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Absolutely Stable Region for Missile Guidance Loop

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Abstract: In this paper, the stable region for missile guidance loop employing an integrated proportional navigation guidance law is derived. The missile guidance loop is formulated as a closed-loop control system consisting of a linear time-invariant feed-forward block and a time-varying feedback gain. By applying the circle criterion to the system, a bound for the time of flight up to which stability can be assured is established as functions of flight time. Less conservative results, as compared to the result by Popov criterion, are obtained.

1. INTRODUCTION

A missile guidance loop is composed of very completed high-order feedback system including digital autopilot, actuator, thrust motor or engine, vehicle dynamics, sensors, and guidance block. Because it has nonlinear characteristics with lots of limiters and generates time-varying guidance command by using output of sensors, a missile guidance loop could be considered as a high-order control system with nonlinear time-varying characteristics. Because finding a systematic and accurate stability analysis method for a general nonlinear time-varying system is very difficult, many works have been focused on stability analysis for nonlinear systems fallen into a special category. As a result of this approach, Lur'e problem (see Mohler, 1991), i.e., absolute stability problem has been studied by many people. The feedback system in Lur'e problem composed a closed-loop including a linear time-invariant block and a nonlinear time-varying block, its input is assumed zero and the stability problem of the system becomes a regulator problem. Fortunately, the missile guidance loop considered here can be transformed into the system of Lur'e problem.

Lur'e problem was originally suggested by assuming a linear time-invariant block and one non-linear block holding time-invariant characteristics and then Lur'e-Postnikov Criterion and Popov criterion had been appeared (see Aizerman *et al.*, 1964, and Vidyasagar, 1978). After that, many studies for stability of Lur'e type system with more expanded nonlinear elements were performed. Thus as the stability analysis method for Lur'e type system including a nonlinear element which has time-varying property and is in a general region, the circle criterion had been suggested, and a paper suggesting the sufficient conditions for unstable parameter region was reported (see Brocket *et al.*, 1967).

Examples which applied the absolute stability analysis methods to missile guidance loop were shown in many references. In Tanaka *et al.*, 1990a, b, the concept of hyperstable range was introduced and derived the stable time region for missile guidance loop using proportional navigation guidance (PNG) command. Guelman, 1990 investigated the finite time absolute stability of PNG systems by employing the Kalman-Yakubovitch-Popov lemma. Gurfil *et al.*, 1998, by using the well-known circle criterion (see Curran, 1993a, b), established an analytic bound for the time of flight up to which stability can be assured. Rew *et al.*, 1996 applied practical stability methods to derive a lower bound on the time-to-go for a PNG system with a single time constant dynamics. As the works for guidance stability of missile with model uncertainties, Impram *et al.*, 2001 suggested the robust circle criterion and the robust Popov criterion that had been extended to systems involving both structured and unstructured uncertainties in the linear plant. And Weiss *et al.*, 2004 studied the stability of modern guidance laws when the missile's actual model differs from the model used in the design, which analysis was performed by means of Lyapunov functions and by means of the multivariable circle criterion.

This paper derives the stable time region for missile guidance system using the integrated proportional navigation guidance (IPNG) law which is an integral form of PNG law. Using the circle criterion, we can establish a lower bound on the system's time-to-go before divergence for a given flight time.

This paper is organized as follows. A mathematical model of IPNG loop is described in Section 2. Section 3 suggests the stable regions be obtained by applying different criteria to the system. In Section 4 an illustrative example is considered, and in Section 5 concluding remarks are given.

2. MISSILE GUIDANCE LOOP

Consider a two-dimensional missile-target engagement geometry as shown in Fig. 1, where the missile M with velocity V_M and the target T with velocity V_T are treated as point masses, R is the relative range between missile and target, θ is line of sight (LOS) angle with respect to a reference line, and θ_M and θ_T are flight-path angles of the missile and the target, respectively.

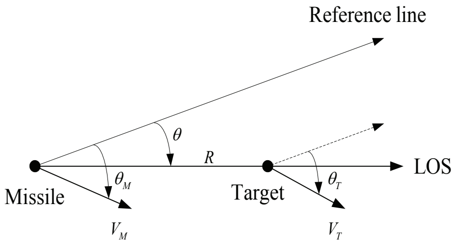

Fig. 1. Missile-target engagement geometry

The kinematic relation between missile and target motions is obtained by resolving velocity components of the missile and the target along and normal to the LOS.

$$\dot{R} = V_T \cos(\theta_T - \theta) - V_M \cos(\theta_M - \theta) \triangleq -V_C , \quad (1)$$

$$R\dot{\theta} = V_T \sin(\theta_T - \theta) - V_M \sin(\theta_M - \theta) \quad (2)$$

where V_C represents the closing velocity.

Assuming that $\theta_M \cong \theta$, $V_M \gg V_T$ and $R(t_f) = 0$ where t_f denotes the total flight time of the engagement, we obtain the expression for the line of sight rate to be

$$\dot{\theta} = \frac{1}{t_{go}}(\theta - \theta_M) \quad (3)$$

where $t_{go} \triangleq t_f - t$ is the time-to-go for the missile to intercept the target.

The IPNG law for a missile having an autopilot of attitude angle controller type is defined as

$$\theta_C = N\theta \quad (4)$$

where N is called the navigation constant.

The missile/autopilot (M/AP) dynamics can be expressed generally in the following transfer function form assuming a linear missile dynamics

$$G(s) = \frac{\theta_M(s)}{\theta_C(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \quad (5)$$

where we assume that $\theta_M(s)$ and $\theta_C(s)$ are coprime and $G(0) = 1$, i.e., $b_0 = a_0$.

The block diagram for the linearized guidance loop is shown in Fig. 2 and note that the guidance loop is a linear system with time-varying property and $1/t_{go}$ increases very fast as t approaches t_f , and so we can not perform the stability analysis by considering this system as a linear time-invariant system or a slowly time-varying system.

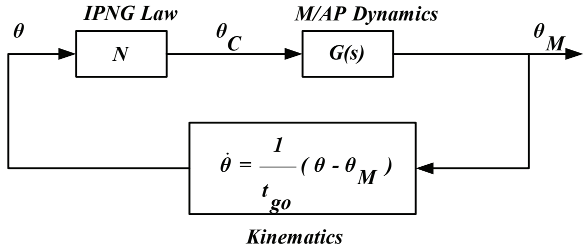

Fig. 2. IPNG loop block diagram

To obtain the stable time region through the absolute stability approach, we have to transform above guidance loop into the form of Lur'e problem. The transfer function $G(s)$ can be realized in the phase-variable canonical form with no loss in generality

$$\dot{x} = Ax + b\theta_C , \quad (6)$$

$$\theta_M = c^T x \quad (7)$$

where $G(s) = c^T(sI - A)^{-1}b$ and x is the system state and A , b and c^T are defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdot & \cdot & -a_{n-1} \end{bmatrix}, \quad (8)$$

$$b = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad (9)$$

$$c^T = \begin{bmatrix} b_0 & \cdot & \cdot & \cdot & b_{n-1} \end{bmatrix} \quad (10)$$

Let us define a new state vector z as

$$z = \dot{x} \quad (11)$$

Then x is rewritten in terms of z and θ_C as

$$x = A^{-1}z - A^{-1}b\theta_C \quad (12)$$

The substitution Eq. (12) in Eq. (6) yields

$$\dot{z} = Az + b\dot{\theta}_C \quad (13)$$

Defining $\varepsilon \triangleq \dot{\theta}_C$ and using Eqs. (3), (4), (6), and (7), we obtain

$$\varepsilon = -\frac{Nc^T A^{-1}z - (Nc^T A^{-1}b + 1)\theta_C}{t_{go}} \quad (14)$$

Defining $\sigma \triangleq Nc^T A^{-1} z - (Nc^T A^{-1} b + 1)\theta_C$ and we rewrite Eqs. (13) and (14) as

$$\dot{z} = Az + b\varepsilon, \tag{15}$$

$$\dot{\theta}_C = \varepsilon, \tag{16}$$

$$\sigma = h^T z + \Gamma \theta_C \tag{17}$$

$$\varepsilon = -\frac{1}{t_{go}} \sigma, \tag{18}$$

where

$$h^T \triangleq N c^T A^{-1}, \tag{19}$$

$$\Gamma \triangleq -(N c^T A^{-1} b + 1) \tag{20}$$

The Lur'e type closed-loop configuration for the linearized two-dimensional IPNG system is shown in Fig. 3. This linear time-varying system consists of a linear time-invariant element in the forward path and a time-varying gain in the feedback. The linear time invariant portion is

$$H(s) = \frac{\sigma(s)}{\varepsilon(s)} = h^T (sI - A)^{-1} b + \frac{\Gamma}{s} \tag{21}$$

and the feedback is the kinematic gain $1/t_{go}$.

From the inverse matrix form of A and $G(0) = 1$, we can easily show that

$$\Gamma = N - 1 \tag{22}$$

and

$$H(s) = \frac{1}{s}(N\,G(s) - 1) \tag{23}$$

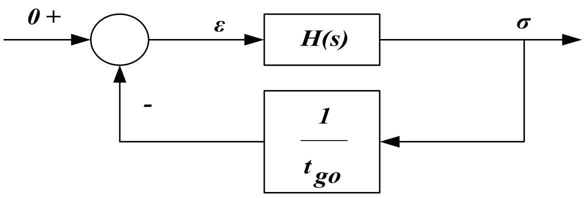

Fig. 3. Lur'e type IPNG loop block diagram

3. STABLE REGIONS

The IPNG loop system expressed in Fig. 3 falls into categories of Lur'e problem and its stability information could be obtained through a well-known absolute stability theory.

First, to find the conditions under which the system remains stable, we suppose that the M/AP dynamics is modeled as a simple first-order system

$$G(s) = \frac{1}{Ts + 1} \tag{24}$$

where T is a time-constant of M/AP dynamics. Then the linear time-invariant portion $H(s)$ of Eq. (23) is expressed as follows

$$H(s) = \frac{(N-1) - Ts}{s(Ts + 1)} \tag{25}$$

3.1 Stable Region by Frozen System Analysis

We first perform the stability analysis with the time-varying gain t_{go} frozen at a value τ. Hence, the closed-loop system becomes a linear time-invariant system and has the characteristic equation as follows

$$\begin{aligned} &1 + \{(N-1) - Ts\}/\{\tau s(Ts+1)\} = 0 \\ &or \quad \tau T s^2 + (\tau - T)s + (N-1) = 0 \end{aligned} \tag{26}$$

To obtain the stable region for τ, we apply well-known Routh-Hurwitz criterion to the characteristic equation as

$$\begin{array}{lll} s^2 & \tau T & N-1 \\ s^1 & \tau - T & 0 \\ s^0 & N-1 & \end{array} \tag{27}$$

From the above Routh table, the necessary conditions for the system to be frozen time stable are

$$i) \quad N > 1, \tag{28}$$

$$ii) \quad \tau > T \quad or \quad SR_H = (T, \infty) \tag{29}$$

where SR_H denotes the stable region by frozen system analysis, is so-called Hurwitz region for system stability and these conditions are only the necessary conditions for system stability.

3.2 Stable Region by Popov Criterion

The frequency response for the transfer function $H(s)$ of the linear time-invariant element is

$$X(\omega) \triangleq R_e H(j\omega) = \frac{-NT}{T^2\omega^2 + 1}, \tag{30}$$

$$Y(\omega) \triangleq I_m H(j\omega) = \frac{T^2\omega^2 - (N-1)}{\omega(T^2\omega^2 + 1)} \tag{31}$$

As the particular case in which the linear time-invariant element has one pole at origin, the conditions for stability-in-the-limit (see Aizerman *et al.*, 1964) required in Popov criterion must be satisfied. For this case to be stable-in-the-limit, it is necessary and sufficient that the following condition be satisfied

$$\lim_{\omega\to+0} I_m H(j\omega) = \lim_{\omega\to+0} Y(\omega) = -\infty \tag{32}$$

Consequently, to satisfy the condition of Eq. (32) it is required that $N > 1$.

Assuming the navigation constant N which is greater than 1, Popov condition is obtained from Popov criterion of the case of time-varying feedback gain as follows

$$X(\omega) + t_{go} > 0 \quad for\ all\ \omega \ge 0 \tag{33}$$

The graph of $H(j\omega)$ is shown in Fig. 4, Popov region satisfying Popov condition of Eq. (33) is $NT < t_{go} < \infty$ and the guidance loop in this Popov region is said to be absolutely stable. In summary, assuming $N > 1$, the absolutely stable region for guidance loop with 1st-order M/AP dynamics is as follows

$$t_{go} > NT \quad or \quad SR_P = (NT, \infty) \tag{34}$$

where the stable region denoted by SR_P is said to be Popov region.

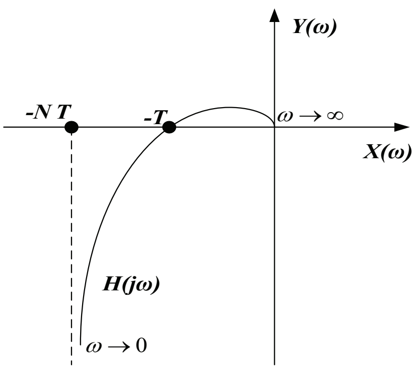

Fig. 4. Popov criterion for $H(j\omega)$

3.3 Stable Region by Circle Criterion

Although t_{go} constrained in the absolutely stable region $NT < t_{go} < \infty$ obtained by using Popov criterion is extended to ∞, the initial value of t_{go} has actually one finite value since missile guidance loop is only defined over a finite time interval. When the maximum value of t_{go} has a positive value α but not ∞, to obtain the minimum of t_{go}, that is, β we apply the circle criterion of the case of time-varying feedback gain. Since $H(s)$ has one pole on the right plane including the imaginary axis, the locus of $H(j\omega)$ has to encircle the critical circle only once in the counter-clockwise direction with increasing frequency where we assume that each pole on the imaginary axis is bypassed by an infinitesimal semicircle to the left of the pole. This is shown in Fig. 5, the locus of $H(j\omega)$ is tangential on the critical circle and the absolutely stable time region becomes

$$\beta < t_{go} \le \alpha \quad or \quad SR_C = (\beta, \alpha] \tag{35}$$

where the stable region obtained by circle criterion is expressed as SR_C.

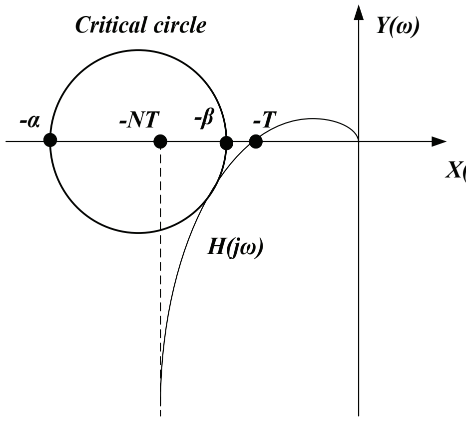

Fig. 5. Illustration of Eq. (35) by circle criterion

Hence β can be calculated in terms of α, T, and N as follows. From Eqs. (30) and (31), we obtain

$$Y^2 = -\{X(X+T)^2\}/(NT+X) \tag{36}$$

The equation of the critical circle which is tangential to $H(j\omega)$ locus and passes through the points $-\alpha$ and $-\beta$ is given by

$$\{X + (\alpha+\beta)/2\}^2 + Y^2 = \{(\alpha-\beta)/2\}^2 \tag{37}$$

Cancelling Y from Eqs. (36) and (37) gives

$$\begin{aligned}&\{(N-2)T + (\alpha+\beta)\}X^2 + \{NT(\alpha+\beta) + \alpha\beta - T^2\}X \\ &+ NT\alpha\beta = 0\end{aligned} \tag{38}$$

In this case, X has equal roots and its discriminant becomes zero as follows.

$$\begin{aligned}&\{NT(\alpha+\beta) + \alpha\beta - T^2\}^2 \\ &-4\{(N-2)T + (\alpha+\beta)\}NT\alpha\beta = 0\end{aligned} \tag{39}$$

From Eq. (39), we obtain a quadratic equation in β

$$\begin{aligned}&(NT-\alpha)^2\beta^2 - 2T\{NT^2 + (N^2-4N+1)T\alpha + N\alpha^2\}\beta \\ &+ T^2(N\alpha - T)^2 = 0\end{aligned} \tag{40}$$

From Eq. (40), the positive β is obtained as follows

$$\begin{aligned}&\beta = \frac{T(\Lambda + \Xi)}{(NT-\alpha)^2} \\ &\Lambda \triangleq NT^2 + (N^2-4N+1)T\alpha + N\alpha^2 \\ &\Xi \triangleq -2(N-1)(\alpha - T)\sqrt{NT\alpha}\end{aligned} \tag{41}$$

Through Eq. (41), we obtained the algebraic equation of β for α and notice also that in the limiting case $\alpha \to \infty$, $\beta \to NT$, so the result of Popov criterion has been obtained. Fig. 6 shows plot of β as function of α from T to $10T$.

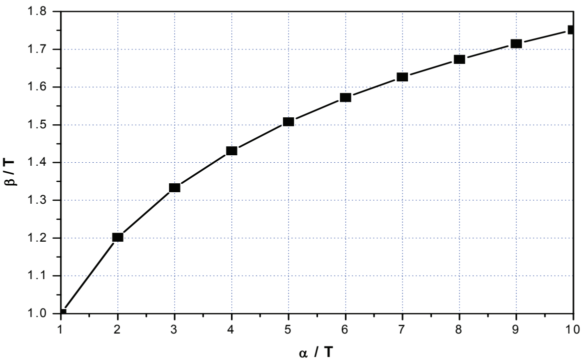

Fig. 6. Plot of β as function of flight time α

5. EXAMPLE

Assuming $N = 3$, $T = 2\sec$, and $\theta_M(0) = -90^O$, Fig. 7-10 show plots of state variables θ, θ_M, $\dot{\theta}_M$, and output σ for $t_{go}(0) = 10T = 20\sec$.

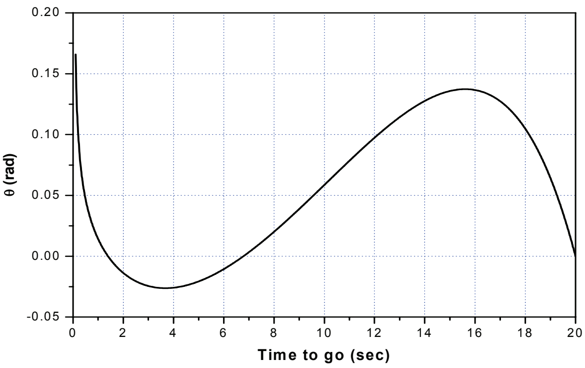

Fig. 7. Time-to-go history of θ

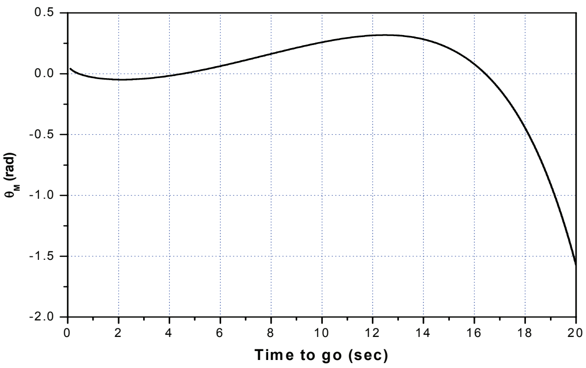

Fig. 8. Time-to-go history of θ_M

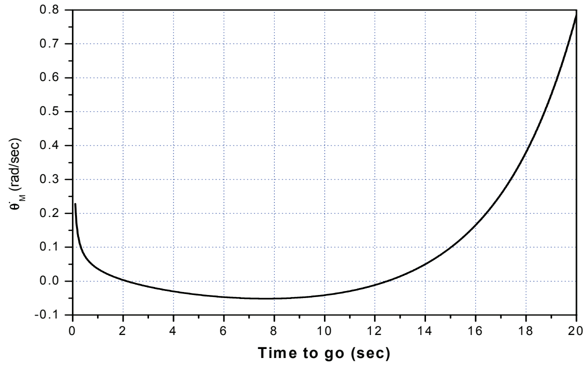

Fig. 9. Time-to-go history of $\dot{\theta}_M$

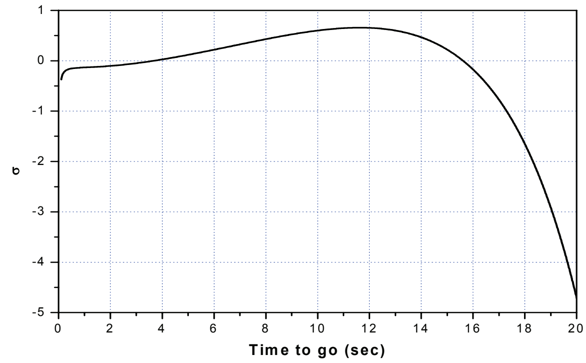

Fig. 10. Time-to-go history of σ

It can be seen that the state variables $\dot{\theta}_M$ and θ of linear time-invariant element tend to increase from $t_{go} = 1.5 \sim 2\ s$ as $t_{go} \to 0$. For $t_{go}(0) = 10T = 20\ s$, the divergence time suggested in Fig. 6 is $1.75T = 1.75 * 2 = 3.5\ s$, which is rather different from the result of simulation. But it is necessary to recall that the result of Fig. 6 is only one sufficient condition for stability. Also note that the lower bound by Popov criterion is $NT = 3 * 2 = 6\ s$.

6. CONCLUSIONS

The circle criterion, one of absolute stability analysis methods was used to obtain the stable region of a missile guidance loop based on an integrated proportional navigation guidance law. This method provides sufficient conditions for the stability of the closed loop system. The lower bounds on the system's time-to-go before divergence were established as functions of flight time. These bounds are less conservative than the bound obtained from the Popov criterion.

REFERENCES

Aizerman, M. A. and F. R. Gantmacher (1964). *Absolute stability of regulator systems*, Holden-day, Inc., San Francisco.

Brocket, R. W., and J. L. Willems (1967). Frequency domain instability criteria for time-varying and nonlinear systems. *Proceedings of the IEEE.*

Curran, P. F. (1993). Proof of the circle criterion for state space systems via quadratic Lyapunov functions---part 1. *International journal of control,* **Vol. 57, No. 4,** 921-955.

Curran, P. F. (1993). Proof of the circle criterion for state space systems via quadratic Lyapunov functions---part 2. *International journal of control,* **Vol. 57, No. 4,** 957-969.

Guelman, M. (1990). The stability of proportional navigation systems. *AIAA paper 90-3380.*

Gurfil, P., M. Jodorkovsky and M. Guelman (1998). Finite time stability approach to proportional navigation systems analysis. *Journal of guidance, control and dynamics,* **Vol. 21, No. 6,** 853-861.

Impram, S. T. and N. Munro (2001). A note on absolute stability of uncertain systems. *Automatica,* **Vol. 37, Issue 4,** 605-610.

Mohler, H. R. (1991). *Nonlinear systems: vol. 1, dynamics and control,* Prentice-hall, Inc. Englewood cliffs, NJ.

Rew, D. Y., M. J. Tahk and H. Cho (1996). Short time stability of proportional navigation guidance loop. *IEEE transactions on aerospace and electronic systems,* **Vol. 32, No. 4,** 1107-1115.

Tanaka, T. and E. Hirofumi (1990). An extended guidance loop and the stability of the homing missiles. *Proceedings of the 27th JSASS aircraft symposium.*

Tanaka, T. and E. Hirofumi (1990). Hyperstable range in homing missiles. *AIAA paper 90-3381.*

Vidyasagar, M. (1978). *Nonlinear systems analysis,* Prentice-hall, Inc. Englewood cliffs, NJ.

Weiss, H. and G. Hexner (2004). Stability of modern guidance laws with model mismatch. *Proceedings of the 2004 American control conference,* Boston, Massachusetts, 3634-3639.