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Controller Design for Minimum-Phase Fractional Systems of Commensurate Order Based on Shaping the Sensitivity Function

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Abstract: This paper is concerned with the problem of designing a controller for certain class of fractional-order systems. The powers of the Laplace variable, s, are limited to rational numbers and the plant transfer function is assumed to be minimum-phase. The approach used in this paper is based on shaping the sensitivity function which is a powerful design algorithm in frequency domain. One advantage of the proposed method is that it does not need heavy computational efforts. It is a well known result that control objectives such as command tracking and noise attenuation can be expressed in terms of the sensitivity function. The notion of coprime factorization is also developed for the systems under consideration and two illustrative examples are presented.

Keywords: Fractional systems; Analytic design; Time-invariant systems.

1. INTRODUCTION

In recent years there has been an increasing attention to fractional-order systems. These systems are of interest for both modelling and control purposes. In the fields of continuous-time modelling, fractional derivatives have been used in linear viscoelasticity, acoustics, rheology, polymeric chemistry etc (Oldham and Spanier, 1974; Hilfer, 2000). In general, fractional-order systems are useful to model various stable physical phenomena (commonly diffusive systems) with anomalous decay, say those that are not of exponential type. For example, Miller and Ross (1993) introduce a system with impulse response

$$h(t) = \frac{\sqrt{2g\pi}}{\Gamma(\frac{3}{2})} t_+^{\frac{1}{2}},$$

which is not of exponential type. Fractional differential systems are also used in control field. Among others, an extension of the classical PID controllers, known as fractional-order PID (FOPID) or $PI^\lambda D^\mu$ (Podlubny, 1994, 1999), and the so-called CRONE (Oustaloup et al., 1996) control are of more interest. Most of the existing controller tuning methods are based on numerical optimization algorithms. Monje et al. (2005) proposed a method for auto-tuning of a fractional order lead-lag compensator using relay feedback tests, which is robust in some sense.

For integer-order systems, the problem of *design for performance* is to find a proper $C(s)$ for which the standard feedback system of Fig. 1 is internally stable and

$$\|W(s)S(s)\|_\infty < 1, \tag{1}$$

where $S(s) \triangleq 1/[1 + C(s)P(s)]$ and $W(s)$ are the sensitivity and the weight functions, respectively. By choosing a suitable weight function and solving (1), the control objectives such as command tracking and noise rejection are achieved. This problem has been fully solved for integer case, i.e. when the transfer functions of the plant $P(s)$ and the controller $C(s)$ are of integer orders (Doyle et al., 1990). This paper is to address the above problem for fractional systems of commensurate order. The powers of the Laplace variable, s, in such transfer functions are rational numbers rather than integer ones. The studies in this paper are restricted to rational powers because this allows the use of some algebraic tools (Miller and Ross, 1993). Note that, in practice, all numbers are rational. It is due to the fact that all numbers are stored with a limited precision in computer. For simplicities, the term “fractional system of commensurate order” is addressed with “fractional-order system” in the rest of this paper.

The rest of this paper is organized as follows. Problem preliminaries are presented in Section 2, controller parametrization is discussed in Section 3, and the controller design algorithms for stable and unstable plants are presented in Sections 4 and 5, respectively. Examples are also provided and, finally, Section 6 concludes the paper.

2. PROBLEM PRELIMINARIES

Consider the standard closed-loop system shown in Fig. 1, where $P(s)$ is a (multi-valued) fractional-order transfer function in the form of

$$P(s) = \frac{b_m s^{\frac{m}{v}} + b_{m-1}s^{\frac{m-1}{v}} + \ldots + b_1 s^{\frac{1}{v}} + b_0}{s^{\frac{n}{v}} + a_{n-1}s^{\frac{n-1}{v}} + \ldots + a_1 s^{\frac{1}{v}} + a_0}. \tag{2}$$

The domain of definition for $P(s)$ is a Riemann surface with v Riemann sheets where origin is the branch point (of order $v-1$) and the branch-cut is assumed at $\mathbb{R}^-$ (LePage, 1961). By definition, (2) is *strictly proper* and *proper* if $n > m$ and $n \geq m$, respectively. Another useful definition is the *relative degree* of (2) which is equal to $n - m$. Note that every fractional-order system can be represented in the form of (2). For example, the transfer function

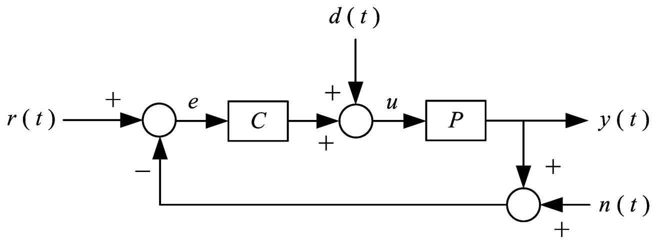

Fig. 1. Standard closed-loop system

$$P(s) = \frac{s^{\frac{1}{2}}+1}{s^{\frac{1}{3}}+1},$$

can be represented in the equivalent form

$$P(s) = \frac{s^{\frac{3}{6}}+1}{s^{\frac{2}{6}}+1},$$

which is in the form of (2). Substituting $s^{\frac{1}{v}}$ with z in (2) leads to another transfer function which is denoted by $\widetilde{P}(z)$ in this paper. The domain of definition of $\widetilde{P}(z)$ is called z-plane.

It is a well-known fact that (2) is BIBO stable if and only if all roots of the equation

$$z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0,$$

lie in the sector

$$|\arg(z)| > \frac{\pi}{2v}, \tag{3}$$

in z-plane, where $z \triangleq s^{\frac{1}{v}}$ (Matignon, 1998). The above condition is equivalent to $P(s)$ having no pole in the closed right half-plane (CRHP) of the first Riemann sheet. Likewise, $P(s)$ is *minimum-phase* if and only if all roots of the equation

$$b_m z^m + b_{m-1}z^{m-1} + \ldots + b_1 z + b_0 = 0,$$

lie in the sector defined by (3). Note that the above definition is equivalent to $P(s)$ having no zero in the CRHP of the first Riemann sheet. It is a natural way to extend the definition of minimum-phase-ness to systems under consideration as it implies that $P^{-1}(s)$ is stable if and only if $P(s)$ is minimum-phase.

As all calculations in this paper are performed in frequency domain we have to evaluate functions like (2) when $s = j\omega$. But the term $s^{\frac{1}{v}}$ in (2) is a multi-valued function of s, i.e., for every $\omega \in [0,\infty)$ there are v possible values for the $(j\omega)^{\frac{1}{v}}$. It is a fact that in multi-valued functions only the first Riemann sheet has its physical significance (Beyer and Kempfle, 1995; Gross and Braga, 1961). So, the principal branch of $(j\omega)^{\frac{1}{v}}$ is considered in this paper.

Let k be a positive integer and τ a positive real number. Define $J(s)$ as

$$J(s) \triangleq \frac{1}{\tau s^{\frac{k}{v}}+1}. \tag{4}$$

Let us discuss the stability properties of J. It follows from (4) that

$$\widetilde{J}(z) = \frac{1}{\tau z^k + 1},$$

the poles of which are calculated as

$$z = \frac{1}{\sqrt[k]{\tau}} e^{j\frac{(2h+1)\pi}{k}}, \quad h = 0, 1, \ldots, k-1.$$

According to (3), J is stable if and only if

$$\frac{\pi}{k} > \frac{\pi}{2v},$$

or equivalently

$$k < 2v. \tag{5}$$

The following lemma will be instrumental in the analysis to follow.

Lemma 1. If $G(s)$ is a stable and strictly proper fractional-order transfer function, and J is the complex function defined by (4) where the stability condition (5) is satisfied, then

$$\lim_{\tau\to 0} \|G(1-J)\|_\infty = 0.$$

Proof. Let $\epsilon > 0$ and $\omega_1 > 0$ be two arbitrarily chosen real numbers. Observing the Bode plot of J, if τ is sufficiently small, then the Nyquist plot of J lies in the disk with center 1 and radius ϵ for $\omega \le \omega_1$, and the disk with center 0 and radius 1 for $\omega > \omega_1$. Now $\|G(1-J)\|_\infty$ is the maximum of

$$\max_{\omega\le\omega_1} |G(j\omega)[1-J(j\omega)]|,$$

and

$$\max_{\omega>\omega_1} |G(j\omega)[1-J(j\omega)]|.$$

Again, using the Nyquist plot of J, it is obvious that the first of these is upper bounded by $\epsilon\|G\|_\infty$, and the second by

$$\|1-J\|_\infty \max_{\omega>\omega_1} |G(j\omega)|.$$

Now considering the fact

$$\|1-J\|_\infty \le \|1\|_\infty + \|J\|_\infty = 2,$$

it is concluded that

$$\|G(1-J)\|_\infty \le \max\{\epsilon\|G\|_\infty, 2\max_{\omega>\omega_1}|G(j\omega)|\}.$$

This holds for τ sufficiently small. But the right-hand side can be made arbitrarily small by suitable choice of ϵ and ω_1 because

$$\lim_{\omega_1\to\infty}\max_{\omega>\omega_1}|G(j\omega)| = |G(j\infty)| = 0.$$

Now it is evident that for every $\delta > 0$, if τ is small enough, then

$$\|G(1-J)\|_\infty \le \delta.$$

This completes the proof.

Note that the form considered for the function J is not unique. For example, one may consider J as

$$J(s) \triangleq \frac{1}{(\tau s^{\frac{1}{v}}+1)^k}, \tag{6}$$

which satisfies all the required conditions for our purpose but obviously it is more complicated than (4) and needs more computational efforts and it leads to controllers with more complicated structures. However, the form of (6) has the advantage that it is stable for every $k \in \mathbb{N}$. We confine our developments in this paper to (4).

3. CONTROLLER PARAMETRIZATION

It can easily be verified that if an integer-order system with transfer function $\widetilde{P}(z)$ be stable and proper, then the corresponding fractional-order transfer function $P(s)$, where $z = s^{\frac{1}{v}}$, is also stable and proper. For example,

$$\widetilde{P}(z) = \frac{1}{z+1},$$

is stable and proper, hence it is concluded that

$$P(s) = \frac{1}{s^{\frac{1}{v}} + 1},$$

is also stable and proper for every $v \in \mathbb{N}$.

Introduce the symbol $\mathcal{S}$ for the family of all stable, proper, real-rational (fractional-order) functions. Notice that $\mathcal{S}$ is closed under addition and multiplication: If $F, G \in \mathcal{S}$, then $F + G$, $FG \in \mathcal{S}$.

Note that a controller satisfying (1) is not necessarily a feasible solution because the resulted closed-loop system is not guaranteed to be stable. To overcome this difficulty, first the controller is parameterized in a way that the resulted closed-loop system be stable (Youla parametrization). Then a solution to (1) is obtained over the set of these stabilizing controllers. In the following, the controller parametrization method is presented for stable and unstable plants, separately.

3.1 Stable Plant

Theorem 2. Assume that $P \in \mathcal{S}$ and C is proper. The set of all C's for which the feedback system of Fig. 1 is internally stable is given by

$$\left\{ \frac{Q}{1 - PQ}, Q \in \mathcal{S} \right\}. \tag{7}$$

Proof. First, we must show that the members of (7) internally stabilize the system of Fig. 1. In order to do that, it must be proved that the transfer functions from all external signals to all internal signals in Fig. (1) are stable. Without loss of generality, we show that the transfer function from r to y is stable. It follows that

$$\frac{Y(s)}{R(s)} = \frac{CP}{1 + CP}. \tag{8}$$

Substituting (7) in (8) yields

$$\frac{Y(s)}{R(s)} = \frac{\frac{QP}{1-PQ}}{1 + \frac{QP}{1-PQ}} = QP, \tag{9}$$

which is stable providing that P and Q are stable. Now, we show that if the system of Fig. 1 is internally stable then the controller can be parameterized as in (7). Let Q denote the transfer function from r to u, that is,

$$Q \triangleq \frac{C}{1 + PC}. \tag{10}$$

Then, obviously $Q \in \mathcal{S}$ and

$$C = \frac{Q}{1 - PQ}. \tag{11}$$

This completes the proof.

Based on the above parametrization, the sensitivity function is given by

$$S(s) = 1 - PQ. \tag{12}$$

3.2 Unstable Plant

In this case, the transfer function P is proper but no longer assumed to be stable. Let $P = N/M$ be a coprime factorization over $\mathcal{S}$ and let X, Y be two functions in $\mathcal{S}$ satisfying the equation

$$NX + MY = 1.$$

Theorem 3. The set of all proper C's for which the feedback system of Fig. 1 is internally stable is given by

$$\left\{ \frac{X + MQ}{Y - NQ}, Q \in \mathcal{S} \right\}. \tag{13}$$

Proof. A procedure similar to the one presented in Theorem 2 can be followed to provide the proof which is omitted here.

Using this parametrization, the sensitivity function is

$$S(s) = M(Y - NQ). \tag{14}$$

4. CONTROLLER DESIGN FOR STABLE PLANTS

4.1 Controller Design Algorithm

The following is the controller design procedure for stable plants. It is an extension for the method presented in Doyle et al. (1990) for integer-order systems.

Step 1: Set k equal to the relative degree of P.

Step 2: If $k < 2v$ then define J as in (4), else

$$J(s) = \frac{1}{\left(\tau s^{\frac{2v-1}{v}} + 1\right)^q \left(\tau s^{\frac{r}{v}} + 1\right)}, \tag{15}$$

where the positive integers q and r are the quotient and reminder of the division $k/(2\nu - 1)$, i.e. $k = (2\nu - 1)q + r$.

Step 3: Choose τ so small that

$$\|W(1 - J)\|_\infty < 1.$$

Step 4: Set $Q = P^{-1}J$.

Step 5: Set $C = Q/(1 - PQ)$.

Note that according to (12) and Step 4 the sensitivity and complementary sensitivity functions are calculated as

$$S(s) = 1 - J = \frac{\tau s^{\frac{k}{v}}}{\tau s^{\frac{k}{v}} + 1}, \quad T(s) = \frac{1}{\tau s^{\frac{k}{v}} + 1}.$$

Hence

$$WS = W(1 - J),$$

which is suitably minimized in Step 3. It is easily verified that the function J defined in (15) is also stable. Obviously, the Q calculated in Step 4 is proper and stable, so it leads to a stabilizing proper controller in Step 5 according to Theorem 2. In Fig. 1, the error signal corresponding to the unit step is calculated as

$$E(s) = S(s) \times \frac{1}{s} = \frac{\tau s^{\frac{k}{v}}}{\tau s^{\frac{k}{v}} + 1} \times \frac{1}{s},$$

which implies that

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{\tau s^{\frac{k}{v}}}{\tau s^{\frac{k}{v}} + 1} = 0,$$

i.e., the proposed controller design algorithm guarantees tracking of the step input without steady-state error. Note that $s = 0$ is the branch point of $E(s)$ and, thus, taking the limit in the latter equation needs further care (LePage, 1961). Indeed, Appendix A shows that the Final-value Theorem can be extended and used in this case.

4.2 Example

According to (Podlubny, 1999), the fractional-order model of a heating furnace is given by

$$P(s) = \frac{1}{14994s^{1.31} + 6009.5s^{0.97} + 1.69}, \quad (16)$$

which is proved to be more exact than the integer-order model. To control this system, Zhao et al. (2005) proposed a tuning method which led to the FOPID controller

$$C_1(s) = 736.8054 - \frac{0.5885}{s^{0.6}} - 818.4204s^{0.35}. \quad (17)$$

Integer-order PID controller (using the Åström-Hägglund tuning algorithm (Åström and Hägglund, 1995)) is also designed for this system by Zhao et al. (2005) and it is shown that the step response of the closed-loop system with FOPID is much faster than the corresponding one with PID and also the overshoot is smaller in FOPID case (Zhao et al., 2005). The main drawback of this method is that it is based on complex numerical optimization algorithms. It is easily verified that the transfer function (16) is stable. Let us design a controller for this system using the proposed algorithm in this section.

Consider the weighting function as

$$W(s) = \frac{0.9}{s+1},$$

which signifies a bandwidth of 1 rad/s.[1] Following the algorithm results:

Step 1: For this system $v = 100$, $m = 0$ and $n = 131$ and the relative degree is $k = n - m = 131$.

Step 2: Since $k < 2v$ then

$$J(s) = \frac{1}{\tau s^{1.31} + 1}.$$

Step 3: Choose τ so that the infinity norm of

$$\frac{0.9}{s+1} \times \frac{\tau s^{1.31}}{\tau s^{1.31} + 1},$$

is less than unity. A value of $\tau \approx 20$ works well.

Step 4:

$$Q(s) = \frac{14994s^{1.31} + 6009.5s^{0.97} + 1.69}{20s^{1.31} + 1}.$$

Step 5:

$$C(s) = \frac{14994s^{1.31} + 6009.5s^{0.97} + 1.69}{20s^{1.31}}. \quad (18)$$

The magnitude plot of WS and the closed-loop system responses are shown in Fig. 2. Figure 2(a) shows that $\|WS\|_\infty \approx 1$. It is observed from Fig. 2(b) that the step response corresponding to the proposed controller (18) competes the one proposed by Zhao et al. (2005) (17). Important feature of (18) is that it cancels all poles of the plant transfer function. Since (17) is non-proper, the corresponding control signal contains impulse due to the discontinuity in step signal. Note that in the proposed algorithm, the closed-loop bandwidth can easily be adjusted by changing $W(s)$ (or equivalently, τ).

[1] Note that, in general, the weighting function can also be of fractional order.

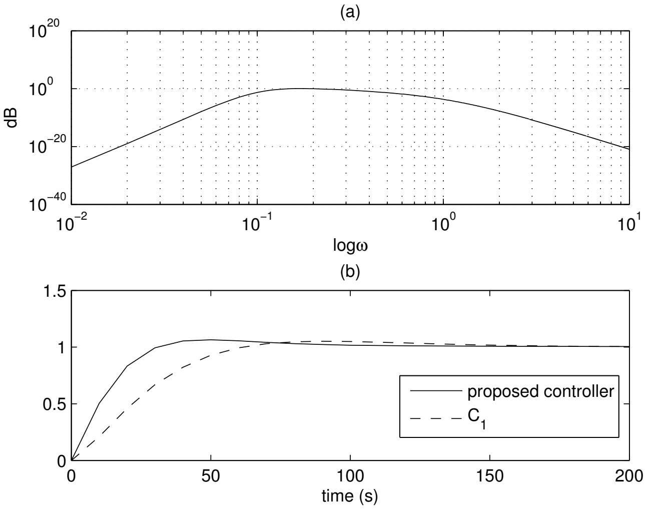

Fig. 2. (a) The magnitude plot of WS, (b) closed-loop step responses corresponding to (17) and (18)

This example shows efficiency of the proposed definition for J. In the case of using the function J as defined in (6), 131 terms appear in the denominator of C which considerably increases the computational effort.

5. CONTROLLER DESIGN FOR UNSTABLE PLANTS

Before introducing the design procedure, we have to develop the concept of coprime factorization for fractional-order systems.

5.1 Coprime Factorization

The coprime factorization for unstable fractional-order systems can be developed as follows.

Step 1: Transform $\widetilde{G}(z)$ to $\widehat{G}(\lambda)$ under the mapping $z = (1-\lambda)/\lambda$. Write $\widehat{G}$ as a ratio of coprime polynomials:

$$\widehat{G}(\lambda) = \frac{n(\lambda)}{m(\lambda)}.$$

Step 2: Using Euclid's algorithm, find polynomials $x(\lambda)$, $y(\lambda)$ such that

$$nx + my = 1.$$

Step 3: Transform $n(\lambda)$, $m(\lambda)$, $x(\lambda)$, $y(\lambda)$ to $\widetilde{N}(z)$, $\widetilde{M}(z)$, $\widetilde{X}(z)$, $\widetilde{Y}(z)$ under the mapping $\lambda = 1/(z+1)$.

Step 4: Transform $\widetilde{N}(z)$, $\widetilde{M}(z)$, $\widetilde{X}(z)$, $\widetilde{Y}(z)$ under the mapping $z = s^{\frac{1}{v}}$.

The mapping used in Step 1 is not unique; the only requirement is that the polynomials in terms of λ are mapped to stable and proper transfer functions in terms of z. It can easily be verified that if e.g. $n(\lambda) = 0$ has no unstable roots then neither $N(s)$ will have.

5.2 Controller Design Algorithm

The following is the controller design algorithm when the plant is unstable.

Step 1: Do a coprime factorization of $P(s)$, i.e., find four stable proper transfer functions satisfying the equations

$$P = \frac{N}{M}, \quad NX + MY = 1.$$

Step 2: Set k equal to the relative degree of P.

Step 3: If $k < 2v$ define J as (4), else use (15).

Step 4: Choose τ so small that

$$\|WMY(1-J)\|_\infty < 1,$$

Step 5: Set $Q = YN^{-1}J$.

Step 6: Set $C = (X + MQ)/(Y - NQ)$.

Note that according to (14) and Step 5 we have

$$WS = WMY(1-J),$$

which is suitably minimized in Step 4. Obviously, the Q calculated in Step 5 is proper and stable (N^{-1} is stable because P is minimum-phase by assumption), so it leads to a stabilizing proper controller in Step 6 according to (13). In Fig. 1, the error signal corresponding to the unit step is calculated as

$$E(s) = S(s) \times \frac{1}{s} = MY(1-J) \times \frac{1}{s},$$

which implies that

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} MY(1-J) = 0,$$

provided that M and Y are stable. As a result, the proposed controller design algorithm guarantees the tracking of the step input without steady-state error when the plant is unstable.

5.3 Example

Consider the closed-loop system of Fig. 1 where

$$P(s) = \frac{s^{\frac{1}{2}} + 2}{s^{\frac{3}{2}} - 3s + s^{\frac{1}{2}} + 5}.$$

Obviously,

$$\widetilde{P}(z) = \frac{z+2}{z^3 - 3z^2 + z + 5},$$

the poles of which are located at $z_1 = -1$, $z_{2,3} = 2 \pm j$. This system is unstable because

$$|\arg(z_{2,3})| \not> \frac{\pi}{2v} = \frac{\pi}{4}.$$

It is also evident that the system is minimum-phase since it has only one (stable) zero at $z = -2$. We will assume that

$$W(s) = \frac{100}{10s+1}.$$

Step 1: Using the coprime factorization algorithm presented in 5.1 we have

$$n(\lambda) = \lambda^2 + \lambda^3, \quad m(\lambda) = 1 - 6\lambda + 10\lambda^2,$$

$$x(\lambda) = \frac{356}{17} - \frac{860}{17}\lambda, \quad y(\lambda) = 1 + 6\lambda + \frac{86}{17}\lambda^2,$$

then

$$\widetilde{N}(z) = \frac{z+2}{(1+z)^3}, \quad \widetilde{M}(z) = \frac{5 - 4z + z^2}{(1+z)^2},$$

$$\widetilde{X}(z) = \frac{4}{17}\frac{-126 + 89z}{1+z}, \quad \widetilde{Y}(z) = \frac{1}{17}\frac{205 + 136z + 17z^2}{(1+z)^2},$$

which implies that

$$N(s) = \frac{s^{\frac{1}{2}} + 2}{(1 + s^{\frac{1}{2}})^3}, \quad M(s) = \frac{5 - 4s^{\frac{1}{2}} + s}{(1 + s^{\frac{1}{2}})^2},$$

$$X(s) = \frac{4}{17}\frac{-126 + 89s^{\frac{1}{2}}}{1 + s^{\frac{1}{2}}}, \quad Y(s) = \frac{1}{17}\frac{205 + 136s^{\frac{1}{2}} + 17s}{(1 + s^{\frac{1}{2}})^2}.$$

Step 2: $k = n - m = 3 - 1 = 2$.

Step 3: Since $k < 2v$ we use

$$J(s) = \frac{1}{\tau s + 1}.$$

Step 4: Choose τ such that the infinity norm of

$$WMY(1-J) = \frac{100}{10s+1}\frac{5 - 4s^{\frac{1}{2}} + s}{(1 + s^{\frac{1}{2}})^2}\frac{1}{17} \times \frac{205 + 136s^{\frac{1}{2}} + 17s}{(1 + s^{\frac{1}{2}})^2}\left(1 - \frac{1}{\tau s + 1}\right), \quad (19)$$

is less than unity. $\tau \approx 0.0058$ is an approximate solution. The magnitude plot of (19) is shown in Fig. 3(a) when $\tau = 0.0058$. It is obvious from the figure that $\|WS\|_\infty \approx 1$.

Step 5:

$$Q(s) = \frac{5000}{17}\frac{(1 + s^{\frac{1}{2}})(205 + 136s^{\frac{1}{2}} + 17s)}{(2 + s^{\frac{1}{2}})(5000 + 29s)}.$$

Step 6:

$$C(s) = \frac{B(s)}{A(s)},$$

where

$$B(s) = 4(1 + s^{\frac{1}{2}}) \times (21250 + 85000s^{\frac{1}{2}} + 120192s + 86508s^{\frac{3}{2}} + 23831s^2),$$

$$A(s) = 29s(2 + s^{\frac{1}{2}})(205 + 136s^{\frac{1}{2}} + 17s).$$

The closed-loop system response to unit step is depicted in Fig. 3(b). Unlike the open-loop system that is unstable, the closed-loop system is stable and gives a desirable transient response. By decreasing the value of τ, the closed-loop system becomes faster at the cost of more noise and larger control.

6. CONCLUSION

An efficient controller design method for a class of fractional-order systems is presented. The proposed method is based on the method of shaping the sensitivity function. Since the control objectives such as command tracking and noise attenuation can be explained in terms of the sensitivity function, the proposed algorithm provides a general approach. The only restriction is that the plant must be minimum-phase. One important feature of the proposed method is that it is an analytic algorithm and needs no complex numerical optimizations. Moreover, all control objectives are explained in terms of a weight function. The

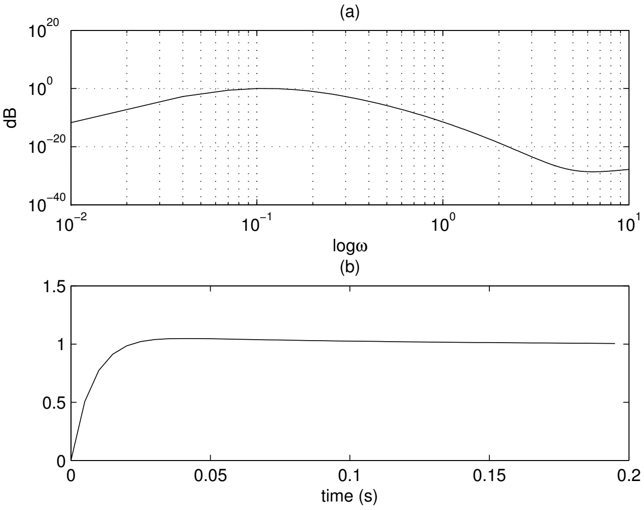

Fig. 3. (a) The magnitude plot of WS, (b) closed-loop step responses

design method is developed for both stable and unstable plants. Two numerical examples are solved to confirm the efficiency of the proposed design algorithms. Similar to the classical case, the proposed method suffers two drawbacks. First, it may lead to conservative solutions, and second, an order reduction algorithm may be needed at the final stage as the controllers obtained using this method are complex.

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Appendix A. THE FINAL-VALUE THEOREM FOR FRACTIONAL CASE

Here, we show that the final-value theorem is applicable when there is a branch point at $s = 0$. Assume that $F(s) = \mathcal{L}\{f(t)\}$ is a multi-valued function of s with a branch point at $s = 0$. Then

$$\int_0^{\infty} f'(t)e^{-st}dt = sF(s) - f(0). \tag{A.1}$$

Now, let s tend to zero in the direction of positive real axis:

$$\lim_{s\to 0}\int_0^{\infty} f'(t)e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0)]. \tag{A.2}$$

Since the Laplace integral is uniformly convergent we can change the order of limit and integral:

$$\int_0^{\infty} \lim_{s\to 0}[f'(t)e^{-st}]dt = \lim_{s\to 0}[sF(s)] - f(0), \tag{A.3}$$

which implies that

$$\int_0^{\infty} f'(t)dt = f(\infty) - f(0) = \lim_{s\to 0}[sF(s)] - f(0), \tag{A.4}$$

or

$$f(\infty) = \lim_{s\to 0}[sF(s)]. \tag{A.5}$$