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A Kalman Decomposition for Possibly Controllable Uncertain Linear Systems ⋆

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Abstract: This paper considers the structure of uncertain linear systems building on concepts of robust unobservability and possible controllability which were introduced in previous papers. The paper presents a new characterization of the possibly controllable states for the case in which a certain system transfer function is non-zero. This result complements the result of a previous paper which presented a characterization of the possibly controllable states for the case in which the transfer function is zero. When combined with previous geometric results on robust unobservability and possible controllability, the results of this paper lead to a general Kalman type decomposition for uncertain linear systems.

1. INTRODUCTION

Controllability and observability are fundamental properties of a linear system; e.g., see Kailath (1980); Rugh (1993). This paper is concerned with extending these notions to the case of uncertain linear systems with the aim of gaining greater understanding of the structure of uncertain linear systems. In particular, the paper presents a new geometric characterization of controllability for uncertain systems in the case in which a certain system transfer function is non-zero. This result complements the result of a previous conference paper, Petersen (2007b) which presented a similar result for the case in which the same transfer function is zero.

One reason for considering the issue of controllability for uncertain systems might be to determine if a robust state feedback controller can be constructed for the system; e.g., see Petersen et al. (2000). In this case, one would be interested in the question of whether the system is "controllable" for all possible values of the uncertainty; e.g., see Petersen (1987, 1990); Savkin and Petersen (1999). Similarly, one reason for considering observability for uncertain systems might be to determine if a robust state estimator can be constructed for the system; e.g., see Petersen and Savkin (1999). In this case, one would be interested in the question of whether the system is "observable" for all possible values of the uncertainty; e.g., see Petersen (2002).

For the case of linear systems, the notions of controllability and observability are central to realization theory; e.g., see Kailath (1980); Rugh (1993). For example, it is known that if a linear system contains unobservable states or uncontrollable states, those states can be removed in order to obtained a reduced dimension realization of the system's transfer function. For the case of uncertain systems, a natural extension of the notion of controllability would be to consider possibly controllable states which are "controllable" for some possible values of the uncertainty. This idea was developed in the papers Petersen (2005b, 2007b) for the case of uncertain linear systems subject to averaged integral quadratic constraints (IQCs). Similarly, a natural extension of the notion of observability to uncertain systems is to consider robustly unobservable states which are "unobservable" for all possible values of the uncertainty. This idea was developed in the papers Petersen (2004, 2005a).

This paper considers the structure of uncertain linear systems building on concepts of "robust unobservability and "possible controllability". The results presented in the paper aim to provide insight into the structure of uncertain systems as it relates to questions of realization theory for uncertain systems; e.g., see Beck and D'Andrea (2004); Beck and Doyle (1999); Petersen (2007a).

We formally define notions of robust unobservability and possible controllability in terms of certain constrained optimization problems. The notion of robust unobservability used in this paper involves extending the standard linear systems definition of the observability Gramian to the case of uncertain systems; see also Gray and Mesko (1999). Also, the notion of possible controllability used in this paper involves extending the standard linear systems definition of the controllability Gramian to the case of uncertain systems; see also Scherpen and Gray (2000). We then apply the S-procedure (e.g., see Petersen et al. (2000)) to obtain conditions for robust unobservability and possible controllability in terms of unconstrained LQ optimal control problems dependent on Lagrange multiplier parameters as in Petersen (2004, 2005b). From this, we develop a geometric characterization for the set of robustly unobservable states (as in Petersen (2005a)) and the set of possibly controllable states. These characterizations imply that the set of robustly unobservable states is in fact a linear subspace. Similarly, we show that the set of possibly controllable states is a linear subspace. These characterizations lead to a Kalman type decomposition for the uncertain systems under consideration. In a previous conference

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paper Petersen (2007b), the Kalman decomposition could only be established for uncertain systems in which the transfer function from the control input to the uncertainty output was zero. In this paper, we complete the picture by establishing the result for the case in which this transfer function is non-zero.

2. PROBLEM FORMULATION

We consider the following linear time invariant uncertain system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \xi(t);\\ z(t) &= C_1 x(t) + D_1 u(t);\\ y(t) &= C_2 x(t) + D_2 \xi(t)\end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the *state*, $y \in \mathbf{R}^l$ is the *measured output*, $z \in \mathbf{R}^h$ is the *uncertainty output*, $u \in \mathbf{R}^m$ is the *control input*, and $\xi \in \mathbf{R}^r$ is the *uncertainty input*.

For the system (1), we define the transfer function $G(s)$ to be the transfer function from the input $\xi(t)$ to the output $y(t)$; i.e.,

$$G(s) = C_2(sI - A)^{-1}B_2 + D_2.$$

Also, we we define the transfer function $H(s)$ to be the transfer function from the input $u(t)$ to the output $z(t)$; i.e.,

$$H(s) = C_1(sI - A)^{-1}B_1 + D_1.$$

System Uncertainty. The uncertainty in the uncertain system (1) is required to satisfy a certain "Averaged Integral Quadratic Constraint".

Averaged Integral Quadratic Constraint. Let the time interval $[0,\ T]$, $T > 0$ be given and let $d > 0$ be a given positive constant associated with the system (1); see also Petersen (2004, 2005b); Savkin and Petersen (1995). We will consider sequences of uncertainty inputs $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \ldots \xi^q(\cdot)\}$. The number of elements q in any such sequence is arbitrary. A sequence of uncertainty functions of the form $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \ldots \xi^q(\cdot)\}$ is an *admissible uncertainty sequence* for the system (1) if the following conditions hold: Given any $\xi^i(\cdot) \in \mathcal{S}$ and any corresponding solution $\{x^i(\cdot), \xi^i(\cdot)\}$ to (1) defined on $[0, T]$, then $\xi^i(\cdot) \in \mathbf{L}_2[0, T]$, and

$$\frac{1}{q}\sum_{i=1}^{q}\int_0^T \left(\|\xi^i(t)\|^2 - \|z^i(t)\|^2\right) dt \le d. \tag{2}$$

The class of all such admissible uncertainty sequences is denoted Ξ. One way in which such uncertainty could be generated is via unstructured feedback uncertainty is shown in the block diagram in Figure 1.

Definition 1. The *robust unobservability function* for the uncertain system (1), (2) defined on the time interval $[0,\ T]$ is defined as

$$L_o(x_0, T) \triangleq \sup_{\mathcal{S}\in\Xi} \frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y(t)\|^2 dt \tag{3}$$

where $x(0) = x_0$ in (1).

This definition extends the standard definition of the controllability Gramian for linear systems.

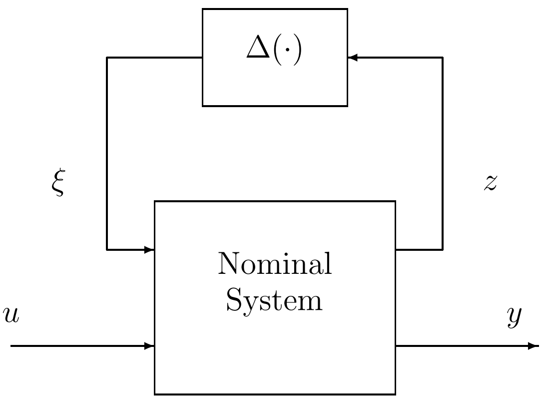

Fig. 1.

Notation.

$$\mathcal{D} \triangleq \{d : d > 0\}.$$

Definition 2. A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *robustly unobservable* for the uncertain system (1), (2) defined on the time interval $[0,\ T]$ if

$$\inf_{d\in\mathcal{D}} L_o(x_0, T) = 0.$$

The set of all robustly unobservable states for the uncertain system (1), (2) defined on the time interval $[0,\ T]$ is referred to as the *robustly unobservable set* $\mathcal{U}$; i.e.,

$$\mathcal{U} \triangleq \left\{x \in \mathbf{R}^n : \inf_{d\in\mathcal{D}} L_o(x, T) = 0\right\}.$$

Definition 3. The *possible controllability function* for the uncertain system (1), (2) defined on the time interval $[0,\ T]$ is defined as

$$L_c(x_0, T) \triangleq \sup_{\epsilon>0}\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[0,T]} \frac{1}{q}\sum_{i=1}^{q}\left[\begin{array}{c}\dfrac{\|x^i(-T)\|^2}{\epsilon}\\ +\displaystyle\int_0^T \|u^i(t)\|^2 dt\end{array}\right] \tag{4}$$

where $x(0) = x_0$ in (1).

This definition extends the standard definition of the observability Gramian for linear systems.

Definition 4. A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *possibly controllable on* $[0,\ T]$ for the uncertain system (1), (2) if

$$\sup_{d\in\mathcal{D}} L_c(x_0, T) < \infty.$$

Definition 5. A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *(differentially) possibly controllable* for the uncertain system (1), (2) if it is possibly controllable on $[0,\ T]$ for all $T > 0$ sufficiently small.

The set of all differentially possibly controllable states for the uncertain system (1), (2) is referred to as the *possibly controllable set* $\mathcal{C}$.

3. EXISTING RESULTS ON ROBUST UNOBSERVABILITY

In this section, we recall some existing results from Petersen (2005a) giving a geometrical characterization of robust unobservability.

For the uncertain system (1), (2) defined on the time interval $[0,\ T]$, we define a function $V_\tau(x_0, T)$ as follows:

$$V_\tau(x_0,T) \triangleq \inf_{\xi(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \left(-\|y\|^2 + \tau\|\xi\|^2 - \tau\|z\|^2 \right) dt. \tag{5}$$

Here $\tau \geq 0$ is a given constant.

$$\bar{\Gamma}(x_0,T) \triangleq \{ \tau : \tau \geq 0 \text{ and } V_\tau(x_0,T) > -\infty \}.$$

Assumption 1. For all $x_0 \in \mathbf{R}^n$, there exists a constant $\tau \geq 0$ such that $V_\tau(x_0, T) > -\infty$.

Remark: The above assumption is a technical assumption required to establish the results of Petersen (2005a). It represents an assumption on the size of the uncertainty in the system relative to the time interval $[0,\ T]$ under consideration. In general, this assumption can always be satisfied by choosing a sufficiently small $T > 0$.

Theorem 1. (See Petersen (2005a) for proof). Consider the uncertain system (1), (2) and suppose that Assumption 1 is satisfied. Also, suppose that $G(s) \equiv 0$. Then a state x_0 is robustly unobservable if and only if it is an unobservable state for the pair (C_2, A).

Remark: From the above theorem and the fact that $G(s) \equiv 0$, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown in Figure 2.

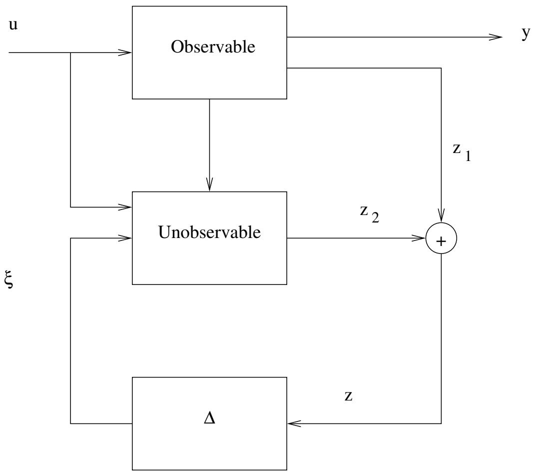

Fig. 2. Observable-Unobservable decomposition for the uncertain system when $G(s) \equiv 0$.

Note that in this case, all of the uncertainty is in the unobservable subsystem and the coupling between the two subsystems.

Theorem 2. (See Petersen (2005a) for proof). Consider the uncertain system (1), (2) and suppose that Assumption 1 is satisfied. Also, suppose that $G(s) \not\equiv 0$. Then a state x_0 is robustly unobservable if and only if it is an unobservable state for the pair $(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A)$.

Remark: The above theorem implies that when $G(s) \not\equiv 0$, the robustly unobservable set is a linear space equal to the unobservable subspace of the pair $(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A)$. From this theorem, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown in Figure 3.

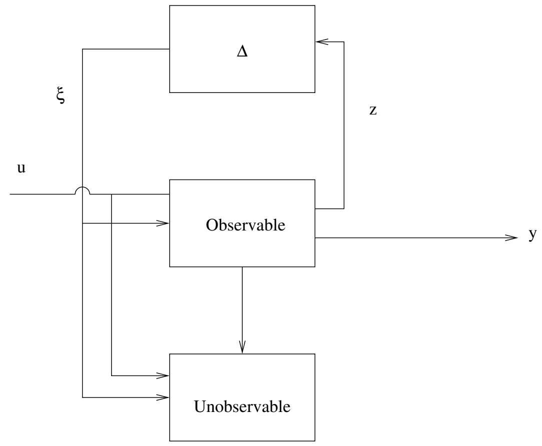

Fig. 3. Observable-Unobservable decomposition for the uncertain system when $G(s) \not\equiv 0$.

In this case, all of the uncertainty is in the observable subsystem or in the coupling between the two subsystems.

4. PRELIMINARY RESULTS ON POSSIBLE CONTROLLABILITY

In this section, we will recall the main results of Petersen (2005b) specialized to the class of uncertain systems considered in this paper.

4.1 A Family of Unconstrained Optimization Problems.

For the uncertain system (1), (2) defined on the time interval $[0,\ T]$, we define functions $W^\epsilon_\tau(x_0, \lambda, T)$, $W^\epsilon_\tau(x_0, T)$ and $W_\tau(x_0, T)$ as follows:

$$W^\epsilon_\tau(x_0,\lambda,T) \triangleq \inf_{[\xi(\cdot),u(\cdot)]\in \mathbf{L}_2[\lambda,T]} \frac{\|x(T)\|^2}{\epsilon} + \int_\lambda^T \left(\|u\|^2 + \tau\|\xi\|^2 - \tau\|z\|^2 \right) dt \tag{6}$$

subject to $x(\lambda) = x_0$;

$$W^\epsilon_\tau(x_0,T) \triangleq W^\epsilon_\tau(x_0,0,T);$$
$$W_\tau(x_0,T) \triangleq \sup_{\epsilon>0} W^\epsilon_\tau(x_0,T).$$

Here $\tau \geq 0$ is a given constant.

4.2 A Formula for the Possible Controllability Function.

Theorem 3. (See Petersen (2005b) for proof). Consider the uncertain system (1), (2) defined on the time interval $[0,\ T]$ and corresponding possible controllability function (4). Then for any $x_0 \in \mathbf{R}^n$,

$$L_c(x_0,T) = \sup_{\epsilon>0}\sup_{\tau\geq 0}\{W_\tau^\epsilon(x_0,T) - \tau d\};$$
$$= \sup_{\tau\geq 0}\{W_\tau(x_0,T) - \tau d\}. \quad (7)$$

Corollary 1. (See Petersen (2005b) for proof). If we define

$$\tilde{L}_c(x_0,T) \triangleq \sup_{d\in\mathcal{D}} L_c(x_0,T)$$

then

$$\tilde{L}_c(x_0,T) = \sup_{\epsilon>0}\sup_{\tau\geq 0} W_\tau^\epsilon(x_0,T) = \sup_{\tau\geq 0} W_\tau(x_0,T).$$

Observation 1. From the above corollary, it follows immediately that a non-zero state $x_0 \in \mathbf{R}^n$ is (differentially) possibly controllable for the uncertain system (1), (2) if and only if

$$\sup_{\epsilon>0}\sup_{\tau\geq 0} W_\tau^\epsilon(x_0,T) = \sup_{\tau\geq 0} W_\tau(x_0,T) < \infty \quad (8)$$

for all $T>0$ sufficiently small.

5. MAIN RESULTS ON POSSIBLE CONTROLLABILITY

In this section we recall some results from the paper Petersen (2007b) for the case in which the $H(s)\equiv 0$ and present some new results for the case in which $H(s)\not\equiv 0$. These results provide a geometric characterization of the differentially possibly controllable states of the uncertain system (1), (2). We first consider the case in which $H(s)\equiv 0$.

Theorem 4. (See Petersen (2007b) for Proof). Consider the uncertain system (1), (2). Also, suppose that $H(s)\equiv 0$. Then a state x_0 is differentially possibly controllable if and only if it is a controllable state for the pair (A,B_1).

Remark: The above theorem implies that when $H(s)\equiv 0$ the possibly controllable set is a linear space equal to the controllable subspace of the pair (A,B_1). From the above theorem and the fact that $H(s)\equiv 0$, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown in Figure 4.

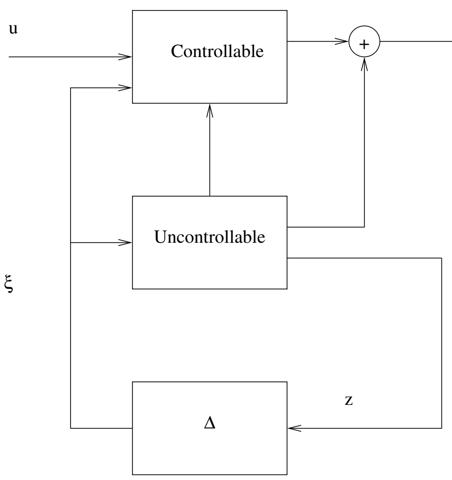

Fig. 4. Control-Uncontrollable decomposition for the uncertain system when $H(s)\equiv 0$.

In this case, we only have uncertainty in the uncontrollable subsystem or in the coupling between the two subsystems.

We now consider the case in which $H(s)\not\equiv 0$. The following theorem is the main new result of this paper.

Theorem 5. Consider the uncertain system (1), (2) and suppose that $H(s)\not\equiv 0$. Then, a state x_0 is differentially possibly controllable if and only if x_0 is a controllable state for the pair $(A,[B_1\ B_2])$.

Proof. Suppose x_0 is a differentially possibly controllable state for the uncertain system (1), (2). Hence, using Observation 1 it follows that

$$\sup_{\epsilon>0}\sup_{\tau\geq 0} W_\tau^\epsilon(x_0,T) < \infty \quad (9)$$

for $T>0$ sufficiently small. Setting $\tau=0$, it follows that there exists a constant $M>0$ such that

$$\inf_{[\xi(\cdot),u(\cdot)]\in\mathbf{L}_2[0,T]} \frac{\|x(T)\|^2}{\epsilon} + \int_0^T \|u\|^2 dt \leq M \quad \forall\epsilon>0$$

where the inf is defined for the system (1) with initial condition $x(0)=x_0$. From this it follows that

$$\inf_{[\xi(\cdot),u(\cdot)]\in\mathbf{L}_2[0,T]} \|x(T)\|^2 \leq \epsilon M \quad \forall\epsilon>0$$

and hence,

$$\inf_{[\xi(\cdot),u(\cdot)]\in\mathbf{L}_2[0,T]} \|x(T)\|^2 = 0.$$

Therefore, the state x_0 must be a controllable state for the pair $(A,[B_1\ B_2])$.

We now suppose the state x_0 is a controllable state for the pair $(A,[B_1\ B_2])$ and show that x_0 is a differentially possibly controllable state for the uncertain system (1), (2). In order to prove that the state x_0 is possibly controllable, we must show that for all $T>0$ sufficiently small $\sup_{\tau\geq 0} W_\tau(x_0,T)<\infty$. In order to show this, we let $T>0$ be given and establish the following claim:

Claim. For the system (1), there exists an input pair $\{u^*(\cdot),\xi^*(\cdot)\}$ defined on $[0,T]$ such that $x(0)=x_0$, $x(T)=0$ and

$$\int_0^T \left(\|\xi^*\|^2 - \|z^*\|^2\right) dt \leq 0.$$

To establish this claim, we first suppose that the standard Kalman decomposition is applied to the pair (A,B_1) to decompose it into controllable and uncontrollable subsystems. That is, we can assume without loss of generality that the system (1) is such that the matrices A, B_1, B_2, C_1 and the vector x are of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix};\quad B_1 = \begin{bmatrix} B_{11} \\ 0\end{bmatrix};$$
$$B_2 = \begin{bmatrix} B_{21} \\ B_{22}\end{bmatrix};\quad C_1 = [\,C_{11}\ C_{12}\,];$$
$$x = \begin{bmatrix} x_1 \\ x_2\end{bmatrix} \quad (10)$$

where the pair (A_{11},B_{11}) is controllable.

Now consider an input pair $\{\bar{u}(\cdot),\bar{\xi}(\cdot)\}$ defined on $[0,\frac{T}{3}]$ such that $x(0)=x_0$ and $x(\frac{T}{3})=0$. Such an input pair exists due to our assumption that x_0 is a controllable state for the pair $(A,[B_1\ B_2])$. Then, we can write

$$J_1 = \int_0^{\frac{T}{3}} \left(\|\bar{\xi}\|^2 - \|\bar{z}\|^2\right) dt < \infty.$$

Now for $t \in (\frac{T}{3}, \frac{2T}{3}]$, consider the input pair $\{\hat{u}(\cdot), \hat{\xi}(\cdot)\}$ defined so that $\hat{\xi}(\cdot) \equiv 0$ and so that $\hat{u}(\cdot)$ is such that the corresponding uncertainty output $\hat{z}(\cdot) \not\equiv 0$. Such an input $\hat{u}(\cdot)$ exists since we have assumed that $H(s) \not\equiv 0$. Then, we let

$$\gamma = \int_{\frac{T}{3}}^{\frac{2T}{3}} \|\hat{z}\|^2 dt > 0.$$

Also, since $x(\frac{T}{3}) = 0$ and $\hat{\xi}(t) = 0$ for $t \in (\frac{T}{3}, \frac{2T}{3}]$, it follows from (10) that $x_2(t) = 0$ for $t \in (\frac{T}{3}, \frac{2T}{3}]$.

Now for $t \in (\frac{2T}{3}, T]$, consider the input pair $\{\check{u}(\cdot), \check{\xi}(\cdot)\}$ defined so that $\check{\xi}(\cdot) \equiv 0$ and so that $\hat{u}(\cdot)$ is such that $x_1(T) = 0$. Such an input $\check{u}(\cdot)$ exists since we have assumed that the pair (A_{11}, B_{11}) is controllable. Also, since $x_2(\frac{2T}{3}) = 0$ and $\check{\xi}(t) = 0$ for $t \in (\frac{2T}{3}, T]$, it follows from (10) that $x_2(t) = 0$ for $t \in (\frac{2T}{3}, T]$. We let $\check{z}(t)$ denote the corresponding uncertainty output for $t \in (\frac{2T}{3}, T]$.

We now consider an input pair $\{u^*(\cdot), \xi^*(\cdot)\}$ defined as follows:

$$u^*(t) = \begin{cases} \bar{u}(t) & \text{for } t \in [0, \frac{T}{3}]; \\ \hat{u}(t) & \text{for } t \in (\frac{T}{3}, \frac{2T}{3}]; \\ \check{u}(t) & \text{for } t \in (\frac{2T}{3}, T]; \end{cases}$$

$$\xi^*(t) = \begin{cases} \bar{\xi}(t) & \text{for } t \in [0, \frac{T}{3}]; \\ 0 & \text{for } t \in (\frac{T}{3}, T]. \end{cases}$$

It follows from this construction that the pair $\{u^*(\cdot), \xi^*(\cdot)\}$ gives $x(T) = 0$ and

$$\begin{aligned} &\int_0^T \left(\|\xi^*\|^2 - \|z^*\|^2\right) dt \\ &= \int_0^{\frac{T}{3}} \left(\|\bar{\xi}\|^2 - \|\bar{z}\|^2\right) dt - \int_{\frac{T}{3}}^{\frac{2T}{3}} \|\hat{z}\|^2 dt \\ &\quad - \int_{\frac{2T}{3}}^{T} \|\check{z}\|^2 dt \\ &\leq J_1 - \gamma. \end{aligned}$$

We now let $\mu > 0$ be a scaling parameter and introduce a modified input pair $\{u^*(\cdot), \xi^*(\cdot)\}$ defined as follows:

$$u^*(t) = \begin{cases} \bar{u}(t) & \text{for } t \in [0, \frac{T}{3}]; \\ \mu\hat{u}(t) & \text{for } t \in (\frac{T}{3}, \frac{2T}{3}]; \\ \mu\check{u}(t) & \text{for } t \in (\frac{2T}{3}, T]; \end{cases}$$

$$\xi^*(t) = \begin{cases} \bar{\xi}(t) & \text{for } t \in [0, \frac{T}{3}]; \\ 0 & \text{for } t \in (\frac{T}{3}, T]. \end{cases}$$

It is straightforward to verify that this input pair also leads to $x(T) = 0$ and

$$\int_0^T \left(\|\xi^*\|^2 - \|z^*\|^2\right) dt \leq J_1 - \mu^2\gamma.$$

Letting,

$$\mu = \sqrt{\frac{J_1}{\gamma}}$$

it follows that

$$\int_0^T \left(\|\xi^*\|^2 - \|z^*\|^2\right) dt \leq 0$$

and hence, the conditions of the claim are satisfied. This completes the proof of the claim.

We now use this claim to complete the proof. Indeed, for any $\tau \geq 0$ and $\epsilon > 0$, we have

$$\begin{aligned} W_\tau^\epsilon(x_0, T) = &\inf_{[\xi(\cdot), u(\cdot)] \in \mathbf{L}_2[0,T]} \frac{\|x(T)\|^2}{\epsilon} \\ &+ \int_0^T \left(\|u\|^2 + \tau\|\xi\|^2 - \tau\|z\|^2 \right) dt \\ \leq &\int_0^T \left(\|u^*\|^2 + \tau\|\xi^*\|^2 - \tau\|z^*\|^2 \right) dt \end{aligned} \tag{11}$$

where the input pair $\{u^*(\cdot), \xi^*(\cdot)\}$ is constructed using the above claim such that $x(0) = x_0$ and $x(T) = 0$ and

$$\int_0^T \left(\|\xi^*\|^2 - \|z^*\|^2 \right) dt \leq 0.$$

Also, $z^*(\cdot)$ is the corresponding uncertainty output for the system (1). Since $\epsilon > 0$ was arbitrary, it follows from (11) that

$$\begin{aligned} W_\tau(x_0, T) = &\sup_{\epsilon > 0} W_\tau^\epsilon(x_0, T) \\ \leq &\int_0^T \|u^*\|^2 dt + \tau \int_0^T \left(\|\xi^*\|^2 - \|z^*\|^2 \right) dt \\ \leq &\int_0^T \|u^*\|^2 dt \end{aligned} \tag{12}$$

for all $\tau \geq 0$. Thus, we can conclude that

$$\sup_{\tau \geq 0} W_\tau(x_0, T) < \infty.$$

Since, $T > 0$ was arbitrary, it follows from Observation 1 that x_0 is differentially possibly controllable. This completes the proof of the theorem. □

Remark: The above theorem implies that when $H(s) \not\equiv 0$ the possibly controllable set is a linear space equal to the controllable subspace of the pair $(A, [B_1\ B_2])$. From the above theorem, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown in Figure 5.

In this case, we only have uncertainty in the controllable subsystem or in the coupling between the two subsystems.

6. KALMAN DECOMPOSITIONS

We can now combine the results of Theorems 1, 2, 4, and 5 to obtain a complete Kalman decomposition for the uncertain system in the following cases:

Case 1 $G(s) \equiv 0$, $H(s) \equiv 0$. In this case, we apply the standard Kalman decomposition to the triple (C_2, A, B_1) to obtain the situation as illustrated in the block diagram shown in Figure 6.

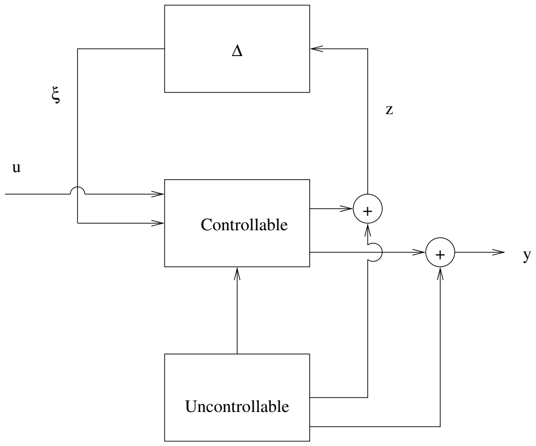

Fig. 5. Control-Uncontrollable decomposition for the uncertain system when $H(s) \not\equiv 0$.

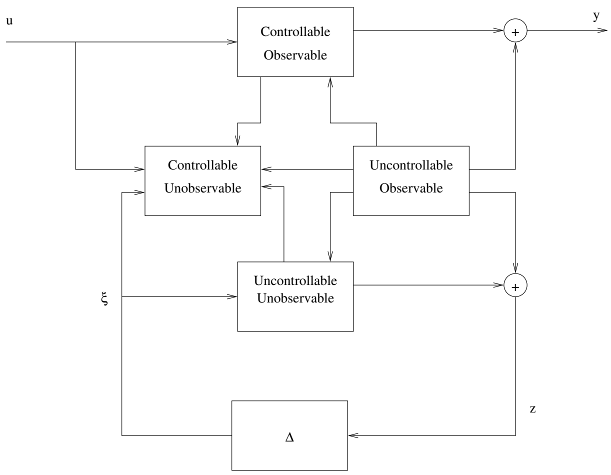

Fig. 6. Kalman decomposition for the uncertain system when $G(s) \equiv 0$, $H(s) \equiv 0$.

This situation corresponds to uncertainty only in the uncontrollable-unobservable block. Also there is uncertainty in the coupling between uncontrollable-observable block and the uncontrollable-unobservable block. Furthermore, there is uncertainty in the coupling between the uncontrollable-unobservable block and the controllable-unobservable block.

Case 2 $G(s) \not\equiv 0$, $H(s) \equiv 0$. In this case, we apply the standard Kalman decomposition to the triple

$$\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A, B_1\right)$$

to obtain the situation as illustrated in the block diagram shown in Figure 7.

This situation corresponds to uncertainty only in the uncontrollable-observable block. Also there is uncertainty in the coupling between uncontrollable-observable block and the uncontrollable-unobservable block. Furthermore, there is uncertainty in the coupling between the uncontrollable observable block and the controllable-unobservable block. As well, there is uncertainty in the coupling between the uncontrollable-observable block and the controllable-observable block.

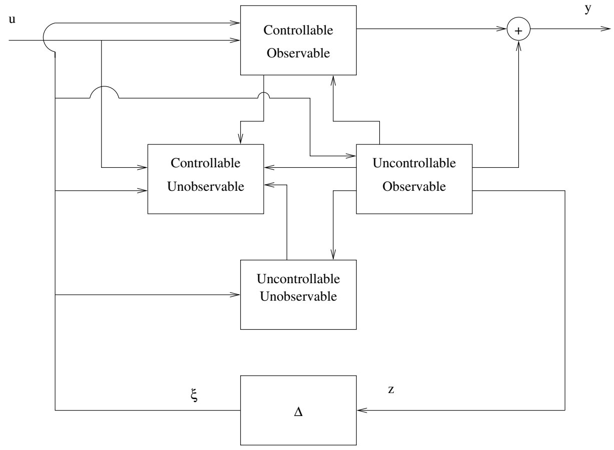

Fig. 7. Kalman decomposition for the uncertain system when $G(s) \not\equiv 0$, $H(s) \equiv 0$.

Note that in order to guarantee that the condition $H(s) \equiv 0$ we needed to make a further restriction on the controllable observable block in the above diagram so that in fact it only has an output y.

Case 3 $G(s) \equiv 0$, $H(s) \not\equiv 0$. In this case, we apply the standard Kalman decomposition to the triple

$$(C_1, A, [B_1\ B_2])$$

to obtain the situation as illustrated in the block diagram shown in Figure 8.

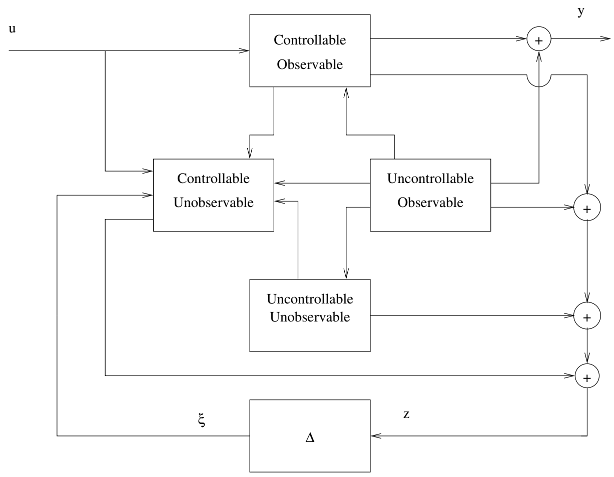

Fig. 8. Kalman decomposition for the uncertain system when $G(s) \equiv 0$, $H(s) \not\equiv 0$.

This situation corresponds to uncertainty only in the controllable-unobservable block. Also there is uncertainty in the coupling between controllable-observable block and each of the other blocks.

Case 4 $G(s) \not\equiv 0$, $H(s) \not\equiv 0$. In this case, we apply the standard Kalman decomposition to the triple

$$\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A, [B_1\ B_2]\right)$$

to obtain the situation as illustrated in the block diagram shown in Figure 9.

This situation corresponds to uncertainty only in the controllable-observable block. Also there is uncertainty in

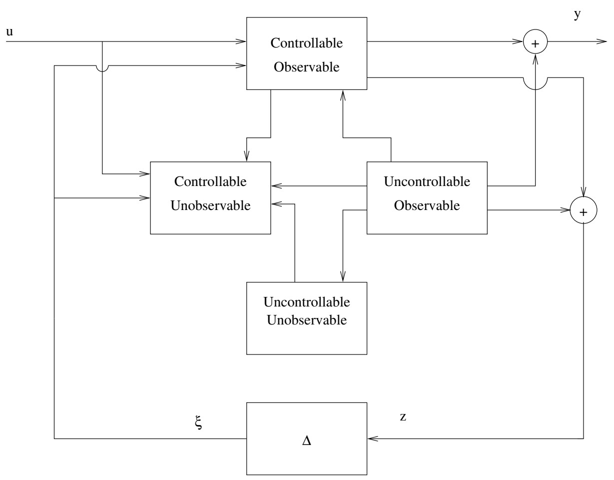

Fig. 9. Kalman decomposition for the uncertain system when $G(s) \not\equiv 0$, $H(s) \not\equiv 0$.

the coupling between controllable-observable block and the uncontrollable-observable block. Furthermore, there is uncertainty in the coupling between the controllable-observable block and the controllable-unobservable block. As well, there is uncertainty in the coupling between the uncontrollable-observable block and the controllable-unobservable block.

Remark Note that each of the four cases considered above corresponds to uncertainty only in one of the four blocks in the Kalman decomposition. It might be conjectured that if structured uncertainty was allowed then we could distribute the uncertainty blocks around the four blocks in the Kalman decomposition rather than the current requirement that the single uncertainty block corresponds to uncertainty in one of the four blocks in the Kalman decomposition.

7. CONCLUSIONS AND FUTURE RESEARCH

The results of this paper have led to a geometric characterization of the notion of possible controllability for a class of uncertain linear systems. These results combined with a corresponding geometric characterization of the notion of robust unobservability have allowed us to present a complete Kalman decomposition for uncertain systems.

Possible areas of future research motivated by the results of this paper include relating the results of this paper to the question of minimum realization for uncertain systems using the results of Petersen (2007a). Another possible area of future research would be to extend the results of the paper to the case of structured uncertainty subject to multiple IQCs.

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