

Approximate feedback linearization using multivariable Legendre polynomials

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Abstract: This paper presents a numerical approach to approximate feedback linearization. By using a Galerkin approach on the basis of multivariable Legendre polynomials an approximate solution to the singular PDE of the feedback linearization technique proposed in Kazantzis and Kravaris (2000a,b) is determined. It is shown that the L_2 -norm of the remaining nonlinearity in the resulting dynamics can be made small on a specified multivariable interval in the state space. Furthermore, a matrix equation is derived for determining the corresponding change of coordinates and feedback such that the proposed design procedure can easily be implemented in a numerical software package. A simple example demonstrates the properties of the new approximate feedback linearization.

Keywords: Nonlinear systems, feedback linearization, Galerkin approach, multivariable Legendre polynomials, L_2 -approximation.

1. INTRODUCTION

The problem of feedback linearization amounts to finding a nonlinear change of coordinates along with a nonlinear feedback such that in the new coordinates the system in question is linear and controllable. The advantage of this approach is that in the new coordinates well developed tools from linear control theory can be applied to control the nonlinear system. Consequently, the problem of feedback linearization has been extensively investigated during the last twenty years. Necessary and sufficient conditions for feedback linearization by static state feedback have been given in Jakubczyk and Respondek (1980) for single-input systems and for multi-input systems in Hunt and Su (1981); Hunt et al. (1983); Jakubczyk and Respondek (1980). Feedback linearization by dynamic state feedback is of interest for multi-input systems where the class of systems exactly linearizable by dynamic state feedback is larger than the class of static state feedback linearizable systems. A sufficient condition for dynamic feedback linearization was first derived in Isidori et al. (1986). However, the conditions for static and dynamic feedback linearization turned out to be restrictive so that the class of feedback linearizable systems is limited.

Recently, in Kazantzis and Kravaris (2000a,b) a new approach for static state feedback linearization of single-input and multi-input systems was proposed that can be applied to wider class of systems than the classical feedback linearization. The main idea of this approach is to transform the nonlinear system via a change of coordinates and static state feedback into a linear system with desired dynamics. This circumvents the need to transform the input vector fields into constant vectors in the new coordinates. Thus, the feedback linearization approach is relaxed from the rather restrictive involutivity conditions. Furthermore, only static state feedback has to be considered for

the feedback linearization of multi-input systems such that the computation of a dynamic extension is not needed. The corresponding change of coordinates and feedback can be computed by solving an initial value problem for a singular PDE where existence and uniqueness can be assured by using Lyapunov's auxiliary theorem. As the corresponding PDE is singular the method of characteristics for first-order PDEs cannot be applied to find a solution of the initial value problem (for details see Kazantzis and Kravaris (2000a,b)). However, since Lyapunov's auxiliary theorem assures that there exists an analytic solution about the origin a Taylor series approximation is proposed in Kazantzis and Kravaris (2000a,b) to solve the initial value problem. The drawback of this approach is that in many cases the local character of this series solution method leads to a small region of attraction for the approximately linear dynamics. Furthermore, the closed loop performance may deteriorate if the initial condition of the system is far away from the origin.

The aim of this paper is to present a Galerkin approach (see e.g. Finlayson (1972); Fletcher (1984)) for determining an approximate solution of the initial value problem that circumvents the drawbacks of the Taylor series solution method. The Galerkin method approximates the solution of the initial value problem by a finite series of orthonormal basis functions up to degree N . In order to determine the coefficients of this approximation the equation error resulting from substituting the approximate solution in the PDE is expanded in a series of the orthonormal basis functions up to degree N with minimal L_2 -error norm. Then, the free parameters in the approximate solution are determined such that the coefficients of the first N terms of the series expansion of the equation error vanish. Since the functions present in the PDE are smooth the series expansion of the equation error decreases very fast with

increasing approximation degree. Thus, by compensating the first terms of this series the L_2 -norm of the equation error will presumably become very small. Thereby, the interval in the state space where the L_2 -norm of the equation error is small can be specified beforehand. Consequently, the domain where the closed loop dynamics becomes nearly linear can be assigned in the design. In this paper multivariable Legendre polynomials are used in the Galerkin approach as basis functions. This has the advantage that by applying the operational matrices for the multiplication and differentiation of multivariable Legendre polynomials matrix equations can be explicitly derived for approximately solving the initial value problem. Hence, the proposed feedback linearization can be easily implemented in MATLAB. An additional property of the Galerkin approach is that already for a small approximation degree rather accurate results may be obtained which reduces the computational effort of the feedback linearization.

The next section reviews the feedback linearization on the basis of Lyapunov's auxiliary theorem. In Section 3 the Galerkin approach for solving the initial value problem is presented. A simple example demonstrates the advantages of the Galerkin approach for approximate feedback linearization relative to the Taylor series method.

2. FEEDBACK LINEARIZATION USING LYAPUNOV'S AUXILIARY THEOREM

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u, \quad x(0) = x_0 \quad (1)$$

with the state $x \in \mathbb{R}^n$ and the inputs $u \in \mathbb{R}^p$. Assume that f and the columns g_i , $i = 1, 2, \dots, p$, of G are analytic mappings in a neighbourhood Ω about $x = 0$, i.e. $f \in C^\omega(\Omega)$ and $g_i \in C^\omega(\Omega)$. Furthermore, let $x = 0$ be an equilibrium point of (1) for $u = 0$, i.e. $f(0) = 0$. In what follows a feedback transformation

$$z = \phi(x), \quad \phi(0) = 0 \quad (2)$$

$$u = \alpha(x) - K\phi(x), \quad \alpha(0) = 0, \quad \frac{\partial \alpha}{\partial x}(0) = 0, \quad \alpha \in C^\omega(\Omega) \quad (3)$$

is determined such that the system (1) is an asymptotically stable and linear system

$$\dot{z} = (A - BK)z, \quad z(0) = \phi(x_0) \quad (4)$$

in the z -coordinates. In (4) the matrices A and B are given by the Jacobian linearization

$$A = \frac{\partial f}{\partial x}(0), \quad B = G(0) \quad (5)$$

of (1) about $x = 0$ and $u = 0$ where it is assumed that $\text{rank} G(0) = p$ and that (A, B) is controllable. Consequently, there exists a feedback gain K so that $A - BK$ is a Hurwitz matrix. Different from the approach in Kazantzis and Kravaris (2000a,b) an additional feedback α is considered in (3) which adds additional degrees of freedom to the feedback transformation (see also Section 3.3).

It is straightforward to show that the feedback transformation (2)–(3) can be determined by solving the initial value problem

$$\frac{\partial \phi(x)}{\partial x} (f(x) + G(x)(\alpha(x) - K\phi(x))) = (A - BK)\phi(x) \quad (6)$$

$$\phi(0) = 0 \quad (7)$$

Existence and uniqueness of a solution of (6)–(7) can be checked by using Lyapunov's auxiliary theorem (for details see Kazantzis and Kravaris (2000a,b)) which leads to the following result.

Theorem 1. The initial value problem (6)–(7) has a unique analytic solution $\phi(x) = x + \phi_{nl}(x)$ with $\phi_{nl}(0) = 0$ and $\frac{\partial \phi_{nl}}{\partial x}(0) = 0$ in a neighbourhood of $x = 0$ if the eigenvalues $\lambda_i(A)$, $i = 1, 2, \dots, n$, of the matrix A in (5) and the eigenvalues $\lambda_i(A - BK)$, $i = 1, 2, \dots, n$, of $A - BK$ in (4) satisfy

$$\lambda_i(A) \neq p_1 \lambda_1(A - BK) + \dots + p_n \lambda_n(A - BK) \quad (8)$$

for all $i = 1, 2, \dots, n$, and for all nonnegative integers p_j such that $p_1 + \dots + p_n \geq 2$.

Since (A, B) is controllable there always exists a choice of the eigenvalues of $A - BK$ such that (8) is satisfied. The corresponding eigenvalues can be determined by using the results in Devanathan (2001).

3. GALERKIN APPROACH FOR SOLVING THE INITIAL VALUE PROBLEM

3.1 Basis functions

The approximate solution of the initial value problem (6)–(7) is represented by a finite series of orthonormal basis functions. A natural choice for the basis functions are multivariable polynomials which are orthonormal on a given n -dimensional interval in the state space. These polynomials can be obtained by considering

$$\varphi_{k_1 \dots k_n}(x) = \varphi_{k_1}(x_1) \cdot \dots \cdot \varphi_{k_n}(x_n) \quad (9)$$

where each $\varphi_{k_\nu}(x_\nu)$ in (9) is a one-dimensional Legendre polynomial

$$\varphi_{k_\nu}(x_\nu) = \sqrt{\frac{2k+1}{2}} \frac{1}{2^k k!} \frac{d^k}{dx_\nu^k} (x_\nu^2 - 1)^k, \quad k \in \mathbb{N}_0 \quad (10)$$

of degree k_ν (for details concerning Legendre polynomials see e.g. Courant and Hilbert (1965)). As a result the multivariable Legendre polynomials $\varphi_{k_1 \dots k_n}(x)$ of degree $k = k_1 + \dots + k_n$ are obtained. The advantage of the Legendre polynomials (10) is, that they are *orthonormal* with the constant weight function 1 on the interval $I = [-1, 1]$, i.e. the *scalar product* is given by

$$\langle \varphi_i, \varphi_j \rangle = \int_{-1}^1 \varphi_i(x_\nu) \varphi_j(x_\nu) dx_\nu = \delta_{ij}, \quad \forall i, j \in \mathbb{N}_0 \quad (11)$$

where

$$\delta_{ij} = \begin{cases} 0 & : i \neq j \\ 1 & : i = j \end{cases} \quad (12)$$

denotes the *Kronecker delta function*. The set of one-dimensional Legendre polynomials is complete, this means that every function $g(x_\nu)$ with finite L_2 -norm $\|g\|_2 = (\langle g, g \rangle)^{\frac{1}{2}}$ can be approximated by $\hat{g}(x_\nu) = \sum_{i=0}^N c_i \varphi_i(x_\nu)$ with arbitrary small error $e = (\langle g - \hat{g}, g - \hat{g} \rangle)^{\frac{1}{2}}$ provided that the approximation degree N is sufficiently large. In Deutscher and Bäuml (2006) it is shown, that the properties of the one-dimensional Legendre polynomials (10) carry over to the multivariable Legendre polynomials (9). Hence, they represent an orthonormal and complete set of functions on the n -dimensional interval $I = [-1, 1]^n$. The

L_2 -approximation of arbitrary intervals can be achieved by using an affine linear transformation, which maps a chosen approximation interval $\bar{I} = [x_{1,min}, x_{1,max}] \times \dots \times [x_{n,min}, x_{n,max}]$ to the interval $I = [-1, 1]^n$ (see Deutscher and Bäuml (2006)).

3.2 Derivation of the matrix equation

Consider the approximate solution

$$\phi_N(x) = T_\Phi \Phi(x) \quad (13)$$

of the initial value problem (6)–(7) where $\Phi(x)$ is the vector of multivariable Legendre polynomials up to degree N and T_Φ is the corresponding matrix of coefficients. The feedback α in (3) is set to

$$\alpha(x) = \alpha_N \Phi(x) \quad (14)$$

In general (13) will be an approximate solution of the PDE (6) for α given by (14) such that substituting (13) in the PDE yields the equation error

$$\begin{aligned} r_N(x) &= \frac{\partial \phi_N(x)}{\partial x} (f(x) + G(x)(\alpha(x) - K\phi_N(x))) \\ &\quad - (A - BK)\phi_N(x) \\ &= \frac{\partial \phi_N(x)}{\partial x} (f(x) + \sum_{i=1}^p g_i(x)(\alpha_i(x) - k_i^T \phi_N(x))) \\ &\quad - (A - BK)\phi_N(x) \end{aligned} \quad (15)$$

in which g_i are the columns of G , α_i are the elements of α and k_i^T are the rows of K . The basic idea of the Galerkin approach is to expand the equation error (15) into multivariable Legendre polynomials with minimal L_2 -error norm on a given interval. To this end, consider the L_2 -approximation

$$\hat{v}(x) = V\Phi(x) \quad (16)$$

of a vector function $v(x) \in \mathbb{R}^n$ where V is given by

$$V = \langle v, \Phi^T \rangle = \begin{bmatrix} \langle v_1, \Phi_1 \rangle & \dots & \langle v_1, \Phi_{n_\Phi} \rangle \\ \vdots & & \vdots \\ \langle v_n, \Phi_1 \rangle & \dots & \langle v_n, \Phi_{n_\Phi} \rangle \end{bmatrix} \quad (17)$$

(for details see Deutscher and Bäuml (2006)). In (17) v_i denotes the i th component of v and Φ_i the i th component of Φ with $n_\Phi = \dim \Phi$. The coefficients in (17) assure that \hat{v} approximates v with minimal L_2 -error norm, i.e.

$$\|v - \hat{v}\|_2 = \left(\int_I (v - \hat{v})^T (v - \hat{v}) dx \right)^{\frac{1}{2}} = \min \quad (18)$$

If v is smooth, a small approximation error (18) can be achieved by relatively low approximation degrees (see Deutscher and Bäuml (2006)). In sequel the notation " \approx " means an approximation in the minimal L_2 -error norm sense (18). Consider the L_2 -approximations

$$\frac{\partial \Phi(x)}{\partial x} f(x) \approx A_\Phi \Phi(x) \quad (19)$$

$$\frac{\partial \Phi(x)}{\partial x} g_i(x) \approx N_{i,\Phi} \Phi(x), \quad i = 1, 2, \dots, p \quad (20)$$

$$\alpha_i(x) \approx \alpha_{i,\Phi}^T \Phi(x), \quad i = 1, 2, \dots, p \quad (21)$$

which exist since $f \in C^\omega(\Omega)$, $g_i \in C^\omega(\Omega)$ and $\alpha_i \in C^\omega(\Omega)$. Then, the remaining terms in the equation error (15) can be approximated by using the L_2 -approximation

$$v(x)s(x) \approx (V \otimes s^T)M\Phi(x) \quad (22)$$

of the product of the vector function $v(x) = V\Phi(x) \in \mathbb{R}^n$ and the scalar function $s(x) = s^T \Phi(x)$. In (22) M is given by $M^T = [M_1^T \dots M_{n_\Phi}^T]$ where M_i are

the *operational matrices of multiplication* of multivariable Legendre polynomials defined by

$$\Phi_i(x)\Phi(x) \approx M_i \Phi(x), \quad i = 1, 2, \dots, n_\Phi \quad (23)$$

and " \otimes " is the *Kronecker product* (see e.g. Deutscher (2005)). Then, the L_2 -approximation

$$\begin{aligned} N_{i,\Phi} \Phi(x)(\alpha_{i,\Phi}^T - k_i^T T_\Phi) \Phi(x) \\ \approx (N_{i,\Phi} \otimes (\alpha_{i,\Phi}^T - k_i^T T_\Phi)) M \Phi(x) \end{aligned} \quad (24)$$

for $i = 1, 2, \dots, p$ is implied by (22). If the change of coordinates (13) and the feedback (14) is determined such that the coefficient matrix of the L_2 -approximation

$$r_N(x) \approx R_N \Phi(x) \quad (25)$$

satisfies

$$R_N \stackrel{!}{=} 0 \quad (26)$$

then the equation error will be small in the least square sense on the specified interval in the state space. Using (19)–(21) and (24) this leads to the nonlinear matrix equation

$$\begin{aligned} R_N &= T_\Phi (A_\Phi + \sum_{i=1}^p (N_{i,\Phi} \otimes (\alpha_{i,\Phi}^T - k_i^T T_\Phi)) M) \\ &\quad - (A - BK) T_\Phi \\ &\stackrel{!}{=} 0 \end{aligned} \quad (27)$$

for T_Φ and $\alpha_{i,\Phi}^T$.

In the new coordinates $z = \phi_N(x) = T_\Phi \Phi(x)$ the system (1) has the representation

$$\dot{z} = (A - BK)z + \rho_N(z), \quad z(0) = T_\Phi \Phi(x_0) \quad (28)$$

with

$$\rho_N(z) = r_N(\phi_N^{-1}(z)) \quad (29)$$

in view of (15). Thus, the linear dynamics (4) are perturbed by the equation error which is small in the least square sense on a given n -dimensional interval in the state space. The system (28) has the equilibrium point $z = 0$ if ρ_N satisfies

$$\rho_N(0) = 0 \quad (30)$$

i.e.

$$r_N(0) = 0 \quad (31)$$

in view of the initial condition

$$\phi_N(0) = 0 \quad (32)$$

By using (15) condition (31) leads to

$$\begin{aligned} r_N(0) &= \frac{\partial \phi_N}{\partial x}(0)(f(0) + G(0)(\alpha(0) - K\phi_N(0))) \\ &\quad - (A - BK)\phi_N(0) = 0 \end{aligned} \quad (33)$$

which is satisfied in view of the assumptions (32), $f(0) = 0$ and $\alpha(0) = 0$ (see (3)). The equilibrium point $z = 0$ of (28) is asymptotically stable by the principle of stability in the first approximation if the Jacobian matrix of (28) at $z = 0$ is given by the Hurwitz matrix $A - BK$. Then, the linear terms in the equation error have to vanish, i.e.

$$\frac{\partial \rho_N}{\partial z}(0) = 0 \quad (34)$$

which by $\phi_N(0) = 0$ is satisfied for

$$\frac{\partial r_N}{\partial x}(0) = 0 \quad (35)$$

In view of

$$\begin{aligned} & \frac{\partial}{\partial x} (M(x)v(x)) \\ &= \left[\frac{\partial M(x)}{\partial x_1} v(x) \quad \frac{\partial M(x)}{\partial x_2} v(x) \quad \dots \quad \frac{\partial M(x)}{\partial x_n} v(x) \right] \\ & \quad + M(x) \frac{\partial v(x)}{\partial x} \end{aligned} \quad (36)$$

and

$$\frac{\partial}{\partial x} v(x)s(x) = v(x) \frac{\partial s(x)}{\partial x} + s(x) \frac{\partial v(x)}{\partial x} \quad (37)$$

with $M(x) \in \mathbb{R}^{n \times n}$, $v(x) \in \mathbb{R}^n$ and a scalar function $s(x)$ it is straightforward to verify that (35) leads to

$$\begin{aligned} \frac{\partial r_N}{\partial x}(0) &= \frac{\partial \phi_N}{\partial x}(0) \left(A + \sum_{i=1}^p g_i(0) \left(\frac{\partial \alpha_i}{\partial x}(0) - k_i^T \frac{\partial \phi_N}{\partial x}(0) \right) \right) \\ & \quad - (A - BK) \frac{\partial \phi_N}{\partial x}(0) = 0 \end{aligned} \quad (38)$$

which is satisfied for $\frac{\partial \phi_N}{\partial x}(0) = I$ and $\frac{\partial \alpha_i}{\partial x}(0) = 0^T$, $i = 1, 2, \dots, p$. Consequently, the change of coordinates (13) and the feedback (14) have to satisfy

$$\phi_N(0) = 0, \quad \frac{\partial \phi_N}{\partial x}(0) = I \quad (39)$$

$$\alpha(0) = 0, \quad \frac{\partial \alpha}{\partial x}(0) = 0 \quad (40)$$

in addition to (27). The latter condition in (39) also assures that ϕ_N is a change of coordinates in a neighbourhood of $x = 0$. By using

$$\frac{\partial \Phi(x)}{\partial x_\nu} = D_\nu \Phi(x), \quad \nu = 1, 2, \dots, n \quad (41)$$

where D_ν is the operational matrix for differentiation of multivariable Legendre polynomials the conditions (39)–(40) can be represented by the linear matrix equations

$$T_\Phi L = [0 \ I] \quad (42)$$

$$\alpha_N L = 0 \quad (43)$$

with

$$L = [\Phi(0) \ D_1 \Phi(0) \ \dots \ D_n \Phi(0)] \quad (44)$$

3.3 Solution of the matrix equation

The matrix equation (27) with auxiliary conditions to be solved reads

$$T_\Phi (A_\Phi + \sum_{i=1}^p (N_{i,\Phi} \otimes (\alpha_{i,\Phi}^T - k_i^T T_\Phi)) M) - (A - BK) T_\Phi = 0 \quad (45)$$

$$T_\Phi L = [0 \ I] \quad (46)$$

$$\alpha_N L = 0 \quad (47)$$

A simple dimension count shows that the set of equations (45)–(47) contains $n \binom{n+N}{n} + p \binom{n+N}{n}$ unknowns and consists of $n \binom{n+N}{n} + (n+p)(n+1)$ equations such that for sufficiently large approximation degree N an undetermined set of equations results. This also demonstrates that the additional feedback α in (3) is needed since otherwise an overdetermined set of equations is always obtained. Since the solutions T_Φ and α_N have to satisfy the conditions (39)–(40) even if only an approximate solution of (45)–(47) exists the general solutions of (46) and (47) are determined. Condition (46) can always be solved for T_Φ since

it is easy to verify that $\text{rank } L = n + 1$. The corresponding solution has the general form

$$T_\Phi = [0 \ I] L^+ + \bar{T}_\Phi L^\perp \quad (48)$$

where \bar{T}_Φ represents an arbitrary matrix, L^+ denotes the Moore-Penrose inverse (see e.g. Ben-Israel and Greville (2003)) and L^\perp is the left annihilator of L , i.e.

$$L^\perp L = 0 \quad (49)$$

where L^\perp has full row rank. The general solution of (47) has the form

$$\alpha_N = \bar{\alpha}_N L^\perp \quad (50)$$

which exists if the $n_\Phi \times (n+1)$ matrix L has more rows than columns, i.e. $\binom{n+N}{n} > n + 1$. Obviously, this can always be achieved if N is sufficiently large. The matrices \bar{T}_Φ and $\bar{\alpha}_N$ in (48) and (50) parameterize all changes of coordinates (13) and feedbacks (14) that fulfil the conditions (46)–(47). By inserting (48) and (50) in (45) one obtains the nonlinear matrix equation

$$\begin{aligned} & ([0 \ I] L^+ + \bar{T}_\Phi L^\perp) (A_\Phi \\ & + \sum_{i=1}^p (N_{i,\Phi} \otimes (\bar{\alpha}_{i,\Phi}^T L^\perp - k_i^T ([0 \ I] L^+ + \bar{T}_\Phi L^\perp)) M) \\ & - (A - BK) ([0 \ I] L^+ + \bar{T}_\Phi L^\perp) = 0 \end{aligned} \quad (51)$$

where $\bar{\alpha}_{i,\Phi}^T$ are the rows of $\bar{\alpha}_N$. A solution \bar{T}_Φ and $\bar{\alpha}_{i,\Phi}^T$, $i = 1, 2, \dots, p$, of (51) can be determined with the aid of numerical software packages. In this paper the routine `fsolve` in MATLAB is used to compute \bar{T}_Φ and $\bar{\alpha}_{i,\Phi}^T$, $i = 1, 2, \dots, p$. If for a given approximation degree N and approximation interval \bar{I} (see Section 3.1) a solution of (51) does not exist then one can try to find a solution by increasing the approximation degree N . This is due to the fact that the number of unknowns increases faster than the number of equations such that the number of degrees of freedom becomes larger which can be used for solving (51). Furthermore, if the conditions of Theorem 1 are satisfied then there exists a local solution of the initial value problem (6)–(7). Thus, the existence of a solution may be achieved by reducing the size of the approximation interval \bar{I} . These guidelines were verified by means of numerous examples where always a solution of (51) was found.

4. EXAMPLE

Consider the nonlinear single-input system

$$\dot{x}_1 = x_2 + x_3^2 \quad (52)$$

$$\dot{x}_2 = x_3 \quad (53)$$

$$\dot{x}_3 = u \quad (54)$$

in extended controller normal form (see Kang (1996)) which has a controllable Jacobian linearization at $(x, u) = (0, 0)$. In Kang (1996) it is shown that this system is not feedback linearizable by using a transformation into controller normal form. However, by the eigenvalue assignment $\tilde{\lambda}_i = -1$, $i = 1, 2, 3$, the condition (8) of Theorem 1 is satisfied such that there locally exists a feedback transformation (2)–(3) into the linear system (4). In the sequel, the feedback controllers resulting from solving the initial value problem (6)–(7) by using the Taylor series

method are compared with the Galerkin approach. To this end, the initial condition $x(0) = [3 \ 0.1 \ 0.1]^T$ is considered for the system which lies relatively far away from the origin. The degrees of freedom in α (see (3)) when applying the Taylor series method are used to keep the transformation ϕ in (2) as close as possible to its globally invertible part x . Thus, one can expect that the size of the neighbourhood where ϕ is a change of coordinates is increased. As approximation interval for the state feedback design using multivariable Legendre polynomials $\bar{I} = [-3.7 \ 3.7] \times [-1.4 \ 1.4] \times [-1.2 \ 1.2]$ is chosen. It can be verified that in all simulations the state trajectories of (52)–(54) remain in this interval for all $t \geq 0$. By using an affine linear transformation this interval is transformed to the unity interval such that the results of the paper are applicable. The L_2 -approximations (19)–(21) and the computation of the operational matrices of multiplication and differentiation are implemented in MATLAB. The change of coordinates ϕ and the feedback α are computed by numerically solving (51) with the MATLAB command `fsolve`. Figure 1 and 2 show the state response of the

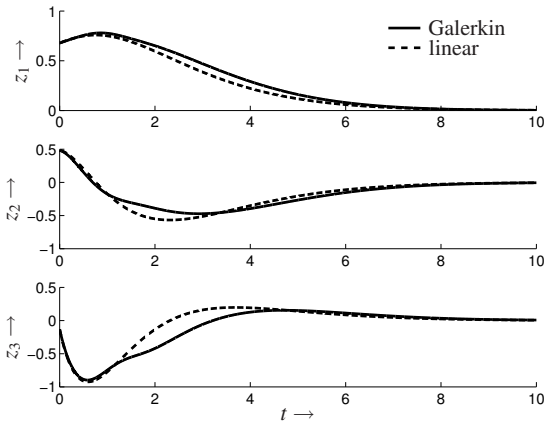


Fig. 1. State response in the z -coordinates for the Galerkin approach (—) with the approximation degree $N = 3$ and exactly linear dynamics (---).

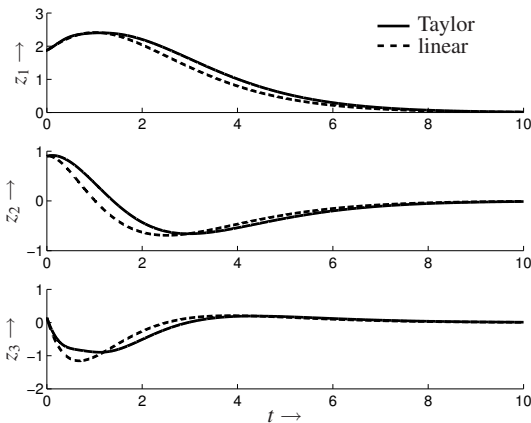


Fig. 2. State response in the z -coordinates for the Taylor series method (—) with the approximation degree $N = 3$ and exactly linear dynamics (---).

nonlinear closed loop system when using the Galerkin approach and the Taylor series method for the approximation degree $N = 3$. For the former approach a solution

Table 1. L_2 -norm of the equation error r_N for different approximation degrees

	$N = 3$	$N = 4$	$N = 5$
$\ r_N\ _2$	0.51679	0.2091	0.090578

of (51) is found with maximum absolute equation error equal to $4.2315 \cdot 10^{-9}$. In both cases the simulations are compared with the exactly linear dynamics (4) to quantify the achieved approximate linearization. In order to improve the linearization results the approximation degree is increased to $N = 5$. Table 1 indicates that the L_2 -norm of the equation error r_N becomes smaller if N is increased. Thus, one expects that the linearization result for the Galerkin approach is improved for larger N . For $N = 5$ a solution of (51) with maximum absolute equation error equal to $7.8733 \cdot 10^{-9}$ is determined. The corresponding

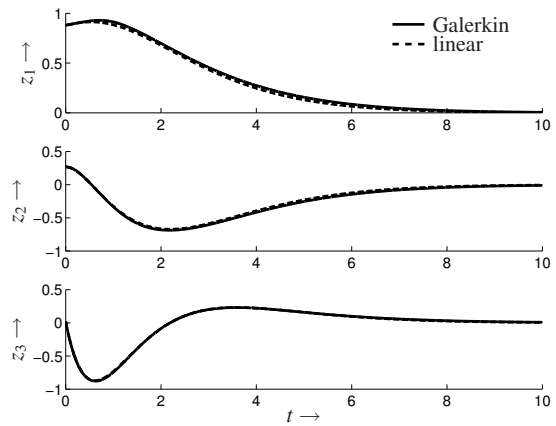


Fig. 3. State response in the z -coordinates for the Galerkin approach (—) with the approximation degree $N = 5$ and exactly linear dynamics (---).

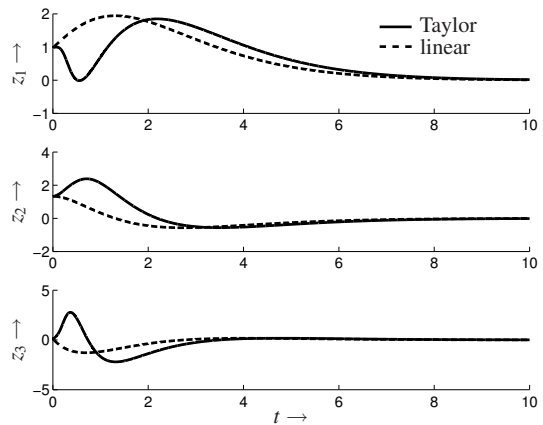


Fig. 4. State response in the z -coordinates for the Taylor series method (—) with the approximation degree $N = 5$ and exactly linear dynamics (---).

simulation is shown in Figure 3 where the linearization error almost vanishes. Consequently, the Galerkin approach achieves a convergent linearization uniformly in the chosen approximation interval. This result is compared to the Taylor series method for $N = 5$ depicted in Figure 4. Obviously, the Taylor series methods does not achieve an

acceptable linearization since this approach only converges in a small neighbourhood about the origin.

5. CONCLUSIONS

This paper considers the state feedback linearization approach on the basis of Lyapunov's auxiliary theorem. By computing an approximate solution of the corresponding initial value problem using the Galerkin approach the dynamics in the new coordinates can be approximately linearized uniformly on a specified interval in the state space. Thus, the proposed design technique circumvents the drawbacks of the previous solution method using a Taylor series approach. Furthermore, the performance of the approximate feedback linearization is quantified by the L_2 -norm of the remaining nonlinearity which is reduced in the design. The corresponding design problem can be solved by computing the solution of a matrix equation which can easily be calculated by a numerical software package. Products of Legendre polynomials in one variable are employed as multivariable orthonormal polynomials for implementing the Galerkin approach. Consequently, existing algorithms and results for Legendre polynomials in one variable can be used for computing the multivariable Legendre polynomials as well as the operational matrices for multiplication and differentiation of polynomials enabling an efficient numerical feedback linearization.

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