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Approximate Reachability Analysis for Linear Discrete Time Systems Using Homothety and Invariance

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Abstract: This paper introduces a method for approximate reachability, for linear discrete time systems, based on homothety and set invariance. The proposed method utilizes two particular families of sets, more precisely their members, and particular forms of the approximation maps to obtain simple inner and outer approximate reachable sets/tubes. The resulting set–dynamics, induced by the uncertainty set, the underlying dynamics in the state space and the approximation maps, are restricted to these particular families of sets and under standard assumptions yield bounded and convergent approximate reachable sets/tubes. A tractable computational procedure is suggested and a few illustrative examples are provided. *Copyright© 2008 IFAC.*

Keywords: Reachability Analysis, Approximations, Homothety, Set Invariance.

1. INTRODUCTION

Reachability analysis is one of the central research topics in the control theory due to its intimate relationship with optimal control, set invariance, set–membership state estimation, safety verification and control synthesis under uncertainty, see the monographs (Aubin, 1991; Kurzhanski and Vályi, 1997), the survey papers (Milanese and Vicino, 1991; Blanchini, 1999) and more recent references (Raković and Mayne, 2005; Artstein and Raković, 2008; Raković, 2007). Initial ideas related to reachability analysis and guaranteed state estimation can be traced back to the pioneering control literature (Witsenhausen, 1968; Hermes and Lasalle, 1969; Bertsekas and Rhodes, 1971; Schweppe, 1973; Kurzhanski, 1977). The research on reachability has recognized that the exact reachability is computationally demanding and has, therefore, focused on approximate reachability. Approximate reachability methods suggested in, for example, (Milanese and Vicino, 1991; Chernousko, 1994; Kurzhanski and Vályi, 1997; Kühn, 1998*a*; Alamo *et al.*, 2005; Kurzhanski and Varaiya, 2006) employ, essentially, ellipsoidal, polytopic or even zonotopic calculus to obtain guaranteed, possibly optimal with respect to a utilized criterion, inner and/or outer estimates of the exact reachable sets/tubes or sets of possible states consistent with acquired information, system dynamics and the uncertainty specification.

Approximate reachability methods result in "optimal approximations" obtained, often, by somehow prioritizing "shape simplicity" and "geometric criteria" over "dynamical aspects" of the approximation procedures. The "dynamical aspects" of the approximation procedure can be addressed more directly using the set invariance theory. Recent advances in the set invariance theory include a theoretical framework for the examination of the minimality of invariant sets for the nonlinear–compact case (Artstein and Raković, 2008) and its specialization to the linear–convex case (Raković, 2007). Following ideas of (Artstein and Raković, 2008), we discuss a version of approximate reachability by utilizing homothety and set invariance. We analyze set–dynamics, induced by the uncertainty set, the underlying dynamics in the state space and the approximation maps, evolving within two particular families of sets whose members are homothetic copies of invariant inner and outer basic shape sets S_Y and S_Z. We establish that, under modest assumptions, the resulting approximate reachable sets/tubes are bounded and convergent.

Paper Structure: Section 2 presents necessary preliminaries. Sections 3 and 4 discuss the use of homothety and invariance in approximate reachability. Sections 5 and 6 provide computational remarks, examples and conclusions.

Basic Nomenclature and Definitions: The sets of non–negative, positive integers and non–negative real numbers are denoted, respectively, by N, N_+ and R_+, i.e. $N := \{0, 1, 2, \ldots\}$, $N_+ := \{1, 2, \ldots\}$ and $R_+ := \{x \in R \,:\, x \geq 0\}$. For two sets $X \subset R^n$ and $Y \subset R^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \,:\, x \in X,\ y \in Y\}$. For a set $X \subset R^n$ and a vector $x \in R^n$ we write $x \oplus X$ instead of $\{x\} \oplus X$. Given the sequence of sets $\{X_i \subset R^n\}_{i=a}^{b}$, $a \in N$, $b \in N, b > a$, we denote $\bigoplus_{i=a}^{b} X_i := X_a \oplus \cdots \oplus X_b$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) we define $MX := \{Mx \,:\, x \in X\}$. Given a matrix $M \in R^{n \times n}$, $\rho(M)$ denotes the largest absolute value of its eigenvalues.

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A set $X \subset R^n$ is *a C set* if it is compact, convex, and contains the origin. A set $X \subset R^n$ is *a proper C set* if it is a C set and the origin is in its non–empty interior. A set $X \subseteq R^n$ is a symmetric set (with respect to the origin in R^n) if $X = -X$. The collection of non–empty compact sets in R^n is denoted by $\mathrm{Com}(R^n)$. The collection of C sets in R^n is denoted by $\mathrm{ComC}(R^n)$. The collection of proper C sets in R^n is denoted by $\mathrm{ComCP}(R^n)$. For $X \in \mathrm{Com}(R^n)$ and $Y \in \mathrm{Com}(R^n)$, the *Hausdorff semi–distance* and the *Hausdorff distance (metric)* of X and Y are, respectively, given by:

$$h_L(X,Y) := \min_{\alpha}\{\alpha \ : \ X \subseteq Y \oplus \alpha L, \ \alpha \geq 0\} \text{ and}$$
$$H_L(X,Y) := \max\{h_L(X,Y), h_L(Y,X)\},$$

where L is a given, symmetric, proper C set in R^n.

2. PRELIMINARIES

Consider the following autonomous discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + w, \ w \in W, \tag{2.1}$$

where $x \in R^n$ is the current state, x^+ is the successor state, $A \in R^{n\times n}$ is the state transition matrix and $w \in R^n$ is an unknown disturbance taking values in the set $W \subset R^n$. The standing assumption in this paper is:

Assumption 1. The disturbance set W is a C set in R^n.

To discuss reachability analysis problem we follow approach of (Artstein and Raković, 2008) and associate the map $\mathcal{R}(\cdot) \ : \ \mathrm{Com}(R^n) \to \mathrm{Com}(R^n)$ with the system (2.1) and the disturbance set W given by:

$$\mathcal{R}(X) := AX \oplus W. \tag{2.2}$$

The function $\mathcal{R}(\cdot)$ maps, indeed, $\mathrm{Com}(R^n)$ to itself since linear transformation of a compact set is a compact set and the Minkowski sum of two compact sets is a compact set. We denote by $\mathcal{R}^k(\cdot)$ the k^{th} iterate of the map $\mathcal{R}(\cdot)$ and notice that for all $k \in N_+$:

$$\mathcal{R}^k(X) = A^kX \oplus \bigoplus_{i=0}^{k-1} A^iW \text{ and } \mathcal{R}^0(X) = X. \tag{2.3}$$

The mapping $\mathcal{R}(\cdot)$ induces the set–dynamics:

$$X^+ = \mathcal{R}(X) = AX \oplus W. \tag{2.4}$$

The exact reach set at time k, given an initial set $X_0 \in \mathrm{Com}(R^n)$ is the set $\mathcal{R}^k(X_0)$, and is, clearly, compact. The standard reachability analysis problem is the determination of *the exact reachable tube* considered henceforth as a countably–infinite sequence of sets $\{X_k\}_{k=0}^{\infty}$ where $X_0 \in \mathrm{Com}(R^n)$ is the initial condition and

$$X_{k+1} = \mathcal{R}(X_k) = \mathcal{R}^{k+1}(X_0), \ k \in N. \tag{2.5}$$

The exact, explicit, solution to the reachability problem requires prohibitive computational effort, in general case, due to the computational complexity of the Minkowski set addition (Gritzmann and Sturmfels, 1993).

Definition 1. Sets $X \subset R^n$ and $Y \subset R^n$ are called homothetic (positively homothetic) if $X = z \oplus \lambda Y$ for some $z \in R^n$ and $\lambda \in R_+$ ($\lambda \in R, \ \lambda > 0$).

Definition 2. A set $\Omega \subseteq R^n$ is an invariant set for the system (2.1) and constraint set (X, W) if and only if $\Omega \subseteq X$ and $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$, i.e. iff $\Omega \subseteq X$ and $\mathcal{R}(\Omega) \subseteq \Omega$ ($A\Omega \oplus W \subseteq \Omega$).

We use the term invariant set rather than the term positively invariant set, no confusion should arise. We denote by $\mathrm{ComInv}(A, W, R^n)$ the family of all (non–empty) compact invariant sets in R^n:

$$\mathrm{ComInv}(A, W, R^n) := \{\Omega \ : \ \Omega \in \mathrm{Com}(R^n), \ \mathcal{R}(\Omega) \subseteq \Omega\}.$$

In order to discuss convergence and boundedness of the exact and approximate reachable sets we invoke:

Assumption 2. The matrix A is strictly stable ($\rho(A) < 1$).

Proposition 4.3 of (Artstein and Raković, 2008), established in a more general nonlinear–compact case, states that when $W \in \mathrm{Com}(R^n)$ and Assumption 2 holds there exists a unique set $X_\infty \in \mathrm{Com}(R^n)$ that solves the set–equation:

$$\mathcal{R}(X_\infty) = X_\infty, \text{ i.e. } AX_\infty \oplus W = X_\infty. \tag{2.6}$$

The set X_∞ is the minimal invariant set i.e. the set X_∞ satisfies $X_\infty \in \mathrm{ComInv}(A, W, R^n)$ and $X_\infty \subset \Omega$ for all $\Omega \in \mathrm{ComInv}(A, W, R^n)$ such that $\Omega \neq X_\infty$. Furthermore, the set X_∞ is the stable attractor for the set–dynamics (2.4) and is given explicitly by:

$$X_\infty = \bigoplus_{i=0}^{\infty} A^iW. \tag{2.7}$$

If Assumptions 1 and 2 both hold, then the set X_∞ is additionally a C set in R^n but, unfortunately, not computable in general case (Kolmanovsky and Gilbert, 1998). However, computationally tractable, invariant approximations of the set X_∞ can be obtained by utilizing results of (Raković *et al.*, 2005), or more recent results of (Artstein and Raković, 2008; Raković, 2007) recalled by the following, somehow summarized, observation:

Proposition 1. Suppose Assumptions 1 and 2 hold. Then there exist a symmetric set $L \in \mathrm{ComCP}(R^n)$ and a scalar $\lambda \in [0, 1)$ such that, for all $k \in N$,

$$A^kL \subseteq \lambda^kL. \tag{2.8}$$

Furthermore, for any symmetric set $L \in \mathrm{ComCP}(R^n)$ and a scalar $\lambda \in [0, 1)$ such that $AL \subseteq \lambda L$, sets S_k given by:

$$S_k := \mathcal{R}^k(\{0\}) \oplus \lambda^k(1 - \lambda)^{-1}\mu L, \tag{2.9}$$

where $\mu := H_L(W, \{0\}) = \min_\gamma\{\gamma \ : \ W \subseteq \gamma L\}$, are invariant sets for any $k \in N$, i.e. $\forall k \in N \ : \ \mathcal{R}(S_k) \subseteq S_k$.

In approximate reachability analysis, the determination of the exact trajectory $\{X_k\}_{k=0}^{\infty}$ of the set–dynamics (2.4) given $X_0 \in \mathrm{Com}(R^n)$ is replaced by a more modest task of the determination of two sequences of sets $\{Y_k\}_{k=0}^{\infty}$ and $\{Z_k\}_{k=0}^{\infty}$ such that:

$$Y_k \subseteq X_k = \mathcal{R}^k(X_0) \subseteq Z_k, \ k \in N \tag{2.10}$$

under restriction that sets $Y_k, \ k \in N$ and $Z_k, \ k \in N$ belong to families of prescribed sets $\mathcal{Y}$ and $\mathcal{Z}$ (typically families of ellipsoidal, polytopic or zonotopic sets) and inner and outer inclusions are tight (Schweppe, 1973; Kurzhanski and Vályi, 1997; Kühn, 1998*b*; Girard, 2005).

This, somehow, vaguely formulated program gives rise to many interesting questions. In this note we treat only one special aspect. Namely, we examine approximate reachability analysis when the employed families of sets $\mathcal{Y}$ and $\mathcal{Z}$ are families of homothetic copies of particular inner and outer basic shape sets S_Y and S_Z, respectively.

3. HOMOTHETY, INVARIANCE & REACHABILITY

We proceed to address approximate reachability by utilizing homothety and invariance. We employ homothety to achieve computational efficiency and exploit set invariance to address "dynamical aspects" of approximate reachability analysis.

We utilize two families of sets $\mathcal{Y}$ and $\mathcal{Z}$ given by:

$$\mathcal{Y} := \{y \oplus \alpha S_Y \ : \ y \in R^n,\ \alpha \in R_+\} \text{ and}$$
$$\mathcal{Z} := \{z \oplus \beta S_Z \ : \ z \in R^n,\ \beta \in R_+\}, \tag{3.1}$$

where inner and outer basic shape sets $S_Y \in \text{ComCP}(R^n)$ and $S_Z \in \text{ComCP}(R^n)$ are constructed/designed off–line. We employ decoupled inner and outer approximation maps $\mathcal{A}_\mathcal{I}(\cdot) \ : \ \text{Com}(R^n) \to \mathcal{Y}$ and $\mathcal{A}_\mathcal{O}(\cdot) \ : \ \text{Com}(R^n) \to \mathcal{Z}$ given by:

$$\mathcal{A}_\mathcal{I}(X) := \arg\inf_Y \{f_\mathcal{I}(X,Y) \ : \ Y \in \mathcal{I}(X)\} \text{ and} \tag{3.2a}$$
$$\mathcal{A}_\mathcal{O}(X) := \arg\inf_Z \{f_\mathcal{O}(X,Z) \ : \ Z \in \mathcal{O}(X)\}, \tag{3.2b}$$

where $f_\mathcal{I}(\cdot,\cdot) \ : \ \text{Com}(R^n) \times \mathcal{Y} \to R$ and $f_\mathcal{O}(\cdot,\cdot) \ : \ \text{Com}(R^n) \times \mathcal{Z} \to R$ are selection criteria for the inner and outer approximation, and where for any $X \in \text{Com}(R^n)$,

$$\mathcal{I}(X) := \{Y \in \mathcal{Y} \ : \ Y \subseteq X\} \text{ and} \tag{3.3a}$$
$$\mathcal{O}(X) := \{Z \in \mathcal{Z} \ : \ X \subseteq Z\}. \tag{3.3b}$$

The inner and outer approximate reachable sets/tubes are obtained, for $k \in N$, by:

$$Y_k(X_0) = \mathcal{A}_\mathcal{I}(\mathcal{R}^k(X_0)),\ Z_k(X_0) = \mathcal{A}_\mathcal{O}(\mathcal{R}^k(X_0)), \tag{3.4}$$

where $X_0 \in \text{Com}(R^n)$ is the given initial condition for the set–dynamics (2.4) and the implicit rather than the explicit form of the exact reachable sets $\mathcal{R}^k(X_0)$, given by (2.3), is preferably employed for computational purposes. By construction it follows that:

Proposition 2. Suppose that the inner and outer approximation maps, $\mathcal{A}_\mathcal{I}(\cdot)$ and $\mathcal{A}_\mathcal{O}(\cdot)$, are non–empty, bounded and single–valued for all $X \in \text{Com}(R^n)$. Then for any $X_0 \in \text{Com}(R^n)$ and all $k \in N$:

$$Y_k(X_0) \subseteq \mathcal{R}^k(X_0) \subseteq Z_k(X_0),$$
$$H_L(\mathcal{R}^k(X_0), Z_k(X_0)) \leq h_L(Z_k(X_0), Y_k(X_0)) \text{ and}$$
$$H_L(\mathcal{R}^k(X_0), Y_k(X_0)) \leq h_L(Z_k(X_0), Y_k(X_0)), \tag{3.5}$$

with $Y_k(X_0) \in \mathcal{Y}$ and $Z_k(X_0) \in \mathcal{Z}$ given by (3.4).

Utilization of set invariance as the second ingredient for approximate reachability is motivated by:

Proposition 3. Suppose Assumption 1 holds and that a set $S \in \text{ComCP}(R^n)$ is an invariant set, i.e. $\mathcal{R}(S) \subseteq S$ and, in addition, that γS is not an invariant set for any $\gamma \in [0, 1)$. Then for any set $X = x \oplus \lambda S,\ x \in R^n,\ \lambda \in R_+$ the inner and outer relations

$$y \oplus \alpha S \subseteq \mathcal{R}(X) \subseteq z \oplus \beta S \tag{3.6}$$

hold for $y = z = Ax$, some $\alpha \in [0, \beta]$ and some $\beta \in R_+$ which, in addition, is such that $1 \leq \beta \leq \lambda$ when $\lambda \geq 1$ and such that $1 \geq \beta \geq \lambda$ when $\lambda \in [0, 1]$.

Motivated by Propositions 2 and 3 we consider the inner and outer approximate reachable sets/tubes given by:

$$Y_{k\bar{N}+l} := \mathcal{A}_\mathcal{I}(\mathcal{R}^l(Y_{k\bar{N}})),\ Z_{k\bar{N}+l} := \mathcal{A}_\mathcal{O}(\mathcal{R}^l(Z_{k\bar{N}})), \tag{3.7}$$

where $\bar{N} \in N_+$, $l \in N_{[1,\bar{N}]} := \{1, 2, \ldots, \bar{N}\}$ and $k \in N$. The integer $\bar{N}$, referred to as the *cycle length*, is introduced to reduce computational complexity. Additionally, we employ initial conditions $Y_0 \in \mathcal{Y}$ and $Z_0 \in \mathcal{Z}$ for *the "Y" and "Z" set–dynamics* (3.7) instead of $X_0 \in \text{Com}(R^n)$ as in (3.4). We require that initial conditions $Y_0 \in \mathcal{Y}$ and $Z_0 \in \mathcal{Z}$ satisfy $Y_0 \subseteq X_0 \subseteq Z_0$ in order to have consistent approximation reachability method. Similarly to Proposition 2, if $Y_0 \subseteq X_0 \subseteq Z_0$ and the inner and outer approximation maps ($\mathcal{A}_\mathcal{I}(\cdot)$ and $\mathcal{A}_\mathcal{O}(\cdot)$) are non–empty, bounded and single–valued for all $X \in \text{Com}(R^n)$, then the inner and outer relations $Y_p \subseteq \mathcal{R}^p(X_0) \subseteq Z_p$ hold for all $p \in N$ when set sequences $\{Y_k\}_{k=0}^\infty$ and $\{Z_k\}_{k=0}^\infty$ are generated by (3.7).

Remark 1. When the cycle length is one, i.e. $\bar{N} = 1$, the considered approximation method reduces to:

$$Y^+ = \mathcal{A}_\mathcal{I}(\mathcal{R}(Y)) \text{ and } Z^+ = \mathcal{A}_\mathcal{O}(\mathcal{R}(Z)). \tag{3.8}$$

The set–dynamics (3.8) form the one–step approximate reachability method. *Facts that set–dynamics* (3.8) *are restricted to families of sets $\mathcal{Y}$ and $\mathcal{Z}$ and are induced by compositions of the inner and outer approximation maps, $\mathcal{A}_\mathcal{I}(\cdot)$ and $\mathcal{A}_\mathcal{O}(\cdot)$, and the map $\mathcal{R}(\cdot)$ provide an explanation, from the dynamical point of view, of the "wrapping effect". Inappropriately chosen families of sets $\mathcal{Y}$ and $\mathcal{Z}$ and approximation maps $\mathcal{A}_\mathcal{I}(\cdot)$ and $\mathcal{A}_\mathcal{O}(\cdot)$ can destabilize even the stable set–dynamics induced by the map $\mathcal{R}(\cdot)$ and, hence, produce the "wrapping effect".*

4. SIMPLE APPROXIMATE REACHABLE SETS

Evidently, computational efficiency of approximate reachability is achieved by utilizing simple inner and outer approximating maps $\mathcal{A}_\mathcal{I}(\cdot)$ and $\mathcal{A}_\mathcal{O}(\cdot)$ and adequate families of sets $\mathcal{Y}$ and $\mathcal{Z}$. However, the simplicity obtained by utilizing homothety requires a careful selection of inner and outer basic shape sets S_Y and S_Z in order to obtain bounded and convergent approximate reachable sets/tubes. We discuss now a more concrete and fairly simple approximation method exploiting invariant inner and outer basic shape sets S_Y and S_Z.

Given any arbitrary $X_0 = x_0 \oplus \tilde{X} \in \text{Com}(R^n)$ with $0 \in \tilde{X}$, let:

$$\alpha_0 := \sup_\alpha \{\alpha \ : \ \alpha S_Y \subseteq \tilde{X},\ \alpha \in R_+\} \text{ and}$$
$$\beta_0 := \inf_\beta \{\beta \ : \ \tilde{X} \subseteq \beta S_Z,\ \beta \in R_+\}, \tag{4.1}$$

so that $Y_0 := x_0 \oplus \alpha_0 S_Y \subseteq X_0 \subseteq x_0 \oplus \beta_0 S_Z =: Z_0$. Since the families of sets $\mathcal{Y}$ and $\mathcal{Z}$, specified in (3.1), are, respectively, families of homothetic copies of inner and outer basic shape sets S_Y and S_Z, the approximate reachable sets/tubes are completely characterized by the *tube center* sequences $\{y_k\}_{k=0}^\infty$ and $\{z_k\}_{k=0}^\infty$ and the *tube cross–section scaling* sequences $\{\alpha_k\}_{k=0}^\infty$ and $\{\beta_k\}_{k=0}^\infty$ and take the form:

$$Y_k = y_k \oplus \alpha_k S_Y \text{ and } Z_k = z_k \oplus \beta_k S_Z,\ \forall k \in N. \tag{4.2}$$

Further computational efficiency is achieved by requiring that the *tube center* sequences $\{y_k\}_{k=0}^\infty$ and $\{z_k\}_{k=0}^\infty$ coincide and by generating the corresponding *tube center* sequence $\{x_k = y_k = z_k\}_{k=0}^\infty$ according to:

$$x^+ = Ax. \tag{4.3}$$

In order to be consistent with (3.7), the *tube cross–section scaling* sequences $\{\alpha_k\}_{k=0}^\infty$ and $\{\beta_k\}_{k=0}^\infty$ are generated via:

$$\alpha_{k\bar{N}+l} := \varphi_l(\alpha_{k\bar{N}}) \text{ and } \beta_{k\bar{N}+l} := \mu_l(\beta_{k\bar{N}}), \tag{4.4}$$

with functions $\varphi_l(\cdot) : R_+ \to R_+$ and $\mu_l(\cdot) : R_+ \to R_+$ given, for $l \in N_{[1,\bar{N}]}$, by:

$$\varphi_l(\alpha) := \sup_{\gamma\geq 0}\{\gamma \; : \; \gamma S_Y \subseteq A^l \alpha S_Y \oplus \bigoplus_{i=0}^{l-1} A^i W\} \text{ and}$$

$$\mu_l(\beta) := \inf_{\gamma\geq 0}\{\gamma \; : \; A^l \beta S_Z \oplus \bigoplus_{i=0}^{l-1} A^i W \subseteq \gamma S_Z\}, \tag{4.5}$$

where $\bar{N} \in N_+$ is the cycle length, $l \in N_{[1,\bar{N}]}$ and $k \in N$. When Assumption 1 holds and when $S_Y \in \text{ComCP}(R^n)$ and $S_Z \in \text{ComCP}(R^n)$, it is, in principle, possible to pre–compute functions $\varphi_l(\cdot)$, $l \in N_{[1,\bar{N}]}$ and $\mu_l(\cdot)$, $l \in N_{[1,\bar{N}]}$ as solutions to a set of, relatively simple, one–dimensional parametric programs (Bank *et al.*, 1983). Additionally, functions $\varphi_l(\cdot)$, $l \in N_{[1,\bar{N}]}$ and $\mu_l(\cdot)$, $l \in N_{[1,\bar{N}]}$ are, respectively, continuous concave and convex functions if Assumption 1 holds and the inner and outer basic shape sets S_Y and S_Z are proper C sets in R^n.

Remark 2. Summarizing, simplified approximate reachable tubes, $\{Y_k\}_{k=0}^{\infty}$ and $\{Z_k\}_{k=0}^{\infty}$, as well as approximate reachable sets, $Y_k = x_k \oplus \alpha_k S_Y$, $k \in N$ and $Z_k = x_k \oplus \beta_k S_Y$, $k \in N$, are completely characterized by the following difference relations:

$$\begin{aligned} &x_{k\bar{N}+l} = A^{k\bar{N}+l} x_0, \\ &\alpha_{k\bar{N}+l} = \varphi_l(\alpha_{k\bar{N}}) \text{ and } \beta_{k\bar{N}+l} = \mu_l(\beta_{k\bar{N}}), \end{aligned} \tag{4.6}$$

where $\bar{N} \in N_+$ is the cycle length and is fixed, $l \in N_{[1,\bar{N}]}$, $k \in N$ and $y_0 = z_0 = x_0$ and α_0 and β_0 given as in (4.1).

In order to discuss boundedness and potential convergence of the approximate reachable sets we also invoke:

Assumption 3. There exists a (finite) integer $\tilde{k} \in N$ such that $\mathcal{R}^{\tilde{k}}(\{0\}) \in \text{ComCP}(R^n)$ and the cycle length $\bar{N}$ satisfies $\infty > \bar{N} \geq \tilde{k}$.

Assumption 4. The inner and outer basic shape sets $S_Y \in \text{ComCP}(R^n)$ and $S_Z \in \text{ComCP}(R^n)$ are invariant sets.

Given the cycle length $\bar{N} \in N$ let:

$$\varphi^*_{\bar{N}} := \sigma^{-1}, \; \sigma := \inf_{\alpha\geq 0}\{\alpha \; : \; S_Y \subseteq A^{\bar{N}} S_Y \oplus \alpha \bigoplus_{i=0}^{\bar{N}-1} A^i W\},$$

$$\mu^*_{\bar{N}} := \rho^{-1}, \; \rho := \sup_{\beta\geq 0}\{\beta \; : \; A^{\bar{N}} S_Z \oplus \beta \bigoplus_{i=0}^{\bar{N}-1} A^i W \subseteq S_Z\},$$

$$\varphi^*_l := \varphi_l(\varphi^*_{\bar{N}}) \text{ and } \mu^*_l := \mu_l(\mu^*_{\bar{N}}), \tag{4.7}$$

where $l \in N_{[1,\bar{N}-1]}$ and $\varphi_l(\cdot)$ and $\mu_l(\cdot)$ are given by (4.5). Under Assumptions 1–4 the tube cross–section scaling sequences $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ contain convergent subsequences $\{\tilde{\alpha}_k\}_{k=0}^{\infty}$ and $\{\tilde{\beta}_k\}_{k=0}^{\infty}$ with, for all $k \in N$, $\tilde{\alpha}_k = \alpha_{\bar{N}k}$ and $\tilde{\beta}_k = \beta_{\bar{N}k}$.

Proposition 4. Suppose Assumptions 1–4 hold. Then scalar sequences $\{\tilde{\alpha}_k\}_{k=0}^{\infty}$ and $\{\tilde{\beta}_k\}_{k=0}^{\infty}$ generated by:

$$\tilde{\alpha}_{k+1} := \varphi_{\bar{N}}(\tilde{\alpha}_k) \text{ and } \tilde{\beta}_{k+1} := \mu_{\bar{N}}(\tilde{\beta}_k), \; k \in N, \tag{4.8}$$

where $\varphi_{\bar{N}}(\cdot)$ and $\mu_{\bar{N}}(\cdot)$ are given by (4.5) converge, exponentially fast, to $\varphi^*_{\bar{N}}$ and $\mu^*_{\bar{N}}$ given by (4.7), respectively, for any arbitrary $\tilde{\alpha}_0 \in R_+$ and $\tilde{\beta}_0 \in R_+$.

Proposition 4 yields the following fact:

Corollary 1. Suppose Assumptions 1–4 hold and consider scalar sequences $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ generated by (4.4) for arbitrary $\alpha_0 \in R_+$ and $\beta_0 \in R_+$. Then, for all $\varepsilon > 0$ there exists a (finite) $\bar{k} \in N$ such that $|\beta_{k\bar{N}+l} - \alpha_{k\bar{N}+l}| \leq |\mu^*_l - \varphi^*_l| + \varepsilon$, for all $k \in N$, $k \geq \bar{k}$ and all $l \in N_{[1,\bar{N}]}$, where φ^*_l and μ^*_l are given by (4.7).

To simplify statements we introduce:

Assumption 5. For a given $X_0 \in \text{Com}(R^n)$, $\{X_k\}_{k=0}^{\infty}$ is the exact infinite time reachable tube, i.e. $X_{k+1} := \mathcal{R}(X_k)$, $\forall k \in N$ and set sequences $\{Y_k\}_{k=0}^{\infty}$ and $\{Z_k\}_{k=0}^{\infty}$ are given by $\{Y_k = x_k \oplus \alpha_k S_Y\}_{k=0}^{\infty}$ and $\{Z_k = x_k \oplus \beta_k S_Z\}_{k=0}^{\infty}$ where $\{x_k\}_{k=0}^{\infty}$, $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ are sequences generated by (4.6) with initial conditions $x_0 \in R^n$, $\alpha_0 \in R_+$ and $\beta_0 \in R_+$ guaranteeing the inner and outer relations $x_0 \oplus \alpha_0 S_Y \subseteq X_0 \subseteq x_0 \oplus \beta_0 S_Z$.

Combining Propositions 2, 3, 4 and Corollary 1 we have:

Proposition 5. Let $X_0 \in \text{Com}(R^n)$ be arbitrary and suppose Assumptions 1–5 hold. Then: (i) $Y_k \in \mathcal{Y}$ and $Z_k \in \mathcal{Z}$, $\forall k \in N$ (ii) $Y_k \subseteq X_k \subseteq Z_k, \forall k \in N$, (iii) $H_L(Y_k, Z_k) \leq (\beta_k - \alpha_k \gamma_{yz}) H_L(S_Z, \{0\})$, $\forall k \in N$, where $\gamma_{yz} := \max_\gamma\{\gamma : \gamma S_Z \subseteq S_Y\}$, and (iv) for all $\varepsilon > 0$ there exists a (finite) $\bar{k} \in N$ such that for all $k \in N$, $k \geq \bar{k}$ and all $l \in N_{[1,\bar{N}]}$ $H_L(Y_{k\bar{N}+l}, Z_{k\bar{N}+l}) \leq (\mu^*_l - \varphi^*_l \gamma_{yz}) H_L(S_Z, \{0\}) + \varepsilon$.

Proposition 5 states that the proposed, and clearly simple, approximate reachability method results in bounded inner and outer approximate reachable sets/tubes. However, introduced computational simplifications may lead to slightly weaker convergence properties. In particular, sets $\{\varphi^*_l S_Y : l \in N_{[1,\bar{N}]}\}$ and $\{\mu^*_l S_Z : l \in N_{[1,\bar{N}]}\}$ might form attractive (set–valued) limit cycles for the "Y" and "Z" set–dynamics, respectively. Let:

$$\bar{\mu} := \max_{l\in N_{[1,\bar{N}]}} \mu^*_l \text{ and } \bar{Z} := \bigcup_{l\in N_{[1,\bar{N}]}} \mu^*_l S_Z = \bar{\mu} S_Z,$$

$$\bar{\varphi} := \min_{l\in N_{[1,\bar{N}]}} \varphi^*_l \text{ and } \bar{Y} := \bigcap_{l\in N_{[1,\bar{N}]}} \varphi^*_l S_Y = \bar{\varphi} S_Y, \tag{4.9}$$

where φ^*_l and μ^*_l are given by (4.7). By utilizing Proposition 5 and (4.9) we obtain:

Proposition 6. Let $X_0 \in \text{Com}(R^n)$ be arbitrary and suppose Assumptions 1–5 hold. Then, for all $k \in N$: (i) $Y_k \in \mathcal{Y}$ and $Z_k \in \mathcal{Z}$, (ii) $Y_k \subseteq X_k \subseteq Z_k$, (iii) $h_L(Z_k, \bar{Z}) \to 0$, $h_L(Y_k, \bar{Z}) \to 0$, $h_L(\bar{Y}, Y_k) \to 0$ and $h_L(\bar{Y}, Z_k) \to 0$ as $k \to \infty$ and, in addition, (iv) for all $\varepsilon > 0$ there exists a (finite) $\bar{k} \in N$ such that $H_L(Y_k, Z_k) \leq H_L(\bar{Y}, \bar{Z}) + \varepsilon = h_L(\bar{Z}, \bar{Y}) + \varepsilon$ for all $k \in N$, $k \geq \bar{k}$.

Our concluding result identifies an exceptional case when the Hausdorff distance between the inner and outer approximate reachable sets converges to 0 *as* $k \to \infty$.

Theorem 1. Let $X_0 \in \text{Com}(R^n)$ be arbitrary, suppose Assumptions 1–5 hold and, in addition, that the inner and outer basic shape sets coincide and are equal to the minimal invariant set, i.e. $S_Y = S_Z = X_\infty = \bigoplus_{i=0}^{\infty} A^i W$. Then for all $k \in N$, $Y_k \in \mathcal{Y}$, $Z_k \in \mathcal{Z}$ and $Y_k \subseteq X_k \subseteq Z_k$ and, furthermore, $H_L(Y_k, X_\infty) \to 0$, $H_L(Z_k, X_\infty) \to 0$ and $H_L(Y_k, Z_k) \to 0$ as $k \to \infty$.

Remark 3. We close this section by pointing out that Theorem 1 identifies, indeed, an exceptional case. Namely, since families of polytopes, zonotopes and ellipsoids are not closed under the limit taking operation, no standard (practical) approximate reachability analysis method can guarantee convergence of the Hausdorff distance between the inner and outer approximate reachable sets to 0 even in the case when Assumptions 1–3 hold and W is, in addition, a polytope/zonotope or an ellipsoid (excluding trivial cases). It is hopefully clear that, results of Proposition 1, 5 and 6 and Theorem 1 can be utilized in a direct way to obtain a computationally efficient method for approximate reachability analysis resulting in bounded and convergent approximate reachable sets/tubes. Finally, it is direct, though with an increase of computational complexity, to design an approximate reachability analysis method using homothety and invariance guaranteeing that the Hausdorff distance between the inner and outer approximate reachable sets is arbitrarily small in the limit.

5. COMPUTATIONS & ILLUSTRATIVE EXAMPLES

If sets S_Y, S_Z and W are bounded and closed polyhedra then functions $\varphi_l(\cdot)$ and $\mu_l(\cdot)$ with $l \in N_{[1,\bar{N}]}$ are, respectively, piecewise affine concave and convex continuous functions and are computable by solving a set of, relatively simple, one dimensional parametric linear programs.

The approximate reachable sets/tube are obtained by:

(i) designing the sets S_Y and S_Z off–line using, for instance, Proposition 1,
(ii) computing $\varphi_l(\cdot)$ and $\mu_l(\cdot)$ with $l \in N_{[1,\bar{N}]}$ off–line,
(iii) computing x_0, α_0 and β_0 by (4.1) for a given $X_0 \in \mathrm{Com}(R^n)$ on–line, and
(iv) evaluating difference relations (4.6) on–line.

We provide three academic examples illustrating the proposed method and, in particular, claims of Theorem 1.

Example 1. The first example is the system with:

$$A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, W = \{x \in R^2 \ : \ -1 \leq x^1 \leq 1, \ x^2 = 0\},$$

and the inner and outer basic shape sets coincide and satisfy $S_Y = S_Z = X_\infty = 2B_\infty$ (hereafter B_∞ is the closed unit ∞–norm ball). Figure 1 illustrates that, when

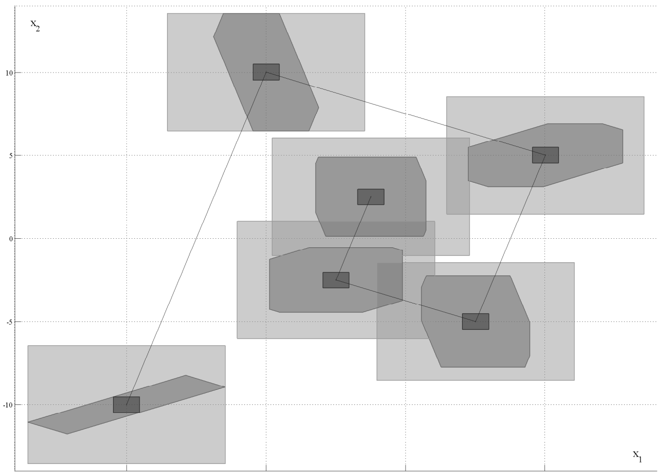

Fig. 1. Simple Approximate Reachable Sets for $\bar{N} = 1$.

the cycle length is $\bar{N} = 1$ the approximate reachable sets Y_k and Z_k, $k = 0, 1, \ldots, 5$, shown in darker and lighter gray–scale shading, inner and outer bound the exact reachable sets X_k, $k = 0, 1, \ldots, 5$, shown in medium gray–scale shading, for a particular $X_0 \in \mathrm{Com}(R^n)$. The approximate reachable sets are only bounded and their Hausdorff distance does not decrease (though, neither it increases) since Assumption 3 is violated for the cycle length $\bar{N} = 1$, as evident from functions $\varphi_1(\cdot)$ and $\mu_1(\cdot)$ given by $\varphi_1(\alpha) = \alpha$ when $\alpha \in [0, 1)$ and $\varphi_1(\alpha) = \frac{1}{2}\alpha + \frac{1}{2}$ when $\alpha \geq 1$, and $\mu_1(\beta) = \frac{1}{2}\beta + \frac{1}{2}$ when $\beta \in [0, 1)$ and $\mu_1(\beta) = \beta$ when $\beta \geq 1$. Figure 2 shows the case when $\bar{N} = 2$ and our assumptions are satisfied. In addition, to $\varphi_1(\cdot)$ and $\mu_1(\cdot)$ we employed functions $\varphi_2(\cdot)$ and $\mu_2(\cdot)$ given by $\varphi_2(\alpha) = \frac{1}{2}\alpha + \frac{1}{2}$ and $\mu_2(\beta) = \frac{1}{2}\beta + \frac{1}{2}$, which are contractive and have 1 as the common fixed point. Direct

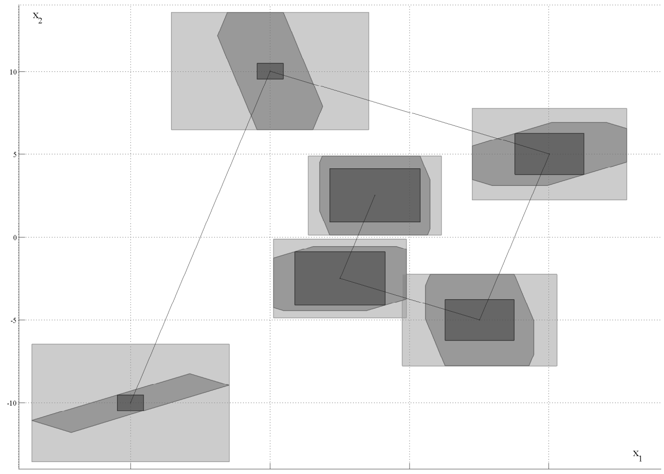

Fig. 2. Simple Approximate Reachable Sets for $\bar{N} = 2$.

inspection of Figure 2 reveals that approximate reachable sets inner and outer bound the exact reachable sets and get closer at every $2k$, $k \in N$ and would, consequently, converge to $2B_\infty$, as expected by Theorem 1.

Example 2. The second example is the system with:

$$A = \begin{bmatrix} 0.9035 & 0.2936 \\ -0.2936 & 0.9035 \end{bmatrix},$$

$W = \{x \in R^2 \ : \ -0.1 \leq x^1 \leq 0.1, \ -1 \leq x^2 \leq 1\}$ and the inner and outer basic shape sets satisfy $S_Y = S_Z = X_\infty$. The minimal invariant set X_∞ is in this example computable and admits representation with either 20 vertices or 20 facets. The cycle length is $\bar{N} = 6$ and

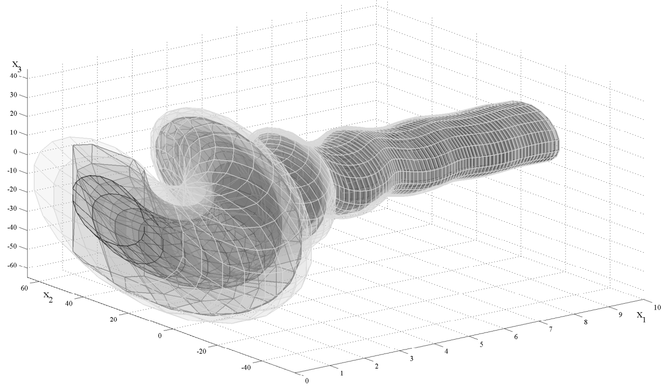

Fig. 3. The Space–Time Evolution of Reachable Sets.

our assumptions are satisfied. The space–time evolution of the inner, exact and outer set–dynamics for a given $X_0 \in \text{Com}(R^n)$ is shown in Figure 3, which also illustrates that sets Y_k and Z_k, shown in darker and lighter gray–scale shading, inner and outer bound sets X_k, shown in medium gray–scale shading, and $H_L(Y_k, Z_k)$ converges to 0 as expected by Theorem 1.

Example 3. The third example is the system with:

$$A = \frac{1}{10}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & -5 & 5 \end{bmatrix},\ W = [-1, 1] \times B_1,$$

where B_1 is the closed unit 1–norm ball. The inner and outer basic shape sets S_Y and S_Z are equal to the minimal invariant set X_∞, which admits finite representation with either 16 vertices or 10 facets. The cycle length is $\bar{N} = 3$ and our assumptions are satisfied. Assertions of Theorem 1 are evidenced in Figure 4 in which sets Y_k and Z_k, shown in darker and lighter gray–scale shading, inner and outer bound sets X_k, shown in medium gray–scale shading and $H_L(Y_k, Z_k)$ converges to 0.

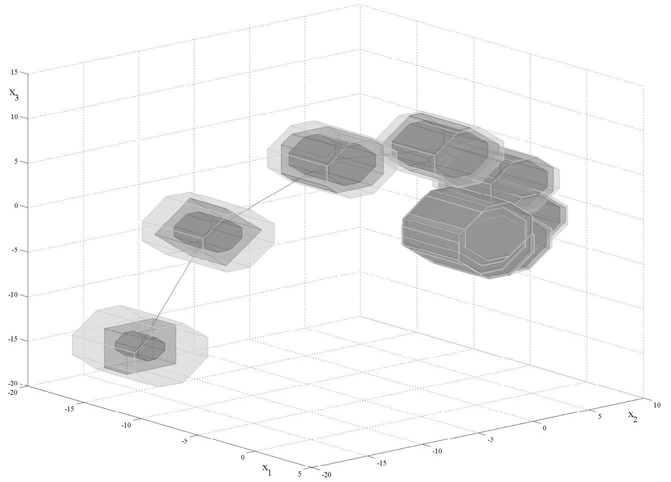

Fig. 4. $3-D$ Simple Approximate Reachable Sets/Tubes.

6. CONCLUSIONS

This note considered a method for approximate reachability utilizing homothety and set invariance. The offered approximate reachability method is computationally efficient and, under relatively modest and standard assumptions, yields bounded and convergent approximate reachable sets/tubes.

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