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Predictive control for linear systems with delayed input subject to constraints

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Abstract: The paper deals with the moving horizon control of systems subject to input delays and affected by input and state and/or output constraints. The robustness of the control law with respect to the uncertainties introduced by the discretization is considered. The stability of the closed-loop system is guaranteed by forcing the state trajectories to attain a robust positively invariant terminal set on the prediction horizon. Illustrative examples complete the paper.

Keywords: input delay; constraints; predictive control; optimization; invariance properties.

1. INTRODUCTION

It is well-known that the reaction of real systems and physical processes to exogenous signals takes never place "instantaneously", and one of the classical way to model such situations and phenomena is by using *time-delays*. Roughly speaking, the delays (constant or time-varying, distributed or not) describe coupling or between the dynamics, propagation and transport phenomena, heredity and competition in population dynamics. Various motivating examples and related discussions ca be found in Niculescu (2001), Gu et al. (2003), Michiels and Niculescu (2007). Networking (congestion mechanisms, consensus algorithms, teleoperation and networked control systems) is one of the classical examples among numerous applications including delays spanning biology, ecology, economy and engineering, where the *delay* is a *critical parameter* in understanding dynamics behavior and/or improving (overall) system's performances.

Independently of the mathematical problems related to the appropriate representation of such dynamics, the delay systems are known to rise challenging control problems due to the instabilities introduced in the closed loop by the presence of delays. Discrete time control of continuous systems affected by delays has to face even more difficulties due to the sampling which introduces an uncertainty in the discrete models.

It is known that predictors can be used to overcome the effects of dead-time (with inherent problems linked to the sensitivity of predictions for unstable models). MPC - "Model Predictive Control" solves at each sampling time a finite-time optimal control problem over a receding prediction horizon and is no surprise that its use in connection with delay systems was proposed from the early approaches (Clarke et al., 1987). At the time of the redaction of the present paper a monograph is in print with a review of the attempts on this direction ranging from dead-time compensation to MPC (Normey-Rico and Camacho, 2007).

Considering the latest advances in MPC design (Maciejowski, 2002; Goodwin et al., 2004) which offer constraints handling capabilities with stability guarantees (Mayne et al., 2000) as well as the possibility of incorporating uncertainties in an explicit manner at the design stage one has the picture of a versatile control strategy with a proved succes among practitioners.

To the best of the authors knowledge, there exists several results in the literature devoted to delay systems and input and or state-constraints, see for instance Tarbouriech et al. (2004), where appropriate (closed-loop) stability conditions have been proposed by using LMIs. Next, various robustness issues of some predictive-based control laws using the discrete dynamics and the uncertainty introduced by small variations of the times between sampling instants can be found in Lozano et al. (2004) (and the references therein). The approach we are proposing here is based on some "minmax" optimization problem that takes into account the "worst-case" performance of the polytopic uncertainty, and it opens interesting perspectives for defining an appropriate methodology, computationally tractable, for handling such class of problems. In other words, the aim of this paper is to develop methods and numerical algorithms for treating *simultaneously* delays and input and state and/or output constraints in a predictive control setting.

Concretely, the present paper emploies a predictive control technique for delay systems by considering the uncertainties introduced at the discretization stage. The obtention of the prediction model is detailed as well as the synthesis of a local state feedback stabilizer for the unconstrained case using convex optimization type of arguments. The invariant set associated to this stabilizing feedback law is constructed in order to impose stability constraints in the MPC synthesis. This can be achieved by adapting the theory of maximal output admissible sets for the system with polytopic uncertainty. Finally a receding horizon optimization problem is solved do drive the state to the origin by robustly satisfying the constraints. By obtaining the explicit formulation of the control law in terms of a piecewise affine control law, the shape of the feasibility domain being available.

The remaining paper is organized as follows: section 2 formulates the control problem and defines the models to be further used in the MPC design. Section 3 deals with the construction of the robust positively invariant sets and section 4 presents the finite-time optimisation problem to be solved at each sampling time in the MPC framework. Finally section 5 presents two illustrative examples whereas section 6 draws the conclusions. The notations are standard.

2. PROBLEM DESCRIPTION

Consider a nominal linear continuous-time system affected by input delay:

$$\dot{x}(t) = A_c x(t) + B_c u(t-h) \tag{1}$$

with $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ and $h > 0$, under appropriate initial conditions.

A corresponding discrete-time model will be constructed upon a chosen sampling period T_e by considering the time instants $t_k = kT_e$. In order to prove the robustness of any discrete-time control scheme with respect to original system, a certain degree of uncertainty being acceptable when dealing with delays:

$$h = dT_e - \epsilon \tag{2}$$

is considered.

In the general case, the variation ϵ can be time-varying but it will be supposed in the following that the choice of d is such that it assures the boundness:

$$0 < \epsilon \leq \bar{\epsilon} \ll T_e \tag{3}$$

where $\bar{\epsilon}$ is the maximal delay variation.

Noting the discrete time instants $x_k = x(t_k)$ one can describe the discrete time model by:

$$x_{k+1} = Ax_k + Bu_{k-d} - \Delta(u_{k-d} - u_{k-d+1}) \tag{4}$$

due to the fact that there is no exact correspondence between the delay in continuous-time and the samples available for the discrete model and this mismatch impose the consideration of an uncertainty.

The matrices A, B, Δ are given by:

$$A = e^{A_c T_e} \tag{5}$$

$$B = \int_0^{T_e} e^{A_c(T_e-\theta)} B_c d\theta \tag{6}$$

$$\Delta = \int_{T_e-|\epsilon|}^{T_e} e^{A_c(T_e-\theta)} B_c d\theta \tag{7}$$

$$= \int_{-|\epsilon|}^{0} e^{-A_c\tau} B_c d\tau \tag{8}$$

obtained by assuming that the control action u is maintained constant between sampling instants, $u(t) = u_k, \forall t \in [t_k, t_{k+1})$.

Remark 1. Equations (2-3) consider a delay uncertainty such that $dT_e \geq h > dT_e - \bar{\epsilon}$. In order to diminish the importance of the uncertainty matrix Δ in (4), the uncertainty can be centered around a delayed input

$$|h - dT_e| \leq \bar{\epsilon} \ll T_e/2$$

thus decreasing the integration limits for (7). The distinction between the case $\epsilon > 0$ and $\epsilon < 0$ can be found in Lozano et al. (2004) as well as a detailed discussion about uncertainties introduced by small variations of the time between sampling instants. In the following we resume our study to the simpler case (3) and observe that the other cases can be treated similarly.

The extreme realizations of the discrete-time model are

For $\epsilon = 0$:

$$x_{k+1} = Ax_k + Bu_{k-d} \tag{9}$$

For $\epsilon = \bar{\epsilon}$:

$$x_{k+1} = Ax_k + (B - \bar{\Delta})u_{k-d} + \bar{\Delta}u_{k-d+1} \tag{10}$$

but all the intermediate realizations have to be considered.

The objective is to design a control law which regulates the system state for any $0 \leq \epsilon \leq \bar{\epsilon} < T_e$ while robustly satisfying a set of constraints:

$$C_c x(t) + D_c u(t) \leq W_c, \quad for \quad t \in [kT_e, (k+1)T_e) \tag{11}$$

which can be rewritten in a linear form function of x_k and u_k as:

$$Cx_k + Du_k \leq W \tag{12}$$

Note that on the given interval $u(t) = u_k$ but precautions have to be taken for $x(t)$ which has the form:

$$x(t) = e^{A_c(t-kT_e)}x_k + \int_{kT_e}^{t} e^{A(t-\theta)} Bu_k d\theta \tag{13}$$

thus depending on x_k and u_k.

This linear type of constraints covers a large class of limitations encountered in practice (input saturations or output constraints for exemple). It is supposed however that the origin is contained in the interior of the polyhedral domain described by (12).

3. PREDICTION MODEL

By rewriting the dynamics (9-10) in a compact form one can obtain the following linear model:

$$\xi_{k+1} = A_\Delta \xi_k + B_\Delta u_k \tag{14}$$

with

$$\xi_k^T = [\, x_k^T \; u_{k-d}^T \; \ldots \; u_{k-1}^T \; u_k^T \,] \tag{15}$$

$$A_\Delta = \begin{bmatrix} A & B-\Delta & \Delta & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & I_m \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{16}$$

$$B_\Delta = [\, 0 \; 0 \; \ldots \; 0 \; I_m \,]^T \tag{17}$$

It can be observed that the matrix Δ is depending on the value of the delay uncertainty ϵ which varies the integration limits in (7). Rigorously speaking, one should use ϵ_k due to the fact that the uncertainty is time-varying (the same for Δ). In the following this explicit dependence on time is omitted for the simplicity of the notation.

Remark 2. For the compact linear model (14) one has $A_\Delta \in \mathbb{R}^{(n+d\cdot m)\times(n+d\cdot m)}$ and $B_\Delta \in \mathbb{R}^{(n+d\cdot m)\times m}$. From (2) it follows that $d \to \infty$ when $T_e \to 0$ which means that the system (14) becomes infinite dimensional when the sampling time decrease to 0.

The idea followed in this paper is to confine Δ in a polytopic set which covers all the possible realizations (thus independent of ϵ). In order to obtain the extreme combinations of this polytopic embedding, the Jordan canonical form can be used $A_c = V\Lambda V^{-1}$. The matrix Λ can be decomposed as $\Lambda = \sum_{i=1}^{n} \Lambda_i$ with Λ_i originated by the terms of the direct sum $\Lambda = \tilde{\Lambda}_1 \oplus \cdots \oplus \tilde{\Lambda}_n$. For the brevity of the paper is assumed that A_c is invertible and not defective and $\tilde{\Lambda}_i$ correspond to the diagonal elements. If this is not the case, the exponential of the Jordan blocks $e^{\Lambda_i \bar{\epsilon}}$ have to be computed separately and each resulting matrix further decomposed upon the upper diagonals in (20) to provide the full set of vertices for the polytopic model (22). Although not explicitly developed, one of the examples in section 6 will fall in this case, proving the generality of the results.

If the integral of the exponential (8) is written as:

$$\Delta = A_c^{-1}(e^{A_c\epsilon} - I_n)B_c \tag{18}$$

then for the limit values of ϵ one can obtain the extreme realizations:

$$\Delta_0 = \mathbf{0}_{n\times m} \tag{19}$$

$$\Delta_i = A_c^{-1}V(e^{\Lambda_i\bar{\epsilon}} - I)V^{-1}B_c,\ \forall i = 1,\ldots,n \tag{20}$$

In order to obtain the desired control objectives for the system (14) one can use a polytopic embedding within the linear models given by:

$$A_{\Delta_0} = \begin{bmatrix} A & B & 0 & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & I_m \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{21}$$

$$A_{\Delta_i} = \begin{bmatrix} A & B - n\Delta_i & n\Delta_i & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & I_m \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, i = 1,\ldots,n \tag{22}$$

The following result resumes the existence of a polytopic model for the system (14).

Theorem 1. For any $0 \le \epsilon \le \bar{\epsilon}$ the state matrix A_Δ satisfies:

$$A_\Delta \in Co\{A_{\Delta_0}, A_{\Delta_1}, \ldots, A_{\Delta_n}\} \tag{23}$$

where $Co\{.\}$ denotes the convex hull [1] and vertices A_i are given by (21-22).

[1] For some nonnegative scalars $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ summing to one

$$A_\Delta = \sum_{i=0}^{n} \alpha_i A_{\Delta_i}$$

Proof: For any $0 \le \epsilon \le \bar{\epsilon}$ and for all $i = 1,\ldots,n$ there exists $0 \le \beta_i \le 1$ such that:

$$\begin{aligned}
&\Delta = V\left(e^{\Lambda\epsilon} - I\right)V^{-1}A_c^{-1}B_c = \\
&= \sum_{i=1}^{n} V\left(e^{\Lambda_i\epsilon} - I\right)V^{-1}A_c^{-1}B_c \\
&= \sum_{i=1}^{n} V\left(\beta_i\left(e^{\Lambda_i\bar{\epsilon}} - I\right) + (1-\beta_i)\left(e^{\Lambda_i 0} - I\right)\right)V^{-1}A_c^{-1}B_c \\
&= \sum_{i=1}^{n} \beta_i\Delta_i + (1-\beta_i)\Delta_0 \\
&= (n - \sum_{i=1}^{n}\beta_i)\Delta_0 + \sum_{i=1}^{n}\beta_i\Delta_i \\
&= \underbrace{\frac{(n - \sum_{i=1}^{n}\beta_i)}{n}}_{\alpha_0} n\Delta_0 + \sum_{i=1}^{n} \underbrace{\frac{\beta_i}{n}}_{\alpha_i} n\Delta_i
\end{aligned} \tag{24}$$

The matrix Δ appears in a linear manner in the structure of A_Δ as it can be seen in (16) and using the scalars $\alpha_i \ge 0, i = 0,\ldots,n$ found before, one can write:

$$A_\Delta = \sum_{i=0}^{n} \alpha_i A_i \tag{25}$$

By observing that $\sum_{i=0}^{n} \alpha_i = 1$ the proof is completed ■

The model (14-17) with the uncertainty (23) will be used as prediction model in the MPC scheme. Before describing the optimization problem to be solved at each sampling time, the next section details the construction of a robust positively invariant set to be further used as terminal constraints for the prediction.

4. ROBUST POSITIVE INVARIANT SET

In the following it is supposed that each pair $(A_i, B), i = 0,\ldots,n$ is controllable. In the first stage a stabilizing control law is found for the polytopic model in the unconstrained case and secondly a positive invariant set is constructed by considering also the constraints (12).

4.1 Stabilizing control law. Unconstrained case.

Consider the linear systems (14) subject to a polytopic uncertainty (23):

$$\begin{aligned} \xi_{k+1} &= A_\Delta\xi_k + B_\Delta u_k \\ A_\Delta &\in \Omega \\ \Omega &= \mathrm{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \ldots A_{\Delta_n}\} \end{aligned} \tag{26}$$

It is supposed that a stabilizing control law

$$u_k = K\xi_k \tag{27}$$

exists and it can be obtained using an LMI (linear matrix inequalities) construction.

Consider an infinite-horizon min-max control problem:

$$\min_{u_k, u_{k+1}, u_{k+2}\ldots} \quad \max_{A_\Delta\in\Omega} J_\infty \tag{28}$$

with

$$J_\infty = \sum_{i=0}^{\infty} \xi_{k+i}^T Q \xi_{k+i} + u_{k+i}^T R u_{k+i} \tag{29}$$

$$u_k = K\xi_k \tag{30}$$

where $Q > 0$, $R > 0$ are suitable weighting matrices.

A quadratic function of the state

$$V(\xi) = \xi^T P \xi, \quad P > 0 \tag{31}$$

will represent an upper bound for J_∞ if the following inequality is satisfied $\forall A_\Delta \in \Omega$:

$$V(\xi_{k+i+1}) - V(\xi_{k+i}) \leq -[\xi_{k+i}^T Q \xi_{k+i} + u_{k+i}^T R u_{k+i}] \tag{32}$$

Rewriting this equation using (30) the following inequality is obtained:

$$\begin{aligned} \xi_{k+i}^T[(A_\Delta + B_\Delta K)^T P (A_\Delta + B_\Delta K) \\ -P + K^T R K + Q]\xi_{k+i} \leq 0 \end{aligned} \tag{33}$$

or equivalently:

$$(A_\Delta + B_\Delta K)^T P (A_\Delta + B_\Delta K) - P + K^T R K + Q \leq 0 \tag{34}$$

Using the ideas in (Boyd et al., 1994), by noting $P = \gamma S^{-1}$, $S \geq I$ and $Y = KS$, the following LMI can be constructed:

$$\begin{bmatrix} S & SA_\Delta^T + Y^T B_\Delta^T & SQ^{1/2} & Y^T R^{1/2} \\ A_\Delta S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2} S & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \succ 0, \tag{35}$$

Using now the fact that $A_\Delta \in \Omega$, a stabilizing control law is given by $K = YS^{-1}$ where Y, S and the scalar γ solutions of the LMI (Kothare et al., 1996):

$$\begin{gathered} \min_{\gamma, S, Y} \gamma \\ \begin{bmatrix} S & SA_{\Delta_i}^T + Y^T B_\Delta^T & SQ^{1/2} & Y^T R^{1/2} \\ A_{\Delta_i} S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2} S & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \succ 0, \\ \text{for all} \quad i = 0, \dots, n \end{gathered} \tag{36}$$

Remark 3. This LMI based procedure is used in (Kothare et al., 1996) to design a MPC law. The LMI in (36) is not depending on the measured state and thus the resulting control law is represented by a fixed feedback control gain. Its stabilizing properties will be used for the construction of a robust positive invariant set.

4.2 Maximal output admissible set

In order to deal with the constraints, the first step is to rewrite (12) in terms of the augmented state variable ξ:

$$\Gamma \xi_k + D u_k \leq W \tag{37}$$

Using the stabilizing control law $u_k = K\xi_k = YS^{-1}\xi_k$ found by solving (36) the following polyhedral domain can be defined in the augmented state space:

$$P = \left\{ \xi \in \mathbb{R}^{(n+d\cdot m)} \mid (\Gamma + DK)\xi \leq W \right\} \tag{38}$$

Definition 1. (Gilbert and Tan, 1991) The *maximal output admissible set*, for a LTI system $\xi_{k+1} = \Phi \xi_k$ and a predefined set P as in (38), is described as:

$$O_\infty = \left\{ \xi_0 \mid \Phi^k \xi_0 \in P, \forall k \in \mathbb{N} \right\} \tag{39}$$

In our case, the generalization of this concept for the polytopic systems is of most interest, the following definition providing the necessary details.

Definition 2. For a system with polytopic uncertainty:

$$\begin{aligned} \xi_{k+1} &= \Phi_i \xi_k \\ \Phi &\in \Omega_K \\ \Omega_K &= \text{Co}\{(A_{\Delta_0} + B_\Delta K);\ \dots;\ (A_{\Delta_n} + B_\Delta K)\} \end{aligned} \tag{40}$$

and a predefined set P, the *maximal output admissible set* O_∞^Ω is defined as the collection of all the initial states ξ_0 for which the state trajectory remains in the interior of P for all future instants $k \geq 0$.

In other words the maximal output admissible set is described readily as:

$$O_\infty^\Omega = \left\{ \xi_0 \mid \Phi^k \xi_0 \in P, \forall \Phi \in \Omega_K, \forall k \in \mathbb{N} \right\} \tag{41}$$

An important problem has to be clarified with respect to this construction: under which conditions the set O_∞^Ω is finitely determined. Taking into account that the control law $u_k = K\xi_k$ was found such that all the extreme realizations in (40) are asymptotically stable, the extension of the Theorem 4.1 in Gilbert and Tan (1991) assures that for bounded P, with $0 \in IntP$, if the pairs $(\Gamma + DK, A_i + B_i K)$, $\forall i \in \{1, \dots, n\}$ are observable, then O_∞^Ω is finitely determined.

Similar to the linear case, the construction algorithm can exploit the fact that O_∞^Ω is finitely determined if and only if $O_N = O_{N+1}$ where:

$$O_N^\Omega = \left\{ \xi_0 \mid \Phi^k \xi_0 \in P, \forall \Phi \in \Omega_K, \forall k \in \{1, \dots, N\} \right\} \tag{42}$$

Observing that the same set can be rewritten as:

$$O_{N+1}^\Omega = \{ \xi \in O_N \mid \Phi \xi \in P, \forall \Phi \in \Omega_K \} \tag{43}$$

and further by noting $\Phi_i = A_{\Delta_i} + B_\Delta K$, $i = 0, \dots, n$ one can obtain a direct computable expression:

$$O_{N+1}^\Omega = \{ \xi \in O_N \mid \Phi_i \xi \in P, \forall \Phi_i \in \{\Phi_0, \dots, \Phi_n\} \} \tag{44}$$

which can be used in a recursive manner to obtain the maximal output admissible set for the class of systems we are interested in.

The set O_∞^Ω enjoys by definition (41) robust positively invariance properties (Blanchini, 1999) and thus it will be further used in the predictive control design.

5. PREDICTIVE CONTROL

A standard MPC strategy for the delay system considered here applies at each sampling instant the first component of the optimal control sequence $\mathbf{k}_u = \{u_k, \dots, u_{k+N-d}\}$ as control action to the system while the tail is discarded. Using the new measurements the optimisation procedure is restarted, thus obtaining a closed-loop control scheme.

As a basic remark, the prediction horizon has to be larger than the delay in order to have an effective measure of its effect at the system output.

$$\begin{aligned} \mathbf{k}_u^* = arg \min_{\mathbf{k}_u} \Bigg\{ & \max_i {\xi_{k+N}^{\{i\}}}^T P \xi_{k+N}^{\{i\}} \\ & + \sum_{j=0}^{N-1} \left[{\xi_{k+j}^{\{i\}}}^T Q \xi_{k+j}^{\{i\}} + u_{k+j}^T R u_{k+j} \right] \Bigg\} \end{aligned} \tag{45}$$

subject to:

$$\begin{cases} \xi^{\{i\}}_{k+j+1} = A_{\Delta_i}\xi^{\{i\}}_{k+j} + B_\Delta u_{k+j} \\ \Gamma\xi^{\{i\}}_{k+j} + Du_{k+j} \leq W, \\ \qquad\qquad i = 1,\ldots,n;\ j = 1,\ldots,N-1 \\ \xi^{\{i\}}_{k+N} \in O^{\Omega}_{\infty};\ \ i = 1,\ldots,n \end{cases}$$

The construction of the predictive control law will be influenced by the choice of the prediction horizon N, and weighting factors Q, R which are respecting the choice made for the stabilizing control law in the previous section (29). In this case the terminal state will be weighted by $P = S^{-1}$.

Remark 4. The formulation (45) is based on a min-max optimization problem which takes into consideration worst-case performance for the polytopic uncertainty $A_\Delta \in \mathrm{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \ldots A_{\Delta_n}\}$. Unfortunately this framework turns to be computationally expensive (Kerrigan and Maciejowski (2004),Olaru and Dumur (2007)), not to mention the version where the optimization is performed using closed-loop predictions which implies a nested min-max optimization to be solved upon dynamic programming principle.

The computational complexity is related only with the cost function, while the constraints are not affected by the way the worst-case is treated. Using this fact, a suboptimal feasible solution can be used for the MPC control scheme, drastically reducing the computational effort:

$$\mathbf{k}_u^* = arg \min_{\mathbf{k}_u}\ {\xi^{\{0\}}_{k+N}}^T P\xi^{\{0\}}_{k+N} + \sum_{j=0}^{N-1}\left[{\xi^{\{0\}}_{k+j}}^T Q\xi^{\{0\}}_{k+j} + u^T_{k+j}Ru_{k+j}\right] \tag{46}$$

subject to:

$$\begin{cases} \xi^{\{i\}}_{k+j+1} = A_{\Delta_i}\xi^{\{i\}}_{k+j} + B_\Delta u_{k+j} \\ \Gamma\xi^{\{i\}}_{k+j} + Du_{k+j} \leq W, \\ \qquad\qquad i = 1,\ldots,n;\ j = 1,\ldots,N-1 \\ \xi^{\{i\}}_{k+N} \in O^{\Omega}_{\infty};\ \ i = 1,\ldots,n \end{cases}$$

It can be observed that the cost function in (46) is based on a nominal model while the constraints take into account all the possible uncertainty realization in order to obtain a robust control scheme. The robust stability is assured by the use of the terminal the terminal constraints and a pseudo-infinite horizon objective function (Mayne et al., 2000).

Remark 5. The optimisation problem in (45) can be reformulated as a multiparametric quadratic problem (Goodwin et al. (2004),Dua et al. (2007))

$$\begin{aligned} &\mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} 0.5\mathbf{k}_u^T H\mathbf{k}_u + \mathbf{k}_u^T F\xi \\ &\text{subject to}:\ \ A_{in}\mathbf{k}_u \leq b_{in} + B_{in}\xi \end{aligned} \tag{47}$$

and further explicit solutions for the MPC law can be obtained by retaining the first component of $\mathbf{k}_u^*(\xi)$, thus expressing the predictive control in terms of a piecewise affine control law (Dua et al., 2007):

$$u_k = K_i^{MPC}\xi + \kappa_i^{MPC}, \quad \text{with} \quad i \text{ s.t. } x \in D_i\,, \tag{48}$$

and the regions D_i convex polyhedra in $\mathbb{R}^{n+dm}$.

6. EXAMPLES

6.1 Unstable system

Consider the unstable system with delay:

$$\dot{x}(t) = \begin{bmatrix} 1.1 & -0.1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h), \quad h \in (0.2,\, 0.3) \tag{49}$$

Sampling at $T_e = 0.1$ a discrete model is obtained with a delay $d = 3$ and the uncertainty $0 < \epsilon \leq 0.1$ which affect the structure of the prediction model (4). Following the procedure described in section 2, a polytopic model can be constructed with three extreme realisations $\{A_{\Delta_0}, A_{\Delta_1}, A_{\Delta_2}\}$.

Using the stabilizing feedback control law

$$u_k = [\,-1.6952\ \ -1.5002\ \ -1.4597\ \ -1.2375\ \ -0.8513\,]\,\xi_k \tag{50}$$

obtained by solving the corresponding LMI (36), one can obtain a robust positive invariant set (figure 1).

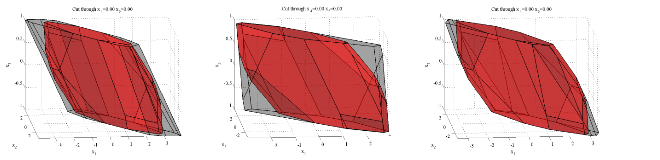

Fig. 1. The robust maximal output admissible set O^{Ω}_{∞} (red) compared from left to right with the maximal output admissible sets of each extreme realization $A_{\Delta_0}, A_{\Delta_1}, A_{\Delta_2}$

The region of the state space where the MPC control law will accomplish the regulation objective is directly related with the length of the prediction horizon (figure 2 presents the case $N = 3$). The MPC synthesis was based on $Q = I$, $R = 1$ and the set of constraints:

$$\begin{gathered} -1 \leq u_k \leq 1 \\ \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{gathered} \tag{51}$$

It can be observed from the shape of the feasible domain that main restriction come from the input constraints activation (not a surprise for a open loop unstable system).

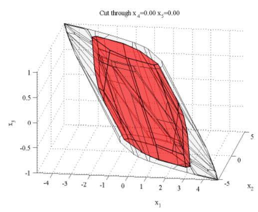

Fig. 2. Feasible domain for the MPC law (wireframe) vs. the the robust maximal output admissible set (solid color).

The time-domain simulation starting from an initial state $\xi^T = [[4 - 4 - 100]]$ (figure 3) proves the effectiveness of the control scheme with constraints satisfaction.

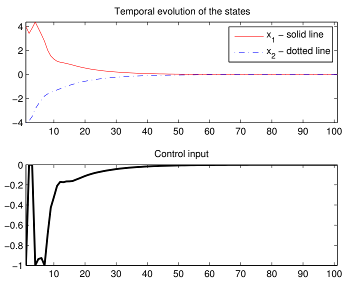

Fig. 3. Time-evolution of the states and inputs.

6.2 Double integrator with time-delay

Consider the double integrator with delay:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h), \quad h \in (0.2, 0.3) \quad (52)$$

A discrete time model has to be constructed for $T_e = 0.1$, the delay being represented by $d = 3$ samples with an uncertainty $0 < \epsilon \leq 0.1$. The model uncertainty has to be expressed in terms of a polytopic model (26). The approach (18) cannot be employed but a simple Taylor decomposition leads to the following extreme realizations:

$$A_{\Delta_0} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0.1 & 1 & 0.05 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_{\Delta_1} = \begin{bmatrix} 1 & 0 & 0.9 & 0.1 & 0 \\ 0.1 & 1 & 0.05 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (53)$$

$$A_{\Delta_2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0.1 & 1 & 0.045 & 0.005 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (54)$$

Solving the LMI (36) with $Q = 50I$ and $R = 0.1$ the stabilizing following control law is obtained:

$$u_k = \begin{bmatrix} -0.0341 & -0.0333 & -0.0073 & -0.0030 & -0.0815 \end{bmatrix} \xi_k \quad (55)$$

Figure **??** presents the robust positive invariant set and the feasible domain for the MPC law synthesized according to (46) with a prediction horizon $N = 7$ and the constraints:

$$-1 \leq u_k \leq 1$$
$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad (56)$$

The feasible domain is represented in fact by the union of 76 regions in the state space for which a fixed affine control law is associated (48).

Finally in figure 4 the simulation in time is presented with the state evolution and the corresponding control action as well as the state space trajectory.

7. CONCLUSION

A control scheme was presented for the systems with time delay and constraints. The predictive control concepts are used for constraints handling, the stability being assured by the construction of a positive invariant set for the system which is embedded in a linear prediction model with polytopic uncertainty.

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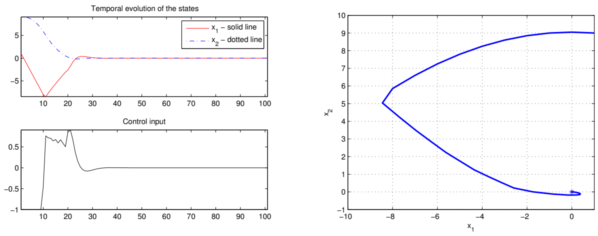

Fig. 4. Time-evolution of the states and inputs (left). Trajectory on the state space of the original system (right).

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