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Conjugation of Hamiltonian Systems in Optimal Control Problems ⋆

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Abstract: The optimal control problem with a functional given by an improper integral is considered for models of economic growth. Properties of concavity of the maximized Hamiltonian are examined and analysis of Hamiltonian systems in the Pontryagin maximum principle is implemented including estimation of steady states and conjugation of domains with different Hamiltonian dynamics. On the basis of this analysis an algorithm is proposed for construction of optimal trajectories by sewing dynamics of Hamiltonian systems. The proposed algorithm is illustrated by computer simulations of optimal trajectories in models of economic growth for real macroeconomic data.

Keywords: optimal control, nonlinear control systems, numerical algorithms, economic systems.

1. INTRODUCTION

The paper deals with analysis of the optimal control problem on infinite horizon. Such statement of the problem arises in models of economic growth (see Arrow [1968], Intriligator [1971], Tarasyev and Watanabe [2001]). Production factors play the role of phase variables in these models. Control parameters are associated with investments into production factors. The level of output is defined by a production function. The discounted integral of logarithmic consumption index stands for the utility function.

The Pontryagin maximum principle (see Pontryagin et al. [1962]) is applied for analysis of the optimal investment problem. Specifically, the research is based on existence results and necessary conditions of optimality in problems with infinite horizon (see Aseev and Kryazhimskiy [2007]).

Analysis of properties of the maximized Hamiltonian is fulfilled for the dynamic problem of investments optimization. It is shown that under general conditions the Hamiltonian is a smooth function of its variables. Under the condition of strict concavity of a production function it is proved on the basis of methods of convex analysis (see Rockafellar [1970]) that the maximized Hamiltonian conserves the property of strict concavity. Namely, it is demonstrated that the maximized Hamiltonian is composed by sewing several smooth strictly concave components in such a way that the result of composition is smooth and strictly concave in phase variables. Description of domains corresponding to different regimes of formation of optimal control is provided and sewing curves of these domains are indicated. Sufficient conditions of optimality are obtained for a class of control systems with concave production functions.

Properties of Hamiltonian systems are examined for different regimes of optimal control. The existence and uniqueness result is proved for a steady state of the Hamiltonian system. Analysis of properties of eigenvalues and eigenvectors is implemented for the linearized system in a neighborhood of the steady state. Description of behavior of the nonlinear Hamiltonian system is provided on the basis of results of the qualitative theory of differential equations (see Hartman [1964]). This analysis allows to outline proportions of the main economic factors and trends of optimal growth in the model.

A numerical algorithm for construction of optimal trajectories of economic growth is elaborated on the basis of constructions of backward procedures and conjugation of an approximate linear dynamics with the nonlinear Hamiltonian dynamics (see Krasovskii and Krasovskii [1995]). The algorithm is simulated basing on the real data for the economy of Japan.

2. MODEL OF ECONOMIC GROWTH

A model is focused on the analysis of GDP of a country which is defined as the market value of all final goods and services produced within a country in a year. Two production factors are considered in a model. If symbols $K(t)$ and $L(t)$ denote stocks of capital and labor, respectively, at time t, then the output $Y(t)$ at time t is given by equation

$$Y(t) = F\big[K(t), L(t)\big]. \qquad (1)$$

Here the symbol F denotes production function. Using the fact that the production function is commonly assumed to be homogenous of degree one it is possible to fix relation between quantity of output per worker and quantities of capital per worker. Introducing per worker notations

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$y = Y/L$ for GDP, and $k = K/L$ for capital, one can consider a per worker production function

$$y(t) = f(k(t)) = F\Big[\frac{K(t)}{L(t)}, 1\Big]. \tag{2}$$

Let symbols $C(t) \geq 0$ and $I(t) \geq 0$ denote rates at time t of consumption and investment, respectively, and the symbol $s(t)$, $0 \leq s(t) \leq 1$, denotes the fraction of output which is saved and invested. Then the national income is defined by the formula

$$Y(t) = C(t) + I(t) = \big(1 - s(t)\big)Y(t) + s(t)Y(t). \tag{3}$$

It is assumed that the model characterizes growth in an aggregative closed economy. The capital stock is accumulated according to equation

$$\dot{K}(t) = s(t)Y(t) - \mu K(t). \tag{4}$$

Here parameter $\mu > 0$ is the rate of capital depreciation. It is assumed that the labor input grows exponentially $\dot{L}(t)/L(t) = n$, with a constant growth rate $n > 0$. Then dynamics of capital per worker is described by equation

$$\dot{k}(t) = s(t)y(t) - \lambda k(t), \tag{5}$$

where $\lambda = \mu + n$ is capital decay, and n is capital dilution.

Let us assume that function $f(k)$ has the following properties

$$f'(k) > 0 \text{ and } f''(k) < 0 \text{ for } k \in K^0 \subset (0,\ +\infty). \tag{6}$$

Here $f'(k)$ is the marginal productivity of capital per worker. The symbol K^0 stands for a nonempty set which is called economic domain (Intriligator [1971]). It is assumed that function $f(k)$ satisfies the "Inada's limit conditions"

$$\begin{cases} \lim\limits_{k\downarrow 0} f(k) = 0, & \lim\limits_{k\uparrow +\infty} f(k) = +\infty, \\ \lim\limits_{k\downarrow 0} f'(k) = +\infty, & \lim\limits_{k\uparrow +\infty} f'(k) = 0. \end{cases} \tag{7}$$

3. OPTIMAL CONTROL SYNTHESIS

3.1 Optimal Control Problem

Let us consider the optimal control problem for growth of the capital stock. Let us introduce the utility functional as the integral of the logarithmic consumption index discounted on the infinite horizon

$$J = \int_0^{+\infty} \Big[\ln f\big(k(t)\big) + \ln\big(1 - s(t)\big)\Big] e^{-\delta t} dt. \tag{8}$$

Here the symbol $\delta > 0$ denotes the constant rate of discount. A central planner starts his investment process with the initial level $k(0) = k^0$ and aims at maximization of the utility functional.

Control Problem. Stated specifically, the problem is to maximize the functional

$$J = \int_0^{+\infty} \Big[\ln f\big(k(t)\big) + \ln\big(1 - s(t)\big)\Big] e^{-\delta t} dt \underset{(k(\cdot),s(\cdot))}{\longrightarrow} \max \tag{9}$$

under the following dynamic constraints

$$\dot{k}(t) = s(t)f(k(t)) - \lambda k(t),\ k(0) = k^0,\ s \in [0, a],\ a < 1, \tag{10}$$

where parameters δ, $\lambda = n + \mu$, k^0 are given positive numbers and $s(t)$ is control variable measurable in time t. Parameter $0 < a < 1$ is a positive number which separates the right bound of control parameter from unit. Let us note that condition of compactness of control restrictions $s \in [0, a]$ is important for accurate application of the Pontryagin maximum principle (Pontryagin et al. [1962]).

The problem is to find the optimal investment level $s^0(\cdot)$ and the corresponding trajectory $k^0(\cdot)$ of the capital per worker stock k subject to dynamics (5) for maximizing the consumption per worker functional (8).

3.2 Hamiltonians in the Pontryagin Maximum Principle

Let us apply the Pontryagin maximum principle to the problem (9)-(10). Introducing the adjoint variable $\tilde{\psi} = \tilde{\psi}(t)$, interpreted in economy as a shadow price of capital, one can compile the Hamiltonian of the problem

$$\tilde{H}(s, k, t, \tilde{\psi}) = \big[\ln(1 - s)f(k)\big]e^{-\delta t} + \tilde{\psi}\big(sf(k) - \lambda k\big). \tag{11}$$

To exclude the exponential term depending on time from the Hamiltonian let us introduce new variables

$$\psi = \tilde{\psi}e^{\delta t}, \quad H(s, k, \psi) = e^{\delta t}\tilde{H}(s, k, t, \psi), \tag{12}$$

and consider the stationary form of the Hamiltonian

$$H(s, k, \psi) = \ln f(k) + \ln(1 - s) + \psi\big(sf(k) - \lambda k\big). \tag{13}$$

3.3 Existence of the Optimal Solution and Necessary Optimality Conditions

Let us mention that for control problem (9)-(10) all conditions of the existence theorem (see Aseev and Kryazhimskiy [2007]) are fulfilled. Moreover, one can formulate the necessary conditions of optimality for problems with infinite horizon in the form of the Pontryagin maximum principle.

Theorem 1. Necessary Optimality Conditions. Let (s^0, k^0) be an optimal process. Then there exists an adjoint variable $\tilde{\psi}$ corresponding to process (s^0, k^0) and satisfying the adjoint equation

$$\dot{\tilde{\psi}}(t) = -\frac{\partial H}{\partial k}(s^0(t), k^0(t), t, \tilde{\psi}(t)), \tag{14}$$

such that

(i) process (s^0, k^0) satisfies the conditions of the Pontryagin maximum principle together with adjoint variable $\tilde{\psi}$

$$H(s^0, k^0, t, \tilde{\psi}) = \max_{s\in[0,a]} H(s, k^0, t, \tilde{\psi}); \tag{15}$$

(ii) process (s^0, k^0) and adjoint variable $\tilde{\psi}$ satisfy the stationarity condition

$$H(s^0, k^0, t, \tilde{\psi}) = \delta \int_t^{\infty} e^{-\delta\tau}\Big[\ln f\big(k^0(\tau)\big) + \ln\big(1 - s^0(\tau)\big)\Big] d\tau$$

(iii) $\tilde{\psi}(t) > 0$ for all $t \geq 0$;

(iv) adjoint variable $\tilde{\psi}(t)$ satisfies the transversality condition

$$\lim_{t\to\infty} k^0(t)\tilde{\psi}(t) = 0. \tag{16}$$

4. CONCAVITY PROPERTIES OF HAMILTONIANS

Let us analyze properties of the Hamiltonian (13).

Lemma 2. The Hamiltonian $H(s, k, \psi)$ (13) is a strictly concave function in variable s.

The proof follows immediately from strict negativity of the second derivative of the Hamiltonian (13) in s.

Let us introduce the necessary maximum condition for the Hamiltonian $H(s, k, \psi)$ (13) in the absence of restrictions

$$\frac{\partial H}{\partial s} = -\frac{1}{1-s} + \psi f(k) = 0. \tag{17}$$

This equation implies the following expression for the optimal investment level

$$s^0 = 1 - \frac{1}{\psi f(k)}. \tag{18}$$

Let us introduce the construction of the maximized Hamiltonian in presence of restrictions on control variable s

$$\hat{H}(k, \psi) = \max_{s \in [0,a]} H(s, k, \psi). \tag{19}$$

Lemma 3. The maximized Hamiltonian $\hat{H}(k, \psi)$ is constructed basing on location of the maximum point s^0 according to the following algorithm:

1. If $s^0 \in [0, a]$ then $\hat{H}(k, \psi) = H(s^0, k, \psi)$.

2. If $s^0 < 0$ then $\hat{H}(k, \psi) = H(0, k, \psi)$.

3. If $s^0 > a$ then $\hat{H}(k, \psi) = H(a, k, \psi)$.

Proof. Since function $s \mapsto H(s, k, \psi)$ is strictly concave according to Lemma 1 then point s^0 (18) is the global maximum point. Therefore, if $s^0 \in [0, a]$ then it is clear that point s^0 is the global maximum point on the interval $[0, a]$.

If $s^0 < 0$ then for $s > s^0$ function $s \mapsto H(s, k, \psi)$ strictly monotonically decreases. Therefore, point $s = 0$ is the global maximum point on the interval $[0, a]$.

If $s^0 > a$ then for $s < s^0$ function $s \mapsto H(s, k, \psi)$ strictly monotonically grows. Therefore, point $s = a$ is the global maximum point on the interval $[0, a]$. □

Hence, the maximized Hamiltonian can be considered as conjugation of three Hamiltonians corresponding to values of the optimal investment plan inside and on the bounds of the interval $[0, a]$.

Let us denote by the symbol $H_1(k, \psi)$ the first branch of the maximized Hamiltonian $\hat{H}(k, \psi)$ that results from the Hamiltonian $H(s, k, \psi)$ (13) at the optimal regime $s = 0$

$$H_1(k, \psi) = \ln f(k) - \psi \lambda k. \tag{20}$$

At the optimal regime $s = s^0$ the Hamiltonian $H(s, k, \psi)$ (13) possesses the value of the second branch $H_2(k, \psi)$ of the maximized Hamiltonian $\hat{H}(k, \psi)$

$$H_2(k, \psi) = -\ln \psi + \psi f(k) - \psi \lambda k - 1. \tag{21}$$

At the optimal regime $s = a$ the Hamiltonian $H(s, k, \psi)$ (13) possesses the value of the third branch $H_3(k, \psi)$ of the maximized Hamiltonian $\hat{H}(k, \psi)$

$$H_3(k, \psi) = \ln f(k) + \ln(1-a) + \psi(af(k) - \lambda k). \tag{22}$$

Let us describe a formation rule for the maximized Hamiltonian $\hat{H}(k, \psi)$ out of its branches $H_i(k, \psi)$, $i = 1, 2, 3$. Let us define the sewing curve of branches $H_1(k, \psi)$ and $H_2(k, \psi)$. To this end, one should compose the difference

$$H_1 - H_2 = \ln(\psi f(k)) - \psi f(k) + 1. \tag{23}$$

One can easily check that this difference is less or equal to zero. Let us note that it vanishes on the unique curve L_1 described by equation

$$L_1 = \{(k, \psi) : \psi = \frac{1}{f(k)},\ k > 0, \psi > 0\}. \tag{24}$$

From Inada's conditions (7) it follows that curve L_1 has a hyperbolic form and the following relations are valid

$$\psi \to +\infty, \quad \text{for} \quad k \to 0; \quad \psi \to 0, \quad \text{for} \quad k \to +\infty.$$

Let us define the sewing curve of branches $H_2(k, \psi)$ and $H_3(k, \psi)$. Consider the difference

$$H_3 - H_2 = \ln\big((1-a)\psi f(k)\big) - (1-a)\psi f(k) + 1. \tag{25}$$

This difference is less or equal to zero and it vanishes at the unique curve L_2 described by equation

$$L_2 = \{(k, \psi) : \psi = \frac{1}{(1-a)f(k)},\ k > 0, \psi > 0\}. \tag{26}$$

The curve L_2 is also a hyperbola which lies beyond the curve L_1.

On the basis of sewing curves L_1 and L_2 one can indicate a formation rule for the maximized Hamiltonian $\hat{H}(k, \psi)$:

1. In domain

$$D_1 = \{(k, \psi) : \psi \leq \frac{1}{f(k)},\ k > 0, \psi > 0\} \tag{27}$$

assume $\hat{H}(k, \psi) = H_1(k, \psi)$.

2. In domain

$$D_2 = \{(k, \psi) : \frac{1}{f(k)} \leq \psi \leq \frac{(1-a)^{-1}}{f(k)}, k > 0, \psi > 0\} \tag{28}$$

assume $\hat{H}(k, \psi) = H_2(k, \psi)$.

3. In domain

$$D_3 = \{(k, \psi) : \psi \geq \frac{1}{(1-a)f(k)},\ k > 0, \psi > 0\} \tag{29}$$

assume $\hat{H}(k, \psi) = H_3(k, \psi)$.

Configuration of domains D_j, $j = 1, 2, 3$, and sewing curves L_i, $i = 1, 2$, is given on Fig. 1-b.

Lemma 4. The maximized Hamiltonian $\hat{H}(k, \psi)$ is smoothly pasted out of branches $H_i(k, \psi)$, $i = 1, 2, 3$, in variables (k, ψ) on sewing curves L_i, $i = 1, 2$.

The result of Lemma 4 is proved by direct calculations of derivatives of the Hamiltonians on sewing curves.

Lemma 5. The maximized Hamiltonian $\hat{H}(k, \psi)$ is a strictly concave function in variable k for all $\psi > 0$.

Proof. Let is analyze properties of branches $H_i(k, \psi)$, $i = 1, 2, 3$. Strict concavity of the branch $k \mapsto H_1(k, \psi)$ follows from the following inequality for the second derivative

$$\frac{\partial^2 H_1}{\partial k^2} = \frac{f''(k)f(k) - (f'(k))^2}{(f(k))^2} < 0.$$

Similarly, strict concavity of the branch $k \mapsto H_2(k,\psi)$ follows from the inequality

$$\frac{\partial^2 H_2}{\partial k^2} = \psi f''(k) < 0,$$

and for the branch $k \mapsto H_3(k,\psi)$ – from the inequality

$$\frac{\partial^2 H_3}{\partial k^2} = \frac{f''(k)f(k) - (f'(k))^2}{(f(k))^2} + a\psi f''(k) < 0.$$

Thus, branches $H_i(k,\psi)$, $i = 1,2,3$, are strictly concave and are smoothly pasted on sewing curves L_i, $i = 1,2$. Hence, the graph of the continuously differentiable maximized Hamiltonian $k \mapsto \hat{H}(k,\psi)$ lies below lines tangent to this graph at any point including points on sewing curves. Therefore, the maximized Hamiltonian $k \mapsto \hat{H}(k,\psi)$ is a strictly concave function according to results of convex analysis (see Rockafellar [1970]). □

The graph of the maximized Hamiltonian (19) is shown on Fig. 1-a.

5. SUFFICIENT OPTIMALITY CONDITIONS IN THE PONTRYAGIN MAXIMUM PRINCIPLE

Theorem 6. Under conditions of Lemmas 2 and 3-5 providing the smoothness property of the maximized Hamiltonian $\hat{H}(k,\psi)$ in variables (k,ψ) and its strict concavity in variable k, the Pontryagin maximum principle ensures sufficient optimality conditions in problem (9)-(10).

Proof. Let us denote by the symbol $(k^*(\cdot), s^*(\cdot))$ the control process satisfying the Pontryagin maximum principle in problem (9)-(10). Such process exists according to Aseev and Kryazhimskiy [2007]. Let us consider an arbitrary control process $(k(\cdot), s(\cdot))$ differing from the optimal process $(k^*(\cdot), s^*(\cdot))$. By virtue of smoothness and concavity properties of the maximized Hamiltonian $\hat{H}(k,\psi)$ the following estimate takes place for positive values of the adjoint variable $\psi(t) > 0$

$$\langle \frac{\partial \hat{H}}{\partial k}\left(t, k^*(t), \psi(t)\right), k^*(t) - k(t)\rangle <$$
$$\hat{H}\left(t, k^*(t), \psi(t)\right) - \hat{H}\left(t, k(t), \psi(t)\right),\ k^*(t) \neq k(t). \quad (30)$$

Here the symbol $\langle x, y\rangle$ denotes the scalar product of vectors x and y.

Let us multiply both parts of the adjoint equation in the Pontryagin maximum principle by $(k(t) - k^*(t))$

$$\langle \dot{\psi}(t) - \delta\psi(t), k(t) - k^*(t)\rangle =$$
$$\langle \frac{\partial \hat{H}}{\partial k}\left(t, k^*(t), \psi(t)\right), k^*(t) - k(t)\rangle. \quad (31)$$

Relations (30)-(31) lead to the inequality

$$\langle \dot{\psi}(t) - \delta\psi(t), k(t) - k^*(t)\rangle <$$
$$\hat{H}\left(t, k^*(t), \psi(t)\right) - \hat{H}\left(t, k(t), \psi(t)\right). \quad (32)$$

Substituting expression for the difference of Hamiltonians into the right-hand side of inequality (32) one obtains the following relation

$$\langle \dot{\psi}(t) - \delta\psi(t), k(t) - k^*(t)\rangle < \langle \psi(t), \dot{k}^*(t) - \dot{k}(t)\rangle +$$
$$\ln\left(f(k^*(t))(1 - s^*(t))\right) - \ln\left(f(k(t))(1 - s(t))\right). \quad (33)$$

Passing to the adjoint variable $\tilde{\psi}(t)$ (12) one can obtain the the following form of inequality (33)

$$\frac{d}{dt}\langle \tilde{\psi}(t), k(t) - k^*(t)\rangle + \Big[\ln f(k(t)) + \ln(1 - s(t))\Big]e^{-\delta t} <$$
$$\Big[\ln f(k^*(t)) + \ln(1 - s^*(t))\Big]e^{-\delta t}.$$

Let us integrate both parts of this inequality over time t and pass to the limit on the half-interval $[t_0, +\infty)$

$$\lim_{t \to +\infty} \langle \tilde{\psi}(t), k(t) - k^*(t)\rangle - \langle \tilde{\psi}(t_0), k(t_0) - k^*(t_0)\rangle +$$
$$\int_{t_0}^{+\infty} \Big[\ln f(k(t)) + \ln(1 - s(t))\Big]e^{-\delta t} dt \leq$$
$$\int_{t_0}^{+\infty} \Big[\ln f(k^*(t)) + \ln(1 - s^*(t))\Big]e^{-\delta t} dt. \quad (34)$$

Substituting the initial condition (10) and the transversality condition on the infinite horizon (16) to relation (34) one obtains the inequality

$$\int_{t_0}^{+\infty} \Big[\ln f(k(t)) + \ln(1 - s(t))\Big]e^{-\delta t} dt \leq$$
$$\int_{t_0}^{+\infty} \Big[\ln f(k^*(t)) + \ln(1 - s^*(t))\Big]e^{-\delta t} dt. \quad (35)$$

The last relation proves that the control process $(k^*(\cdot), s^*(\cdot))$ satisfying the Pontryagin maximum principle is the optimal process. □

Corollary 7. If the maximized Hamiltonian $\hat{H}(k)$ has strictly negative second derivative then one can prove that the control process satisfying the Pontryagin maximum principle is the strict global maximum.

Proof. The following lemma plays the key role in proof of this fact.

Lemma 8. Let function $\hat{H}(k)$ be twice continuously differentiable and its second derivative be strictly negative $\hat{H}''(k) < 0$ in compact domain K^1. Then there exists a positive constant $\alpha > 0$ such that the following relation takes place

$$\hat{H}(k) < \hat{H}(k^*) + \frac{\partial \hat{H}}{\partial k}(k^*)(k - k^*) - \alpha(k - k^*)^2,$$
$$k^*, k \in K^1,\ k \neq k^*. \quad (36)$$

Proof. Let us present function $\hat{H}$ according to the Taylor expansion of second order with the Cauchy remainder term

$$\hat{H}(k) = \hat{H}(k^*) + \frac{\partial \hat{H}}{\partial k}(k^*)(k - k^*) + \frac{\partial^2 \hat{H}}{\partial k^2}(z)\vartheta(k - k^*)^2,$$
$$z = \vartheta k^* + (1 - \vartheta)k,\ 0 < \vartheta < 1,\ \vartheta = \vartheta(k). \quad (37)$$

Let us show that there exists a positive constant α such that the following inequality takes place

$$\Big|\frac{\partial^2 \hat{H}}{\partial k^2}(z)\Big|\vartheta(k-k^*)^2 > \alpha(k-k^*)^2. \qquad (38)$$

On the contrary, assume that

$$\inf_{k\in K^1} \vartheta(k) = 0. \qquad (39)$$

Let sequence $k_n \in K^1$ be such that $\vartheta_n = \vartheta(k_n) \to 0$ when $n \to \infty$. Since K^1 is a compact set then without loss of generality one can assume that sequence k_n converges, $\lim_{k_n\to\infty} = k^1 \in K^1$. Two cases are possible. In the first case, $k^1 \neq k^*$. In this case, passing to the limit in formula (37) one obtains the relation

$$\hat{H}(k^1) = \hat{H}(k^*) + \frac{\partial \hat{H}}{\partial k}(k^*)(k^1 - k^*), \qquad (40)$$

which contradicts to strict concavity of function $\hat{H}(k)$. In the second case, $k^1 = k^*$. Let us present the Cauchy remainder term in the Taylor form

$$\frac{\partial^2 \hat{H}}{\partial k^2}(z_n)\vartheta_n(k_n-k^*)^2 = \frac{\partial^2 \hat{H}}{\partial k^2}(k^*)(k_n-k^*)^2 + o\big((k_n-k^*)^2\big). \qquad (41)$$

Dividing the last formula by $(k_n - k^*)^2$ and passing to the limit, one obtains the relation

$$\frac{\partial^2 \hat{H}}{\partial k^2}(k^*) = 0, \qquad (42)$$

which contradicts to assumptions of Lemma. Finally, one concludes that

$$\inf_{k\in K^1} \vartheta(k) > 0. \qquad (43)$$

The last inequality proves the Lemma. □

To finalize the proof of Corollary, one should substitute inequality (32) by estimate (36). Then integration of the quadratic term $\alpha(k - k^*)^2$ in relation (34) provides the addition strictly positive term which gives the strict inequality for global maximum in relation (35). □

6. QUALITATIVE ANALYSIS OF THE HAMILTONIAN SYSTEM

6.1 Hamiltonian System in the Steady State Domain

Consider the Hamiltonian system in the steady state domain D_2 (28)

$$\begin{cases} \dot{\psi} = \psi\left(\delta + \lambda - f'(k)\right), \\ \dot{k} = f(k) - \lambda\, k - \dfrac{1}{\psi}\,. \end{cases} \qquad (44)$$

Let us introduce the new variable $z = \psi k$ and express the Hamiltonian system in variables (k, z)

$$\begin{cases} \dot{z} = z\Big(\dfrac{f(k)}{k} + \delta - f'(k)\Big) - 1, \\ \dot{k} = f(k) - \lambda k - \dfrac{k}{z}. \end{cases} \qquad (45)$$

A steady state of the Hamiltonian system (45) is defined by the system of equations

$$\begin{cases} \Big(\dfrac{f(k)}{k} + \delta - f'(k)\Big) - 1 = 0, \\ f(k) - \lambda k - \dfrac{k}{z} = 0. \end{cases} \qquad (46)$$

Lemma 9. There exists the unique steady state (k^*, z^*) for which the following estimates are valid

$$k^* > 0, \quad 0 < z^* < \frac{1}{\delta}. \qquad (47)$$

The proof of the lemma follows directly from the property of strict concavity of the production function $f(k)$.

Lemma 10. At the steady state (k^*, z^*) the optimal investment level s^0 *(18)* is bounded below by zero and above by a number strictly less than unit.

Proof. Indeed, the following estimate takes place

$$0 < s^0(k^*) < \lambda\frac{1}{f'(k^*)} = \frac{\lambda}{\lambda+\delta} < 1. \quad \square \qquad (48)$$

6.2 Saddle Character of the Steady State

Analysis of properties of the steady state (k^*, z^*) is based on characterization of eigenvalues and eigenvectors of the linearized Hamiltonian system.

Lemma 11. Eigenvalues of the linearized Hamiltonian system are real numbers. One of them is positive and another one is negative. Moreover, the positive eigenvalue is larger than the discount coefficient δ.

Lemma is proved by direct calculations of coefficients of the Taylor expansion for the Hamiltonian system (45).

Remark 1. Lemma 11 implies that the steady state (k^*, z^*) is a saddle point. According to the Grobman-Hartman theorem (see Hartman [1964]) the nonlinear system (45) admits a trajectory the same as the linear system. This trajectory converges to equilibrium and is tangent to the eigenvector corresponding to the negative eigenvalue.

6.3 Hamiltonian System in the Zero Control Domain

Consider the Hamiltonian system in the zero control domain D_1 (27) defined in variables (k, z)

$$\begin{cases} \dot{z} = \delta z - \dfrac{kf'(k)}{f(k)}, \\ \dot{k} = -\lambda k. \end{cases} \qquad (49)$$

The Hamiltonian system (49) has no any steady state.

6.4 Hamiltonian System in the Intensive Control Domain

Consider the Hamiltonian system in the intensive control domain D_3 (29) defined in variables (k, z)

$$\begin{cases} \dot{z} = z\Big(\delta + a\dfrac{f(k)}{k} - af'(k)\Big) - \dfrac{kf'(k)}{f(k)}, \\ \dot{k} = af(k) - \lambda k. \end{cases} \qquad (50)$$

Let us note that the Hamiltonian system (50) has no steady state in the domain D_3.

7. ALGORITHM FOR CONSTRUCTION OF OPTIMAL TRAJECTORY

Let us design an algorithm for construction of optimal trajectory in control problem (9)-(10) based on the conjugation of Hamiltonian systems (45), (49) and (50). To follow the algorithm one should represent domains D_j, $j = 1, 2, 3$, and sewing curves L_i, $i = 1, 2$, in variables (k, z). Let us remind that the unique steady state (k^*, z^*) belongs to domain D_2.

Numerical Algorithm. The algorithm for construction of the optimal trajectory includes the following steps.

1. Numerical estimation of the steady state (k^*, z^*).

2. Linearization of the Hamiltonian system (45) in the neighborhood of the steady state (k^*, z^*).

3. Calculation of eigenvalues and eigenvectors of the linearized Hamiltonian system.

4. Fixation of the precision parameter $\varepsilon > 0$ and calculation of the characteristic point $(k_\varepsilon, z_\varepsilon)$ at the ε-neighborhood of the steady state (k^*, z^*) in the direction of the eigenvector corresponding to the negative eigenvalue.

5. Integration of the Hamiltonian system (45) in the reverse time starting from the characteristic point $(k_\varepsilon, z_\varepsilon)$. Integration is performed until one of two alternatives: 1) if the integrated trajectory reaches the initial point k^0 in domain D_2 then the algorithm is stopped and the trajectory is built; 2) if the integrated trajectory reaches sewing curves L_i, $i = 1, 2$, before it reaches the initial point k^0 then the Hamiltonian system (45) is switched either to the Hamiltonian system (49) at points of the sewing curve L_1, or to the Hamiltonian system (50) at points of the sewing curve L_2.

6. Expansion of the integrated trajectory in the direct time and its time scaling.

8. SIMULATION OF THE MODEL

The numerical algorithm is realized in the elaborated software for construction of optimal trajectories of economic growth and optimization of investment level. Numerical experiments are presented for parameters identified from the real data on the Japan's economy. Parameters of the Cobb-Douglas production function $f(k) = Ak^\alpha$ are identified in the econometric package "SPSS Sigma Stat 3.0". Their values are estimated as follows: $A = 1.677$, $\alpha = 0.588$. The discount coefficient δ and the rate λ are given by estimates: $\delta = 0.1$, $\lambda = 0.02$. Parameter a of restrictions on control $s(t)$ is defined by inequality (48) and is chosen at the level $a = 0.17$. Parameters of numerical integration are: precision parameter $\varepsilon = 0.001$, time step $\Delta t = 0.0001$. The values of the steady state are calculated as $(k^*, z^*) = (155.897, 5.259)$. The system is integrated in the reverse time until the stopping criterion $k^0 = 7.5$. Results of construction of the synthetic model trajectory is shown on Fig. 1-b.

On Fig. 1-c the investment plan $s^0(t)$ is given. The graph of investments show that, firstly, in the period 1962-1997 the optimal investment level stays at the intensive level $s^* = a = 0.17$; secondly, on sewing curve L_2 a switch happens

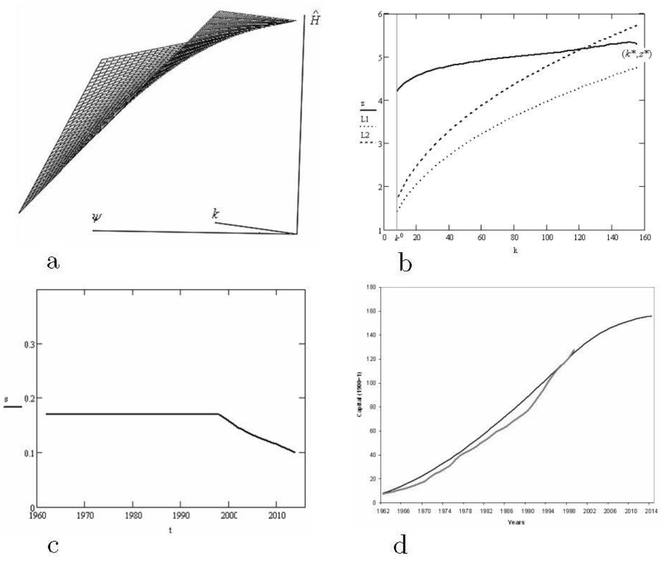

Fig. 1. Results of numerical simulations.

from the intensive level $s^* = a$ to the transition level $s^* = s^0$ which is implemented in domain D_2; thirdly, while time t grows to infinity the investment level monotonically decreases to the level $s^* = 0.095$ defined by the steady state (k^*, z^*). On Fig. 1-d the obtained optimal synthetic trajectory of capital growth shown by the heavy line is compared with the time series of the macroeconomic data for Japan depicted by the grey line. The comparison shows that the synthetic trajectory adequately reflects trends of the real data and can be used for forecasting future scenarios of growth. The character of the graph has an S-shape which is provided by the restriction parameter a on investment $s(t)$. Moreover, the graph demonstrates the saturation tendency of growth. Experiments show that in the absence of the restriction parameter a or its overstating the graph has a concave form and does not reflect the data trends.

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