1^3 - \frac{1}{2}x_2^2)^2 - \frac{3}{4}x_2^4 - 2x_4^4 - x_3^2 x_4^2 \leq 0,$$

$$Q_2(x) := \dot{V}(x)|_{f_2} = -(x_3^2 - x_4)^2 - 2x_4^2 \leq 0,$$

$$Q_3(x) := \dot{V}(x)|_{f_3} = -2(x_2^2 - \frac{1}{4}x_3^2)^2 - \frac{7}{8}x_3^4 \leq 0.$$

Obviously, $\sum_{i=1}^{3} Q_i(x) < 0, \ \forall\, x \neq 0$. Thus, V is a CJLF. In a addition,

$$\ker(f_1) = Z_1 = \{x \mid x_1 = x_2 = x_4 = 0\}$$
$$\ker(f_2) = Z_2 = \{x \mid x_3 = x_4 = 0\}$$
$$\ker(f_3) = Z_3 = \{x \mid x_2 = x_3 = 0\}.$$

According to Theorem 15 we conclude that system (22) is globally asymptotically stable if the switching signals satisfy $A1$ and $A2'$. □

5. CONCLUSION

In this paper, we investigated the stability of switched nonlinear systems. By introducing common joint Lyapunov function, two extensions of LaSalle's invariance principle were obtained. Unlike traditional extensions, our results do not require individual switching modes to be asymptotically stable, while certain ergodicity restrictions are imposed on the switching signals. It has been shown that in a practical dynamic process, such as joint connection of multi-agent systems (Jadbabaie (2003); Moreau (2005)), ergodicity assumption is reasonable.

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