

# A New Iterative Procedure to Obtain H-infinity, L-infinity Optimal Regulators

L. Keviczky and Cs. Bányász

Computer and Automation Research Institute and Control Engineering Research Group  
Hungarian Academy of Sciences  
H-1111 Budapest, Kende u 13-17, HUNGARY  
e-mail: keviczky@sztaki.hu; banyasz@sztaki.hu

**Abstract:** The sensitivity function in a generic two-degree of freedom (*TDOF*) control system can be decomposed into three major parts: design-, realizability- and modeling-loss. The paper investigates the optimality of the second term in infinite norm spaces and proposes a new iterative algorithm for the solution. Copyright©2008 IFAC.

Keywords: Optimal regulator, error decomposition, Youla-parametrization

## 1. INTRODUCTION

Control system optimization is usually based on the error signal or the error transfer function of the closed-loop (Doyle, *et al.*, 1992). The last one is called sensitivity function (*SF*), so any such optimization procedures is strongly connected to the sensitivity or the robustness of control systems (Kucera, 1981). One widely applied possibility to optimize a proper norm (Doyle, *et al.*, 1992) formulated for the closed-loop *SF* is to consider the criterion as a function of the loop-parameters (design, regulator, constraints, etc.) and to solve the strongly nonlinear constrained mathematical programming problem. These methods do not analyze the internal properties of the control error and the different contributing parts of the sensitivity. Therefore a special decomposition of the sensitivity function (control error) is used in the paper to understand and explain the main three contributions of the different performance components.

An important component, the realizability degradation is used to optimize the control system. This part depends only on the selected reference model, the nominal plant and the designed controller.

*Youla-parametrization* (Maciejowski, 1989; Youla, *et al.*, 1976) will be used for regulator design, which has a limitation for open-loop plants only, however, this parametrization allows explicit algebraic design of the controller, except the optimization step. The paper investigates the  $\mathcal{H}_\infty$ ,  $\mathcal{L}_\infty$  optimality of such systems, which is generally a nonlinear problem. A special combined role of the  $\mathcal{H}_\infty$ ,  $\mathcal{L}_\infty$  norms is also shown, resulting structural requirement for the form of the cost function if integrating regulator is expected. After some low order simple examples, when the regulator can be computed by explicit formulas an iterative procedure is introduced for higher order general cases.

## 2. CONTROL ERROR DECOMPOSITION IN *TDOF* SYSTEMS

Assume that the plant to be controlled is factorable as

$$P = P_+ P_- = \frac{\mathcal{B}}{\mathcal{A}} = \frac{\mathcal{B}_+ \mathcal{B}_-}{\mathcal{A}} \quad (1)$$

where  $P_+ = \mathcal{B}_+ / \mathcal{A}$  means the *inverse stable (IS)* and  $P_- = \mathcal{B}_- / \mathcal{A}$  the *inverse unstable (IU)* factors, respectively. In a practical case only the model  $\hat{P}$  of the process is known. Assume that the model  $\hat{P}$ , is similarly factorable as the process in (1)

$$\hat{P} = \hat{P}_+ \hat{P}_- = \frac{\hat{\mathcal{B}}}{\hat{\mathcal{A}}} = \frac{\hat{\mathcal{B}}_+ \hat{\mathcal{B}}_-}{\hat{\mathcal{A}}} \quad (2)$$

where  $\hat{P}_+ = \hat{\mathcal{B}}_+ / \hat{\mathcal{A}}$  means the *IS*,  $\hat{P}_- = \hat{\mathcal{B}}_- / \hat{\mathcal{A}}$  the *IU* factors, respectively. Introduce the additive and relative model errors

$$\Delta = P - \hat{P} \quad ; \quad \ell = \frac{\Delta}{\hat{P}} = \frac{P - \hat{P}}{\hat{P}} \quad (3)$$

The complementary sensitivity function (*CSF*) of a one-degree of freedom (*ODOF*) control system is

$$T = \frac{C P}{1 + C P} = \hat{T} \frac{1 + \ell}{1 + \hat{T} \ell} \quad ; \quad \hat{T} = \frac{\hat{C} \hat{P}}{1 + \hat{C} \hat{P}} \quad (4)$$

where  $\hat{T}$  is the *CSF* of the model based *ODOF* system. For a two-degree of freedom (*TDOF*) control system (Keviczky, 1995) it is reasonable to request the design goals by two stable and usually strictly proper transfer functions  $R_r$  and  $R_n$ , that are partly capable to place desired poles in the tracking and the regulatory transfer functions. They can even be called as reference models, so usually  $R_r(\omega = 0) = 1$  and  $R_n(\omega = 0) = 1$  are selected.

An important new observation that the *SF* ( $S = 1 - T$ ) can be decomposed into additive components according to different principles:

$$S = \underbrace{(1 - R_n)}_{S_{des}} + \underbrace{(R_n - \hat{T})}_{S_{real}} - \underbrace{(T - \hat{T})}_{S_{id}} = \overbrace{(R_n - \hat{T})}^{S_{perf}} - (T - \hat{T})$$

$$\begin{aligned}
 &= S_{\text{des}} + S_{\text{real}} + S_{\text{id}} = \underbrace{(1 - R_n)}_{S_{\text{des}}} + \underbrace{(R_n - T)}_{S_{\text{perf}}} = \\
 &= S_{\text{des}} + S_{\text{perf}} = \underbrace{(1 - \hat{T})}_{S_{\text{cont}}} + S_{\text{id}} = \underbrace{(1 - \hat{T})}_{S_{\text{des}} + S_{\text{real}}} + S_{\text{id}} = \quad (5) \\
 &= S_{\text{cont}} + S_{\text{id}}
 \end{aligned}$$

Here  $S_{\text{des}} = (1 - R_n)$  is the design,  $S_{\text{real}} = (R_n - \hat{T})$  is the realizability and  $S_{\text{id}} = -(T - \hat{T}) = \hat{T} - T$  is the modeling (or identification) degradation, respectively. Furthermore  $S_{\text{cont}} = (1 - \hat{T})$  and  $S_{\text{perf}} = (P_n - T)$  are the overall control and performance degradations, respectively. The  $SF$  depends on the model-based  $SF$  ( $\hat{S} = 1 - \hat{T}$ ) as

$$S = \frac{1}{1 + C P} = \hat{S} \frac{1}{1 + \hat{T} \ell} = \hat{S} + S_{\text{id}} \quad ; \quad \hat{S} = \frac{1}{1 + C \hat{P}} \quad (6)$$

(The realizability and identification degradations can be called as systematic ( $S_{\text{sys}}$ ) and random ( $S_{\text{rand}}$ ) components, too.)

In a general case the overall  $CSF$  of a  $TDOF$  control system has the form  $T_r = F T$  and a similar decomposition can be introduced for the tracking error function  $S_r = 1 - T_r$  as for  $S$  in (5):

$$\begin{aligned}
 S_r &= (1 - R_r) + (R_r - \hat{T}_r) - (T_r - \hat{T}_r) = \\
 &= S_{\text{des}}^r + S_{\text{real}}^r + S_{\text{id}}^r \quad (7)
 \end{aligned}$$

The overall transfer function of the  $TDOF$  system is

$$T_r = \hat{T}_r \frac{1 + \ell}{1 + \hat{T} \ell} \quad (8)$$

In an ideal control system it is required to follow the transients required by  $R_r$  and  $R_n$  (more exactly  $(1 - R_n)$ ), i.e., the ideal overall transfer characteristics of the  $TDOF$  control system would be

$$y^o = R_r y_r - (1 - R_n) y_n = y_r^o + y_n^o \quad (9)$$

while a practical, realizable control can provide only

$$\begin{aligned}
 y &= T_r y_r - S y_n = T_r y_r - (1 - T) y_n \\
 \hat{y} &= \hat{T}_r y_r - \hat{S} y_n = \hat{T}_r y_r - (1 - \hat{T}) y_n \quad (10)
 \end{aligned}$$

for the true ( $y$ ) and model-based ( $\hat{y}$ ) closed-loop control output signals. Here  $y_r, y$  and  $y_n$  are the reference, process output and disturbance (or output noise) signals, respectively.

Express the deviation between the ideal ( $y^o$ ) and the realizable best ( $y$ ) closed-loop output signals as

$$\begin{aligned}
 \Delta y &= y^o - y = (R_r - T_r) y_r - (R_n - T) y_n = \\
 &= S_{\text{perf}}^r y_r - S_{\text{perf}}^n y_n \quad (11)
 \end{aligned}$$

where  $S_{\text{perf}}^r$  is the performance degradation for tracking and  $S_{\text{perf}}^w = S_{\text{perf}}$  is the performance degradation for the disturbance rejection (or control) behaviors, respectively. Similar equation can be obtained for the deviation between the ideal ( $y^o$ ) and the model based ( $\hat{y}$ ) closed-loop outputs

$$\begin{aligned}
 \Delta \hat{y} &= y^o - \hat{y} = (R_r - \hat{T}_r) y_r - (R_n - \hat{T}) y_n = \\
 &= S_{\text{real}}^r y_r - S_{\text{real}}^n y_n \quad (12)
 \end{aligned}$$

One can see that the realizability degradation(s) play an important role forming  $\Delta \hat{y}$  (and indirectly  $\Delta y$ ).

It is important to note that the term  $S_{\text{real}}$  (and  $S_{\text{real}}^r$ ) can be made zero for  $IS$  processes only, however, for  $IU$  plants the reachable minimal value of  $S_{\text{real}}$  (and  $S_{\text{real}}^r$ ) always depends on the invariant factors and never becomes zero. In the sequel  $YP$  based control system will be discussed.

### 3. DECOMPOSITION IN YOULA-PARAMETRIZED SYSTEMS

If the applied regulator design is based on the *Youla-parametrization* ( $YP$ ) (Maciejowski, 1989; Youla, *et al.*, 1976) then the realizable best all stabilizing and the model based regulators are

$$C = \frac{Q}{1 - Q P} \quad ; \quad \hat{C} = \frac{Q}{1 - Q \hat{P}} \quad (13)$$

where the "parameter"  $Q$  ranges over all proper ( $Q(\omega = \infty)$  is finite), stable transfer functions. The  $CSF$ 's of the true and model-based  $ODOF$  control systems are

$$T = \frac{\hat{C} P}{1 + \hat{C} P} = \frac{Q \hat{P} (1 + \ell)}{1 + Q \hat{P} \ell} \quad ; \quad \hat{T} = \frac{\hat{C} \hat{P}}{1 + \hat{C} \hat{P}} = Q \hat{P} \quad (14)$$

Only in case of  $YP$  one can also compute the realizable best  $CSF$

$$T_* = \frac{C P}{1 + C P} = Q P = Q \hat{P} (1 + \ell) = \hat{T} (1 + \ell) \quad (15)$$

The  $SF$  of the model based and true closed-loops are now

$$\hat{S} = \frac{1}{1 + \hat{C} \hat{P}} = 1 - Q \hat{P} \quad (16)$$

and

$$S = \frac{1}{1 + C P} = \frac{1 - Q \hat{P}}{1 + Q \hat{P} \ell} = \frac{\hat{S}}{1 + \hat{T} \ell} \quad (17)$$

The realizable best  $SF$ , corresponding to  $T_*$  is

$$S_* = \frac{1}{1 + C P} = 1 - Q P = 1 - Q \hat{P} (1 + \ell) = \hat{S} - \hat{T} \ell \quad (18)$$

The decomposition of the  $SF$  is

$$S = (1 - R_n) + (R_n - \hat{T}) - (T - \hat{T}) = S_{\text{des}} + S_{\text{real}} + S_{\text{id}}$$

$$= (1 - R_n) + (R_n - Q\hat{P}) - \frac{Q\hat{P}(1 - Q\hat{P})}{1 + Q\hat{P}\ell} \quad (19)$$

The decomposition of the tracking error function for the *YP* is

$$S_r = 1 - T_r = (1 - R_r) + (R_r - Q_r\hat{P}) - (T_r - \hat{T}_r) = S_{des}^r + S_{real}^r + S_{id}^r \quad (20)$$

where

$$S_{id}^r = -\frac{Q_r\hat{P}(1 - Q\hat{P})}{1 + Q\hat{P}\ell} \Big|_{\ell=0} \approx -Q_r\hat{P}(1 - Q\hat{P}) \quad (21)$$

#### 4. A *GTDOF* CONTROLLER FOR STABLE LINEAR PLANTS

In many practical cases the plant to be controlled is stable, and a *TDOF* control system is required because of the high performance dual tracking and regulatory requirements (Keviczky, 1995; Keviczky and Bányász, 2001). An ideal solution for this task is the *generic two-degree of freedom (GTDOF)* scheme introduced in (Keviczky and Bányász, 1999). This framework and topology is based on the *YP* (Maciejowski, 1989) providing stabilizing regulators for open-loop stable plants.

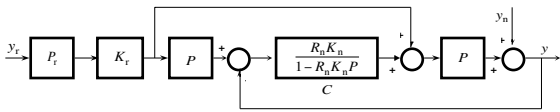


Fig. 1. The *generic TDOF (GTDOF)* control system

A *GTDOF* control system is shown in Fig. 1. The realizable best regulator of the *GTDOF* scheme can be given by an explicit form for  $\ell = 0$

$$C_* = \frac{Q_*}{1 - Q_* P} = \frac{R_n K_n}{1 - R_n K_n P} = \frac{R_n G_n P_+^{-1}}{1 - R_n G_n P_-} \quad (22)$$

where

$$Q_* = Q_n^* = R_n K_n = R_n G_n P_+^{-1} \quad (23)$$

is the associated optimal *Y-parameter* furthermore

$$Q_r^* = R_r K_r = R_r G_r P_+^{-1} \quad ; \quad K_n = G_n P_+^{-1} \quad (24)$$

$$K_r = G_r P_+^{-1}$$

It is interesting to see how the transfer characteristics of the closed-loop look like:

$$y = R_r K_r P y_r + (1 - R_n K_n P) y_n = T_r y_r + S y_n = y_* = R_r G_r P_- y_r + (1 - R_n G_n P_-) y_n = y_t + y_d \quad (25)$$

where  $y_t$  is the tracking (servo) and  $y_d$  is the regulating (disturbance rejection or control) independent behavior of the closed-loop response, respectively.

So the invariant factor  $P_-$  can not be eliminated, consequently the ideal design goals  $R_r$  and  $R_n$  are biased by  $G_r P_-$  and  $G_n P_-$ . We can not reach the ideal tracking  $y_r^o = P_r y_r$  and regulatory  $y_n^o = (1 - R_n) y_n$  behaviors (see (12)), because of the invariant factor (mainly zeros) in the *IU* factor  $P_-$ . (In a general case the time delay should also be considered here as an invariant factor.) The realizable best transients, corresponding to (10) and (25), are given by  $R_r G_r P_-$  and  $(1 - R_n G_n P_-)$  respectively, where  $G_r$  and  $G_n$  can optimally attenuate the influence of  $P_-$ . The unity gain of  $R_n$  ensures integral action in the regulator, which is maintained if and only if the applied optimization provides  $G_n P_-(\omega = 0) = 1$  (or including  $R_n$  the condition is  $R_n G_n P_-(1) = 1$ ).

The model based version of the *YP* regulator  $\hat{C} = C(\hat{P})$  in the *GTDOF* scheme means that  $P$  is substituted by  $\hat{P}$  in equations (22)-(24).

The decomposition of the *SF* in the true *GTDOF* control system by (19) is

$$S = S_{des} + S_{real} + S_{id} = (1 - R_n) + R_n (1 - G_n \hat{P}_-) - \frac{R_n G_n \hat{P}_- (1 - R_n G_n \hat{P}_-)}{1 + Q\hat{P}\ell} \quad (26)$$

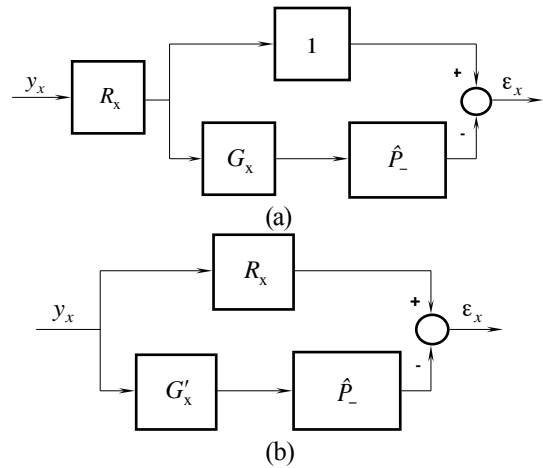


Fig. 2. Reduced form of  $\mathcal{H}_\infty$  optimality for  $\|S_{real}^x\|_\infty$

#### 5. THE $\mathcal{H}_\infty, \mathcal{L}_\infty$ OPTIMAL REALIZABILITY LOSS

It is obvious from the previous sections that the optimization of the realizability loss requires the minimization of some  $J_{real}^x = \|S_{real}^x\|_{x=r,n}$  norms for tracking (r) and the disturbance rejection (n), respectively. These *SF* components have the general form of  $\|S_{real}^x\| = \|R_x (1 - G_x \hat{P}_-)\|$ , which can be simply rearranged equivalently to  $\|S_{real}^x\| = \|R_x - G'_x \hat{P}_-\|$  as the Fig. 2 (a) and (b) show: consequently  $G'_x = R_x G_x$ . Because the

reference models have unity gains, it is enough to ensure the condition  $G'_x \hat{B}_- \Big|_{s \rightarrow 0} = 1$  to the integral behavior of the regulator. If the excitation is not a Dirac impulse or a white noise it is reasonable to use the more general form of  $\|S_{\text{real}}^r\| = \|Y_x (R_x - G'_x \hat{P}_-)\|$ , where  $Y_x = L(y_x) = s^{-j}$  is the Laplace transform of the well known test signals. In the sequel the infinite norms are investigated.

The goal of this optimization step is to minimize the realizability loss  $J_{\text{real}}^x$  using the optimal embedded filters  $G'_x = G_x^{\text{opt}}$  attenuating the influence of the invariant model factor  $\hat{P}_-$  as

$$G_x^{\text{opt}} = \arg \left\{ \min_{G'_x} (J_{\text{real}}^x) \right\} = \arg \left\{ \min_{G'_x} \left\| \frac{1}{s^j} (R_x - G'_x \hat{P}_-) \right\|_{\infty} \right\} \quad (27)$$

Here an  $s^{-j}$  form for the excitation  $Y_x$  was applied and  $R_x$  is a unity gain reference model. This formulation corresponds to a generalized approach for infinite norms, i.e.

$$J_x^{\infty} = \|Y_x (R_x - G'_x \hat{P}_-)\|_{\infty} = \left\| \frac{1}{s^j} (R_x - G'_x \hat{P}_-) \right\|_{\infty} = \|W\|_{\infty} = \begin{cases} \mathcal{H}_{\infty} \text{ norm} ; & j = 0 \\ \mathcal{L}_{\infty} \text{ norm} ; & j \geq 1 \end{cases} \quad (28)$$

This norm is the  $\mathcal{H}_{\infty}$  system (or operator) norm for  $j = 0$  and is the  $\mathcal{L}_{\infty}$  signal norm for  $j \geq 1$ . One must know that the  $\mathcal{L}_{\infty}$  norm is usually formulated for real functions and not for complex functions as here. It is also important to note that this "generalized supremum norm" is bounded for  $j = 0$  and for  $j \geq 1$  if the low-pass  $Y_x$  is multiplied by a high-pass control error term  $(R_x - G'_x \hat{P}_-)$ , which is satisfied for closed-loop system with type number higher than zero.

The optimal solution lies in optimal interpolation theory and is known as the Nevanlinna-Pick problem (Wang and Chen, 1988). Assuming  $m_-$  number of unstable zeros in  $\hat{P}_-$  the optimal  $W^o$  minimizing  $\|W\|_{\infty}$  is an all-pass form

$$W^o = \begin{cases} \mu \frac{h^{\#}}{h} & , \text{ if } m_- \geq 1 \\ 0 & , \text{ if } m_- = 0 \end{cases} \quad (29)$$

where  $h$  is a Hurwitz polynomial of degree at most  $(m_- - 1)$ .

The computation of  $h^{\#}$  can be obtained by mirroring the zeros of  $h$  through the imaginary axis. The constant  $\mu$  and the coefficients of  $h$  are real and are uniquely determined by the following - so-called - interpolation constraints

$$W(z_i) = \mu \frac{h^{\#}(z_i)}{h(z_i)} = \frac{1}{s^j} (R_x - G'_x \hat{P}_-) \Big|_{z_i} =$$

$$= \frac{R_x(z_i)}{z_i^j} = r_i \quad ; \quad i = 1, 2, \dots, m_- \quad (30)$$

where  $z_1, z_2, \dots, z_{m_-}$  denote the distinct zeros of  $\hat{P}_-$ . (The multiplicity of zeros can be easily considered by additional differential interpolation constraints.) Because of the interpolation constraints (30), the  $m_-$  number of unknown parameters -  $(m_- - 1)$  is in  $h$  and the  $m_-$ -th is the  $\mu$  itself - can be obtained from the  $m_-$  number of equations. It can also be observed that  $\hat{P}_-$  is certainly the divisor of  $R_x - s^j W$ , thus

$$R_x - s^j W = \frac{\mathcal{B}_x}{\mathcal{A}_x} - \mu \frac{s^j h^{\#}}{h} = \frac{\mathcal{N}}{\mathcal{D}} \hat{P}_- = \frac{\mathcal{N}}{\mathcal{D}} \hat{B}_- \quad (31)$$

where the notation of  $R_x = \mathcal{B}_x / \mathcal{A}_x$  and  $\hat{P}_- = \hat{B}_-$  are used again and the form

$$G'_x = \frac{\mathcal{N}}{\mathcal{D}} \quad (32)$$

was introduced for the embedded filter. Comparing the two sides of (31) the polynomials  $\mathcal{D}$  and  $\mathcal{N}$  can be obtained by

$$\mathcal{D} = \mathcal{A}_x h \quad (33)$$

and

$$\mathcal{N} = \frac{\mathcal{B}_x h - \mathcal{A}_x \mu s^j h^{\#}}{\hat{B}_-} \quad (34)$$

where the last division is with no residue. The equation (30) can be rearranged into a "quasi" linear equation system in case of  $m_-$  disjunct real zeros  $z_i$

$$\mu h^{\#}(z_i) = r_i h(z_i) \quad ; \quad i = 1, 2, \dots, m_- \quad (35)$$

for the computation of  $h$  and  $\mu$ . The minimum of the cost function is given by

$$\mu = \min \left\{ \|W(j\omega)\|_{\infty} \right\} = \|W^o(j\omega)\|_{\infty} \quad (36)$$

After some straightforward manipulations one can obtain that

$$G'_x \hat{B}_- = \frac{\mathcal{B}_x h - \mathcal{A}_x \mu s^j h^{\#}}{\mathcal{A}_x h} = \frac{\mathcal{B}_x}{\mathcal{A}_x} - \frac{\mu h^{\#}}{h} s^j \quad \text{or} \quad (37)$$

$$G'_x \hat{B}_- = \frac{\mathcal{A}_x \mathcal{B}_x h - \mathcal{A}_x \mu s^j h^{\#}}{\mathcal{B}_x \mathcal{A}_x h} = 1 - \frac{\mathcal{A}_x \mu h^{\#}}{\mathcal{B}_x} s^j$$

so it is easy to see that  $G'_x \hat{B}_- \Big|_{s \rightarrow 0} = G_x \hat{B}_- \Big|_{s \rightarrow 0} = 1$ , providing integrating regulator, can be obtained if, and only if  $j \geq 1$ , i.e. only for  $\mathcal{L}_{\infty}$  optimality. The original formulation of the  $\mathcal{H}_{\infty}$  optimality of  $\|S_{\text{real}}^x\|_{\infty}$  using the classical operator norm ( $j = 0$ ) can not provide an integrating regulator.

*Example 1.*

Assume a first order reference model  $R_x = 1/(1+sT_w)$  and be  $\hat{B}_- = 1-sT$ . Because  $m_- = 1$  the interpolation polynomial  $h$  is of zero order and the trivial  $h = h^\# = 1$  constant can be selected. So  $\mu$  can be easily obtained from the interpolation constraint (35):  $\mu = R_x(z_1 = 1/T) = T/(T_w + T)$ . The denominator polynomial of  $G'_x$  is  $D = 1 + sT_w$  from (33). Apply  $j = 0$  first. Because of the orders in the right side of (34) a scalar  $\mathcal{N} = k$  can be used now

$$1 - k(1 - sT) = (1 + sT_w)\mu \quad (38)$$

The solution gives  $\mathcal{N} = k = T_w/(T_w + T)$  and  $\mu$  is the same as obtained before. The  $\mathcal{H}_\infty$  optimal filter is

$$G'_x = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\mathcal{N}}{\mathcal{A}_x h} = \frac{T_w}{T_w + T} \frac{1}{1 + sT_w} \quad (39)$$

and it is easy to check that  $G'_x \hat{B}_- \Big|_{s \rightarrow 0} \neq 1$ , i.e. the optimal regulator is not integrating.

*Example 2.*

Apply  $j = 1$  now, consequently (33) remains and only (38) will change to

$$1 - \mathcal{N}(1 - sT) = (1 + sT_w)\mu s^j \quad (40)$$

Searching a first order  $\mathcal{N} = k(1 + s\tau)$  polynomial (40) will have the form

$$1 - k(1 + s\tau)(1 - sT) = (1 + sT_w)\mu s^j \quad (41)$$

Comparing the coefficients of the two sides the solution gives

$$k = 1 \quad ; \quad \mu = \frac{T}{T_w + T} T \quad ; \quad \tau = \frac{T_w}{T_w + T} T \quad (42)$$

The  $\mathcal{L}_\infty$  optimal filter is

$$G'_x = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\mathcal{N}}{\mathcal{A}_x h} = \frac{1 + s\tau}{1 + sT_w} \quad (43)$$

and it is easy to check that  $G'_x \hat{B}_- \Big|_{s \rightarrow 0} = 1$ , i.e. the optimal regulator is integrating.

It is important to note that examples 1 and 2 are for a low order case ( $m_- = 1$ ), when  $h = 1$ , just to demonstrate the computations. For higher order cases this optimality (the solution of Eqs. (30), (35)) requires to solve a nonlinear equations system. The original nonlinear task can be decomposed into a nonlinear (to determine  $\mu$  and  $h$ ) and a linear problem (to compute  $\mathcal{N}$  by a simple polynomial division). Investigate first (30) and (35), which can also be written in a matrix form

$$r = F(\mu, r, z) h(\mu, h) \quad (44)$$

where

$$r = [r_1 \quad \dots \quad r_{m_-}]^T \quad ; \quad z = [z_1 \quad \dots \quad z_{m_-}]^T \quad (45)$$

$$h(\mu, h) = [\mu, h_1, \dots, h_{m_1}]^T \quad ; \quad m_1 = m_- - 1 \quad (46)$$

$$F(r, \mu, z) = \begin{bmatrix} 1 & f_{11} & \dots & f_{1m_1} \\ 1 & f_{22} & \dots & f_{2m_1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_{m_-1} & \dots & f_{m_-m_1} \end{bmatrix} \quad (47)$$

Here

$$f_{ik} = \sum_{k=1}^{m_1} \left[ \mu (z_i^*)^k - r_i (z_i)^k \right] \quad ; \quad z_i^* = \text{sign}(h_i) z_i \quad (48)$$

The "quasi" linear term is used because the  $f_{ik}$  elements depend on  $\mu$ , too. In such cases only iterative solutions can be formulated. One of the simplest method is the relaxation type iterative algorithm

$$h(\mu_{l+1}, h_{l+1}) = [F(\mu_l, r, z)]^{-1} r \quad (49)$$

Having obtained the iterative solution for  $\mu$  and  $h$  the polynomial  $\mathcal{N}$  is computed from (34).

It is interesting to note that the order of the embedded filter  $G'_x$  does not depend on  $j$  only on the invariant factor  $\hat{B}_-$  (i.e.,  $m_-$ ) and the reference model  $R_x$ .

*Example 3.*

Consider a second order  $IU$  process polynomial  $B_- = (1 - sT_1)(1 - sT_2)$  with  $T_1 = 1$  and  $T_2 = 2$ . Select a unity gain first order reference model with  $T_w = 0.5$ . In this case  $h(s) = 1 + h_1 s$  is first order because  $m_- = 2$  and  $m_1 = 1$ . This is already a nonlinear problem as indicated above. The solution of (44), however still does not need the iterative algorithm, instead a second order polynomial equation system

$$\mu^2 + \mu \frac{T_1 + T_2}{T_1 - T_2} (r_2 - r_1) - r_1 r_2 = 0 \quad (50)$$

$$h_1^2 + h_1 \frac{r_2 + r_1}{r_2 - r_1} (T_2 - T_1) - T_1 T_2 = 0$$

can be formulated which has explicit analytical solution. Investigating the roots (both  $\mu$  and  $h_1$  must be positive) the following optimal solution is obtained for  $j = 0$ :

$$\mu = 0.9572 \quad \text{and} \quad h_1 = 0.1789 \quad (51)$$

Then  $\mathcal{N}$  can be computed from the division (34)

$$\mathcal{N} = 0.0428 \quad (52)$$

The  $\mathcal{L}_\infty$  optimal filter is

$$G'_x = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\mathcal{N}}{\mathcal{A}_x h} = \frac{0.0428}{(1 + 0.5s)(1 + 0.1789s)} \quad \text{and}$$

$$G_x = \frac{0.0428}{1 + 0.1789s} \quad (53)$$

so the regulator is not integrating. Applying the iterative algorithm (49) for  $j=1$  the following numerical solutions are obtained

$$\mu = 3.1397 \quad \text{and} \quad h_1 = 0.6498 \quad (54)$$

The convergence of the iteration was very fast. Then  $\mathcal{N}$  can be computed from the division (34)

$$\mathcal{N} = 1 + 0.5s \quad (55)$$

and the  $\mathcal{L}_\infty$  optimal filter is

$$G'_x = \frac{\mathcal{N}}{\mathcal{A}_x h} = \frac{1 + 0.5s}{(1 + 0.5s)(1 + 0.6498s)} = \frac{1}{1 + 0.6498s} \quad \text{and}$$

$$G_x = \frac{1 + 0.5s}{1 + 0.6498s} \quad (56)$$

It is important to investigate the realizability of the optimal regulator (22) based on the above  $\mathcal{H}_\infty$ ,  $\mathcal{L}_\infty$  optimal embedded filter  $G'_x$ , i.e.

$$C_* = \frac{G'_x P_+^{-1}}{1 - G'_x P_-} \quad (57)$$

Simple calculations give that the order of  $\mathcal{N}$  is

$$n_{\mathcal{N}} = n_x + j - 1 \quad (58)$$

where  $n_x$  is the order of the denominator of  $R_x$ , so the pole access of  $C_*$  is

$$\Delta n_C = m - n \quad (59)$$

where  $m$  and  $n$  are the orders of polynomials  $\mathcal{B}$  and  $\mathcal{A}$ , respectively. This means that the optimal regulator is realizable, iff.  $n = m$ , when the pole access of the process is zero. For the first example if the process is given by  $P = (1 - sT)/(1 + sT_1)$  the optimal regulator is

$$C_* = \frac{T_w}{T} \frac{1 + sT_1}{1 + sT_w(2 + T_w/T)} \quad (60)$$

## 6. CONCLUSIONS

The relatively easy and reasonably optimal solution of a generally very sophisticated control problem strongly depends on the proper decomposition of the original paradigm. These decompositions correspond to a natural control engineering practice, where the best reachable design goal and the way how to obtain it appear in a generally iterative sequential procedure.

The  $\mathcal{H}_\infty$ ,  $\mathcal{L}_\infty$  optimality of the realizability loss, which is a major component of the sensitivity function, is investigated.

If any external excitation is assumed, then a combined application of the  $\mathcal{H}_\infty$ ,  $\mathcal{L}_\infty$  norms is necessary. An interesting result that the optimization provides integrating regulator, iff an excitation form  $Y_x(s) = s^{-j}$  is assumed with  $j \geq 1$  which corresponds to  $\mathcal{L}_\infty$  optimality. *The classical  $\mathcal{H}_\infty$  optimality does not provide integrating regulator.*

Simple, easy to repeat low order examples, when the optimal parameters can be calculated explicitly, are first presented to demonstrate the computation of the optimal embedded filters for different cases.

Finally an iterative method is also introduced to solve the  $\mathcal{H}_\infty$ ,  $\mathcal{L}_\infty$  optimality of the realizability loss for higher order general cases.

The results can be easily applied for discrete time systems, too, where  $\mathcal{B}^-$  contains the unstable zeros of  $\mathcal{B}_-$  mirrored on the unit circle and  $Y_x(z) = (z-1)^{-j}$ . The major advantage of the application to discrete time system is, that the inclusion of the process time delay is relatively easy, because the transfer functions remain in the class of rational functions.

## REFERENCES

- Doyle J.C., Francis B.A. and Tannenbaum A.R., *Feedback Control Theory*, Macmillan P.C., 1992, pp. 227.
- Keviczky L., *Combined identification and control: another way*, (Invited plenary paper.) 5th IFAC Symp. on Adaptive Control and Signal Processing, ACASP'95, Budapest 1995, pp.13-30.
- Keviczky L. and Bányász Cs., *Optimality of two-degree of freedom controllers in  $\mathcal{H}_2$ - and  $\mathcal{H}_\infty$ -norm space, their robustness and minimal sensitivity*, 14th IFAC World Congress, Beijing 1999, Vol. F, pp. 331-336.
- Keviczky L. and Bányász Cs., *Iterative identification and control design using K-B parametrization*, In: Control of Complex Systems, Eds: K.J. Åström, P. Albertos, M. Blanke, A. Isidori, W. Schaufelberger and R. Sanz, Springer 2001, pp. 101-121.
- Kucera V., *Exact model matching, polynomial equation approach*, Int. J. Systems Sci., Vol. 12 (1981), pp. 1477-1484.
- Maciejowski J.M., *Multivariable Feedback Design*, Addison Wesley 1989, pp. 424.
- Youla D.C., Jabri H.A. and Bongiorno J.J., *Modern Wiener-Hopf design of optimal controllers: part II: the multivariable case*, IEEE Transaction Automatic Control, AC-21 (1976), pp. 319-338.
- Wang S. and Chen B., *Optimal model matching control for time-delay systems*, Int. J. Control, Vol. 47(1988), No. 3, pp. 883-894.

This work was supported in part by the Hungarian Scientific Research Fund (OTKA) and the Control Engineering Research Group of the HAS, at the Budapest University of Technology and Economics.