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High-Gain Observers in the Presence of Measurement Noise: A Switched-Gain Approach ⋆

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Abstract:
This paper considers output feedback control using high-gain observers in the presence of measurement noise for a class of nonlinear systems. We study stability in the presence of measurement noise and illustrate the tradeoff when selecting the observer gain between state reconstruction speed and robustness to model uncertainty on the one hand versus amplification of noise on the other. Based on this tradeoff we propose a high-gain observer that switches between two gain values. This scheme is able to quickly recover the system states during large estimation error and reduce the effect of measurement noise in a neighborhood of the origin of the estimation error. We argue boundedness and ultimate boundedness of the closed-loop system under switched-gain output feedback.

1. INTRODUCTION

It is well known from observer theory (Kwakernaak and Sivan [1972]) that a tradeoff exists between the speed of state reconstruction and the immunity to measurement noise. The high-gain observer (HGO) is known for having the ability to quickly reconstruct the system states and reject modeling disturbances (see Esfandiari and Khalil [1992]). In this paper we study output feedback control using high-gain observers in the presence of measurement noise for a class of nonlinear systems. We explore the trade-off between fast reconstruction of the states and rejection of modeling error versus the immunity to measurement noise. Based on this, we introduce a high-gain observer design where the gain matrix is switched between two values. The idea is to use high gain during the transient to quickly recover the state estimates. Then once the estimation error has reached a steady-state threshold, we switch to a second gain to reduce the effect of measurement noise. Observer designs that employ switching schemes can be found in Mayne et al. [1997] and Elbeheiry and Elmaraghy [2003]. An estimator with continuous gain transition is presented in Tilli and Montanari [2001]. The switched-gain scheme proposed in Section 3 uses high gain during the transient period followed by switching to a low gain. The switching event takes place when the output estimation error reaches a predetermined zone containing the origin. Due to the observer transient response, the design contains a few special features. First, the observer eigenvalues are assigned to ensure that the output estimation error decays monotonically towards the switching zone and reaches it in finite time. Second, a delay time is incorporated into the scheme that delays switching till after the observer transient period, in order to prevent multiple gain switchings. Third, to avoid peaking after the switching event takes place, the ratio of the two gains is restricted.

We begin in the next section by quantifying the tradeoffs associated with using a high-gain observer in the presence of bounded measurement noise. We study the impact of the noise on the closed-loop stability by showing boundedness and ultimate boundedness, where the size of the ultimate bound is limited by the magnitude of the noise. Also, we examine closeness of trajectories under output feedback to the ones under state feedback. Previous results for high-gain observers in the presence of measurement noise and disturbances can be found in Ahrens and Khalil [2004], Atassi [1999], Dabroom and Khalil [1999], [Atassi, 1999, Chapter 4], and Vasiljevic and Khalil [2006]. In Section 3 we introduce the switched-gain high-gain observer design. In Section 4 we provide a numerical example to illustrate the switched-gain observer performance. Section 5 contains the concluding remarks.

2. PERFORMANCE RECOVERY IN THE PRESENCE OF MEASUREMENT NOISE

Consider the nonlinear system

$$\dot{x} = Ax + B\phi(x, z, d, u) \tag{1}$$

$$\dot{z} = \psi(x, z, d, u) \tag{2}$$

$$y = Cx + v \tag{3}$$

$$w = \Theta(x, z, d) \tag{4}$$

where $u \in \mathbb{R}$ is the control input, $x \in \mathbb{R}^r$ and $z \in \mathbb{R}^\ell$ are the states, $y \in \mathbb{R}$ and $w \in \mathbb{R}^s$ are the measured outputs,

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$d(t) \in \mathbb{R}^p$ is a vector of exogenous signals, and $v(t) \in \mathbb{R}$ is measurement noise. The $r \times r$ matrix A, the $r \times 1$ matrix B, and the $1 \times r$ matrix C are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\, 1 \;\, 0 \, \cdots \cdots 0 \,]$$

Assumption 1.

(1) $d(t)$ is continuously differentiable, both $d(t)$ and $\dot{d}(t)$ are bounded, and $d(t) \in \mathcal{D}$ (a compact subset of $\mathbb{R}^p$);
(2) $v(t)$ is measurable and bounded, with $|v(t)| \leq \mu$;
(3) ϕ, ψ, and Θ are locally Lipschitz functions in x, z, and u, uniformly in d, over the domain of interest; that is, for each compact subset of (x, z, u) in the domain of interest, the function ϕ, ψ, or Θ satisfies the Lipschitz inequality with a Lipschitz constant independent of d for all $d \in \mathcal{D}$

The state feedback controller takes the form

$$\dot{\vartheta} = \Gamma(\vartheta, x, w, d) \tag{5}$$

$$u = \gamma(\vartheta, x, w, d) \tag{6}$$

and the closed-loop system under (5)–(6) is represented by

$$\dot{\chi} = f_r(\chi, d) \tag{7}$$

where $\chi = (x, z, \vartheta) \in \mathbb{R}^N$ and

$$f_r(\chi, d) = \begin{bmatrix} Ax + B\phi(x, z, d, \gamma(\vartheta, x, w, d)) \\ \psi(x, z, d, \gamma(\vartheta, x, w, d)) \\ \Gamma(\vartheta, x, w, d) \end{bmatrix}$$

Assumption 2.

(1) Γ and γ are locally Lipschitz functions in ϑ, x, and w, uniformly in d, over the domain of interest;
(2) Γ and γ are globally bounded functions of x;
(3) The closed-loop system (7) is uniformly asymptotically stable with respect to a compact positively invariant set $\mathcal{A}$, uniformly in d;
(4) $\phi(x, z, d, \gamma(\vartheta, x, w, d))$ is zero in $\mathcal{A}$, uniformly in d.

We work with the notion of uniform asymptotic stability with respect to a set as discussed in Atassi [1999] and Lin et al. [1996]. The set $\mathcal{A}$ takes different forms, depending on the problem formulation. For stabilization problems where the objective is to stabilize the origin $\chi = 0$, $\mathcal{A} = \{0\}$. For regulation or tracking problems where the objective is to asymptotically regulate y to zero, $\mathcal{A} = \{x = 0\} \times \{(z, \vartheta) \in \mathcal{B}\}$ for some compact set $\mathcal{B}$. For practical regulation or tracking problems, $\mathcal{A} = \{x \in \mathcal{U}\} \times \{(z, \vartheta) \in \mathcal{B}\}$ where the size of $\mathcal{U}$ is controlled by some design parameters.

The high-gain observer has the form

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, w, d, u) + H(y - C\hat{x}) \tag{8}$$

where the observer gain H is given by

$$H^T = \left[\frac{\alpha_1}{\varepsilon} \;\; \frac{\alpha_2}{\varepsilon^2} \;\; \cdots \;\; \frac{\alpha_r}{\varepsilon^r} \right] \tag{9}$$

ε is a small positive parameter, and the roots of

$$s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r = 0 \tag{10}$$

have negative real parts. The function $\phi_0(x, w, d, u)$ is a nominal model of $\phi(x, z, d, u)$, which satisfies the following assumption.

Assumption 3. ϕ_0 is locally Lipschitz in x, w, and u, uniformly in d, over the domain of interest, globally bounded in x, and zero in $\mathcal{A}$.

The output feedback controller is obtained by replacing x in (5)–(6) by $\hat{x}$. For the purpose of analysis, we replace the observer dynamics by the equivalent dynamics of the scaled estimation error

$$\eta_i = \varepsilon^{i-1}(x_i - \hat{x}_i) \tag{11}$$

for $i = 1, \ldots, r$. This scaling differs from the one used in previous work on high-gain observers; e.g. Atassi and Khalil [1999], due to the presence of measurement noise. With the scaling (11), we have $\hat{x} = x - D^{-1}(\varepsilon)\eta$, where $D(\varepsilon) = diag[1, \varepsilon, \cdots, \varepsilon^{r-1}]$. The closed-loop system under the output feedback controller can be written in the form

$$\dot{x} = Ax + B\phi(x, z, d, \gamma(\vartheta, x - D^{-1}(\varepsilon)\eta, w, d)) \tag{12}$$

$$\dot{z} = \psi(x, z, d, \gamma(\vartheta, x - D^{-1}(\varepsilon)\eta, w, d)) \tag{13}$$

$$\dot{\vartheta} = \Gamma(\vartheta, x - D^{-1}(\varepsilon)\eta, w, d) \tag{14}$$

$$\varepsilon\dot{\eta} = A_0\eta + \varepsilon^r B\tilde{\varphi}(x, z, \vartheta, D^{-1}(\varepsilon)\eta, d) + B_2 v \tag{15}$$

where $\tilde{\varphi}(x, z, \vartheta, D^{-1}(\varepsilon)\eta, d) = \phi(x, z, d, \gamma(\vartheta, \hat{x}, w, d)) - \phi_0(\hat{x}, w, d, \gamma(\vartheta, \hat{x}, w, d))$

$$A_0 = \begin{bmatrix} -\alpha_1 & 1 & \cdots & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -\alpha_{r-1} & \cdots & \cdots & 0 & 1 \\ -\alpha_r & 0 & \cdots & \cdots & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} -\alpha_1 \\ -\alpha_2 \\ \vdots \\ -\alpha_r \end{bmatrix}$$

The matrix A_0 is Hurwitz by design. Let $f(\chi, d, D^{-1}(\varepsilon)\eta)$ denote the right-hand side of (12)–(14), $g(\chi, d, D^{-1}(\varepsilon)\eta) = \tilde{\varphi}(x, z, \vartheta, D^{-1}(\varepsilon)\eta, d)$, and rewrite (12)–(15) as

$$\dot{\chi} = f(\chi, d, D^{-1}(\varepsilon)\eta) \tag{16}$$

$$\varepsilon\dot{\eta} = A_0\eta + \varepsilon^r Bg(\chi, d, D^{-1}(\varepsilon)\eta) + B_2 v \tag{17}$$

Equations (16)–(17) appear in the standard singularly perturbed form (Kokotović et al. [1986]), except for the presence of negative powers of ε in the term $D^{-1}(\varepsilon)\eta$. Notice, however, that the functions f and g are globally bounded functions in $D^{-1}(\varepsilon)\eta$ because they are globally bounded functions in $\hat{x}$ and the term $D^{-1}(\varepsilon)\eta$ results from substituting $x - D^{-1}(\varepsilon)\eta$ for $\hat{x}$. This property will enable us to extend to (16)–(17) behavior associated with standard singularly perturbed systems. With $\eta = 0$, (16) reduces to

$$\dot{\chi} = f(\chi, d, 0) = f_r(\chi, d) \tag{18}$$

which is the closed-loop system (7) under the state feedback controller (5)–(6). This system is uniformly asymptotically stable with respect to the compact positively invariant set $\mathcal{A}$. By a converse Lyapunov theorem [Atassi, 1999, Theorem 3.10], if $\mathcal{R}$ is an open connected subset of the region of attraction, which contains $\mathcal{A}$, then there is a smooth Lyapunov function $V(\chi)$ in $\mathcal{R}$ and three positive definite, with respect to $\mathcal{A}$, functions U_1, U_2, and U_3, all defined in $\mathcal{R}$, such that

$$V(\chi) = 0 \ \Leftrightarrow \chi \in \mathcal{A} \tag{19}$$

$$U_1(\chi) \leq V(\chi) \leq U_2(\chi) \tag{20}$$

$$\lim_{\chi \to \partial \mathcal{R}} U_1(\chi) = \infty \tag{21}$$

$$\frac{\partial V}{\partial \chi} f(\chi, d, 0) \leq -U_3(\chi), \quad \forall\, d \in \mathcal{D} \tag{22}$$

Theorem 4. Let Assumptions 1 to 3 hold and consider the closed-loop system (12)-(15). Let $\mathcal{M}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{N}$ be any compact subset of $\mathbb{R}^r$, and suppose that $\chi(t_0) \in \mathcal{M}$ and $\hat{x}(t_0) \in \mathcal{N}$. Then

- There exist positive constants c_a and μ^* such that for each $\mu < \mu^*$ there is a constant $\varepsilon_a = \varepsilon_a(\mu) > c_a \mu^{1/r}$, with $\lim_{\mu \to 0} \varepsilon_a(\mu) = \varepsilon_a^* > 0$, such that for each $\varepsilon \in (c_a \mu^{1/r}, \varepsilon_a]$ the trajectories of the closed-loop system are bounded for all $t \geq 0$;
- There exist $\mu_1^* > 0$ and a class $\mathcal{K}$ function ρ_1 such that for every $\mu < \mu_1^*$ and every $\xi_1 > \rho_1(\mu)$, there are constants $T_1 = T_1(\xi_1) \geq 0$ and $\varepsilon_b = \varepsilon_b(\mu, \xi_1) > c_a \mu^{1/r}$, with $\lim_{\mu \to 0} \varepsilon_b(\mu, \xi_1) = \varepsilon_b^*(\xi_1) > 0$, such that for each $\varepsilon \in (c_a \mu^{1/r}, \varepsilon_b]$ we have
$$\max\{|\chi(t)|_{\mathcal{A}}, \|x(t) - \hat{x}(t)\|\} \leq \xi_1, \qquad \forall\, t \geq T_1 \tag{23}$$
- There exist $\mu_2^* > 0$ and a class $\mathcal{K}$ function ρ_2 such that for every $\mu < \mu_2^*$ and every $\xi_2 > \rho_2(\mu)$, there is a constant $\varepsilon_c = \varepsilon_c(\mu, \xi_2) > c_a \mu^{1/r}$, with $\lim_{\mu \to 0} \varepsilon_c(\mu, \xi_2) = \varepsilon_c^*(\xi_2) > 0$, such that for each $\varepsilon \in (c_a \mu^{1/r}, \varepsilon_c]$ we have
$$\|\chi(t) - \chi_r(t)\| \leq \xi_2, \quad \forall\, t \geq 0 \tag{24}$$
where $\chi_r(t)$ is the solution of (7) with $\chi_r(t_0) = \chi(t_0)$.

Remark 5. We make the following remarks on Theorem 1:

(1) The three bullets of the theorem show, respectively, boundedness of all trajectories, ultimate boundedness where the trajectories come close to the set $\mathcal{A} \times \{x - \hat{x} = 0\}$ as time progresses, and closeness of the trajectories under output feedback to the ones under state feedback.
(2) Comparison of Theorem 4 with the corresponding results in Atassi [1999], Atassi and Khalil [1999, 2001], for the case without measurement noise, shows that the presence of measurement noise is manifested in three points, which are intuitively expected:
   - The amplitude of measurement noise μ is limited by the restriction $\mu < \mu^*$.
   - There is a lower bound on ε, which is of the order $O(\mu^{1/r})$.
   - The constants ξ_1 and ξ_2, which measure ultimate boundedness and closeness of trajectories, respectively, cannot be made arbitrarily small. Instead, they are bounded from below by class $\mathcal{K}$ functions of μ.

Due space limitations, the proof is left for the full paper Ahrens and Khalil [2008].

3. SWITCHED-GAIN OBSERVER

There exists a tradeoff in the choice of the observer parameter ε in the presence of measurement noise. It can be shown that the estimation error satisfies the ultimate bound

$$\|x(t) - \hat{x}(t)\| \leq \varepsilon c_1 + \frac{\mu}{\varepsilon^{r-1}} c_2 \triangleq F_r(\varepsilon, \mu) \tag{25}$$

for some positive constants c_1 and c_2. This inequality shows a tradeoff between the error due to model uncertainty, εc_1, and the error due to measurement noise, $\mu c_2 / \varepsilon^{(r-1)}$. This inequality puts a lower bound on ε of the order $O(\mu^{1/r})$. Hence, we cannot choose ε arbitrarily small. On the other hand, recovering the performance of the state feedback controller can be achieved by choosing ε small, for fast reconstruction of the state estimates. To relax this tradeoff, we propose a switched-gain observer. Switching is based on the output error $(y - \hat{x}_1)$ and a known upper bound μ on the measurement noise. The idea is to use a smaller value of ε when the output error is large. This will provide fast reconstruction of the state estimates at the expense of increased error due to measurement noise during the transient period. When the output error has reduced to a small value, we switch to a larger value of ε to achieve a better balance between the error due to model uncertainty and the error due to measurement noise. The switching criterion is based upon the output error reaching a particular zone. To avoid repeated switching, the observer gain should be designed such that the output error decays monotonically towards the switching zone and does not overshoot it. Considering that estimates of the higher order derivatives will exhibit peaking, we will have to exercise some care in determining when to switch. If we switch before the estimates of the higher order derivatives have recovered from peaking, it could drive the output error outside the switching zone. We define the switching zone as $\mathcal{I}_\delta = [-\delta, \delta]$ for some design parameter $\delta > 0$. We will discuss the choice of δ later on. We use the same observer as before:

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, w, d, u) + H(y - C\hat{x}) \tag{26}$$

but with the gain matrix H taken as

$$H^T = H_1^T = \begin{bmatrix} \frac{\alpha_1^1}{\varepsilon_1} & \frac{\alpha_2^1}{\varepsilon_1^2} & \cdots & \frac{\alpha_r^1}{\varepsilon_1^r} \end{bmatrix} \tag{27}$$

before switching and

$$H^T = H_2^T = \begin{bmatrix} \frac{\alpha_1^2}{\varepsilon_2} & \frac{\alpha_2^2}{\varepsilon_2^2} & \cdots & \frac{\alpha_r^2}{\varepsilon_2^r} \end{bmatrix} \tag{28}$$

after switching, where $0 < \varepsilon_1 < \varepsilon_2$. The constants α_i^j, $j = 1, 2$, and $i = 1, \cdots, r$, are chosen such that the roots of the corresponding polynomial (10) have negative real parts. The different sets of parameters, α_i^1's and α_i^2's allow for the flexibility of choosing the observer poles at different locations. In the analysis we will consider the closed-loop system under output feedback for two cases. For the case when the gain $H = H_2$ we use the same rescaling as before, $\eta_i = \varepsilon_2^{i-1}(x_i - \hat{x}_i)$. This will yield the same system of equations as (16)-(17) with ε replaced by ε_2. When the gain is given by H_1 we have, using the rescaling $\theta_i = \varepsilon_1^{i-1}(x_i - \hat{x}_i)$,

$$\dot{\chi} = f(\chi, d, D^{-1}(\varepsilon_1)\theta) \tag{29}$$

$$\varepsilon_1 \dot{\theta} = A_0 \theta + \varepsilon_1^r B g(\chi, d, D^{-1}(\varepsilon_1)\theta) + B_2 v \tag{30}$$

We will focus on (29)-(30) for the moment. We would like switching of ε to be based on detection of the output error entering the switching zone. We need to include a delay between the time $(y - \hat{x}_1)$ enters the switching zone $\mathcal{I}_\delta$ and the time the gain is switched. A delay timer will be initiated upon detection of $(y-\hat{x}_1)$ entering $\mathcal{I}_\delta$. However, the transient response of the observer may cause $(y-\hat{x}_1)$ to overshoot the switching zone. Our switching scheme will reset the delay timer whenever $(y-\hat{x}_1)$ exits the switching zone $\mathcal{I}_\delta$ and restart the timer upon re-entry of $(y-\hat{x}_1)$ into $\mathcal{I}_\delta$. Thus, overshoot of $\mathcal{I}_\delta$ may cause starting, resetting, and restarting of the delay timer. We can avoid this scenario by designing the observer poles so that $(y-\hat{x}_1)$ does not overshoot $\mathcal{I}_\delta$. To see this, write the observer polynomial (10) as

$$(s^{r-1}+\beta_1 s^{r-2}+\cdots+\beta_{r-2}s+\beta_{r-1})(s+\kappa)=0 \quad (31)$$

where the first polynomial is Hurwitz with $O(1)$ roots and $\kappa \gg 1$. With this choice of polynomial roots, the observer dynamics will exhibit a two-time scale behavior. It will have a fast component that corresponds to the pole located at $-\kappa$ and $(r-1)$ slow components that correspond to the roots of

$$s^{r-1}+\beta_1 s^{r-2}+\cdots+\beta_{r-2}s+\beta_{r-1}=0 \quad (32)$$

Hence, we can represent the estimation error in the singularly perturbed form. Toward that end, rewrite A_0 and B_2 in the following way:

$$A_0 = A_{01}\kappa + A_{02} \quad (33)$$

and

$$B_2 = B_{20}\kappa + B_{21} \quad (34)$$

where

$$A_{01}=\begin{bmatrix} -1 & 0 & \cdots\cdots & 0 \\ -\beta_1 & 0 & 0 \;\cdots & 0 \\ \vdots & & & \vdots \\ -\beta_{r-2} & \cdots\cdots & 0 & 0 \\ -\beta_{r-1} & 0 & \cdots\cdots & 0 \end{bmatrix}, B_{20}=\begin{bmatrix} -1 \\ -\beta_1 \\ \vdots \\ -\beta_{r-2} \\ -\beta_{r-1} \end{bmatrix}$$

and

$$A_{02}=\begin{bmatrix} -\beta_1 & 1 & \cdots\cdots & 0 \\ -\beta_2 & 0 & 1 \;\cdots & 0 \\ \vdots & & & \vdots \\ -\beta_{r-1} & \cdots\cdots & 0 & 1 \\ 0 & 0 & \cdots\cdots & 0 \end{bmatrix}, B_{21}=\begin{bmatrix} -\beta_1 \\ -\beta_2 \\ \vdots \\ -\beta_{r-1} \\ 0 \end{bmatrix}$$

To transform the system into the singularly perturbed form, we follow the procedure of [Kokotović et al., 1986, Section 1.6]. First, notice that the direct sum of the range and null spaces of A_{01} spans $\mathbb{R}^r$. Let the $r\times(r-1)$ matrix M and the $r\times 1$ matrix N be given by

$$M=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, N=\begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{r-1} \end{bmatrix}$$

The columns of M and N are the bases for the null-space and range-space of A_{01}, respectively. We define the inverse of a transformation matrix T as $T^{-1}=[M \;\; N]$. Then, $T=\begin{bmatrix} P \\ Q \end{bmatrix}$, where the $1\times r$ matrix Q is given by $Q=[1 \;\; 0 \;\; \cdots \;\; 0]$ and the $(r-1)\times r$ matrix P satisfies $PA_{01}=0$. According to [Kokotović et al., 1986, Proposition 6.1], the change of variables

$$\begin{bmatrix} \zeta \\ \theta_1 \end{bmatrix} = T\theta = \begin{bmatrix} P \\ Q \end{bmatrix}\theta$$

transforms the system (30) into

$$\varepsilon_1\dot{\zeta} = PA_{02}M\zeta + PA_{02}N\theta_1 + \varepsilon_1^r PBg + PB_{21}v$$
$$\varepsilon_1\dot{\theta}_1 = QA_{02}M\zeta + (\kappa QA_{01}N + QA_{02}N)\theta_1 - (\kappa+\beta_1)v$$

where we have used the relation $QA_{01}M=0$. It is easy to show that $QA_{01}N=-1$, $QA_{02}M=Q$, and $A_{02}N=0$. Therefore, we have

$$\varepsilon_1\dot{\zeta} = PA_{02}M\zeta + \varepsilon_1^r PBg(\chi,d,x-\hat{x}) + PB_{21}v \quad (35)$$
$$\varepsilon_1\dot{\theta}_1 = \zeta_1 - \kappa\theta_1 - (\kappa+\beta_1)v \quad (36)$$

Note that $\theta_1 = x_1 - \hat{x}_1$ and $PA_{02}M$ is a Hurwitz matrix. From singular perturbation theory Kokotović et al. [1986] we see that the solution of (36) is $O(1/\kappa)$ close to the solution of

$$(\varepsilon_1/\kappa)\dot{\theta}_1 = -\theta_1 - v$$

which decays monotonically towards the zone I_δ provided $\delta > \mu$. Hence, by choosing κ large enough we can ensure that $(y-\hat{x}_1)$ will enter, and remain in, the switching zone during a time period $[t_0, t_0+T_{12}(\varepsilon_1/\kappa)]$, for some $T_{12}>0$, where $T_{12}(\varepsilon_1/\kappa)\to 0$ as $(\varepsilon_1/\kappa)\to 0$. We note that if ε is switched before the transient response of the estimates of the higher order derivatives has settled, it may cause the output error $(y-\hat{x}_1)$ to leave the switching zone. This could result in repeated switching of ε until the remaining trajectories recover from peaking. To avoid this scenario, once $(y-\hat{x}_1)$ enters the switching zone we delay switching by a time period T_d that depends upon the peaking period of the observer to ensure that switching takes place after the trajectories of the estimation error θ have reached a positively invariant set.

3.1 Switching Scheme

Based on the foregoing discussion, we use the following gain switching scheme for the observer (26):

1. Choose $H=H_1$ and reset the delay timer whenever $|y-\hat{x}_1|>\delta$.
2. Once $(y-\hat{x}_1)$ enters (or begins in) $[-\delta,\delta]$ start the delay timer; keep $H=H_1$.
3. After the delay time T_d, and while $(y-\hat{x}_1)\in[-\delta,\delta]$, switch to $H=H_2$.

Analysis of the closed-loop system under the switched-gain observer is relegated to the full paper.

3.2 Choice of ε_1, ε_2, and T_d

The ultimate bound on the estimation error $\|x-\hat{x}\|$ is given by (25), where the constants c_1 and c_2 may be different before and after switching due to different choices of the observer eigenvalues. The function $F_r(\varepsilon,\mu)$ attains a minimum at $\varepsilon = c_a\mu^{1/r}$, is strictly increasing for $\varepsilon > c_a\mu^{1/r}$ and approaches infinity as ε tends to zero. To avoid the increase of $F_r(\varepsilon,\mu)$ with decreasing values of ε, in Theorem 4 we restricted ε to the range $\varepsilon > c_a\mu^{1/r}$. Because ε_2 determines the steady-state behavior of the observer, it is chosen according to Theorem 4, with a lower bound $\varepsilon_2 > c_a\mu^{1/r}$. For ε_1, we would like to choose $\varepsilon_1 < \varepsilon_2$ to allow for a faster decay of the transient response. In other words, we would like to work in the range $\varepsilon_1 < c_a\mu^{1/r}$. However, the choice of ε_1 has to be limited by a lower bound because of two factors. First, we have to ensure boundedness of the slow variable χ during the transient period. Second, we have to ensure that $|y(t) - \hat{x}_1| \leq \delta$ after the switching time. These concerns can be addressed by choosing

$$\varepsilon_1 = k_\varepsilon \varepsilon_2^{r/(r-1)} \tag{37}$$

for some positive constant k_ε. The delay time T_d is chosen to satisfy a lower bound to ensure that at the switching time the estimation error (ζ, θ_1) would have reached a positively invariant set in order to prevent multiple gain switchings. On the other hand, to show closeness of trajectories for all time we need to show that the time it takes until $\eta(t)$ enters a positively invariant set can be made arbitrarily small. This conditions imposes an upper bound on the choice of T_d. We summarize our conclusions in the following theorem.

Theorem 6. Let Assumptions 1 to 3 hold and consider the closed-loop system formed of the plant (1)–(4), the output feedback controller (5)–(6), with x replaced by $\hat{x}$, and the switched-gain observer (26) with the switching scheme described in Section 3.1. Let $\varepsilon_1 = k_\varepsilon \varepsilon_2^{r/(r-1)}$ and let $\mathcal{M}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{N}$ be any compact subset of $\mathbb{R}^r$, and suppose that $\chi(t_0) \in \mathcal{M}$ and $\hat{x}(t_0) \in \mathcal{N}$. Then we can choose T_d and δ such that

- There exist positive constants c_a and μ^*, κ^*, and k_ε^* such that for each $\mu < \mu^*$, $\kappa \geq \kappa^*$, and $k_\varepsilon \geq k_\varepsilon^*$, there is a constant $\varepsilon_a = \varepsilon_a(\mu) > c_a \mu^{1/r}$, with $\lim_{\mu \to 0} \varepsilon_a(\mu) = \varepsilon_a^* > 0$, such that for each $\varepsilon_2 \in (c_a \mu^{1/r}, \varepsilon_a]$ the trajectories of the closed-loop system are bounded for all $t \geq 0$;
- There exist $\mu_1^* > 0$ and a class $\mathcal{K}$ function ρ_1 such that for every $\mu < \mu_1^*$ and every $\xi_1 > \rho_1(\mu)$, there are constants $T_1 = T_1(\xi_1) \geq 0$ and $\varepsilon_b = \varepsilon_b(\mu, \xi_1) > c_a \mu^{1/r}$, with $\lim_{\mu \to 0} \varepsilon_b(\mu, \xi_1) = \varepsilon_b^*(\xi_1) > 0$, such that for each $\varepsilon_2 \in (c_a \mu^{1/r}, \varepsilon_b]$ we have
$$\max\{|\chi(t)|_{\mathcal{A}}, \|x(t) - \hat{x}(t)\|\} \leq \xi_1, \qquad \forall\, t \geq T_1$$
- There exist $\mu_2^* > 0$ and a class $\mathcal{K}$ function ρ_2 such that for every $\mu < \mu_2^*$ and every $\xi_2 > \rho_2(\mu)$, there are constants $\varepsilon_c = \varepsilon_c(\mu, \xi_2) > c_a \mu^{1/r}$, with $\lim_{\mu \to 0} \varepsilon_c(\mu, \xi_2) = \varepsilon_c^*(\xi_2) > 0$, and $T_d^* = T_d^*(\xi_2)$ such that for $T_d \leq T_d^*$ and $\varepsilon_2 \in (c_a \mu^{1/r}, \varepsilon_c]$ we have
$$\|\chi(t) - \chi_r(t)\| \leq \xi_2, \quad \forall\, t \geq 0$$
where $\chi_r(t)$ is the solution of (7) with $\chi_r(t_0) = \chi(t_0)$.

4. EXAMPLE

We consider a field controlled DC motor Khalil [2002] and design a controller based on feedback linearization so that the shaft angular velocity tracks the reference trajectory shown in Figure 1. The motor equations are given by

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = \phi(x, u), \;\; \dot{x}_3 = \psi(x, u) \tag{38}$$

$$y = x_1 + v, \;\; \theta_1 = x_3 \tag{39}$$

where x_1 is the rotor position, x_2 is the rotor angular velocity, x_3 is the armature current, and control u is the field current. The functions ϕ and ψ are given by $\phi(x, u) = -0.1x_2 + 0.1x_3 u$ and $\psi(x, u) = -2x_3 - 0.2x_2 u + 200$. The estimates, $\hat{x}$, are saturated outside $[-100, 100]$. For the observer, we have $\phi_0(\hat{x}, u) = -0.11x_2 + 0.1x_3 u$, and we use the following gains

$$H_1^T = \begin{bmatrix} \frac{71}{\varepsilon_1} & \frac{70}{\varepsilon_1^2} \end{bmatrix}, \;\; H_2^T = \begin{bmatrix} \frac{2}{\varepsilon_2} & \frac{1}{\varepsilon_2^2} \end{bmatrix} \tag{40}$$

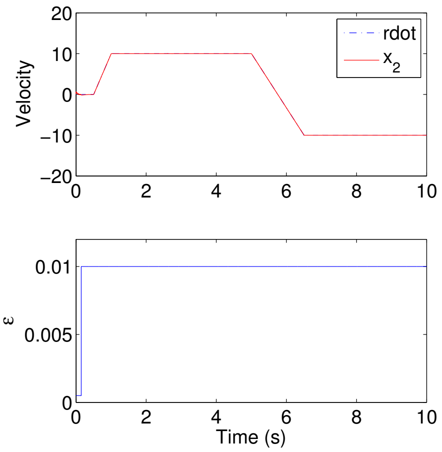

Fig. 1. The velocity reference trajectory $(\dot{r})$(dotted) and x_2 under the switched observer (solid). Bottom: Switching behavior of the gain.

where $\varepsilon_1 = 0.0005$ and $\varepsilon_2 = 0.01$. The gain H_1 was chosen, using simulation, to ensure that the estimation error does not over shoot the switching zone. For the switching threshold we use $\delta = 0.05$ and a delay time $T_d = 0.15s$. The initial conditions for the system and observer are $\hat{x}_1(0) = \pi$, $x_1(0) = x_2(0) = \hat{x}_2(0) = 0$. The measurement noise is generated by Simulink's "Uniform Random Number" block with magnitude limited within $[-0.0016, 0.0016]$ and sampling time set at 0.0008 seconds. This error magnitude is consistent with a 1000 c/r encoder. Figure 1 shows the velocity reference $\dot{r}$ (dotted) and the trajectory x_2 (solid) for the closed-loop system under the switched-gain observer. The two plots are indistinguishable. The bottom figure plots ε versus time, illustrating the switching behavior. Figure 2 plots the velocity tracking error, $e_2 = x_2 - \dot{r}$, for the closed-loop system under the switched-gain observer ($\varepsilon = \varepsilon_i$, top), a fixed gain observer with $\varepsilon = \varepsilon_2 = 0.01$ (middle), and a fixed gain observer with $\varepsilon = \varepsilon_1 = 0.0005$ (bottom). The switched-gain observer has better velocity tracking during the initial transient than the fixed-gain case with $\varepsilon = 0.01$ due to the faster state reconstruction. Figure 3 zooms in on the steady-state behavior of $e_2 = x_2 - \dot{r}$ showing that more of the measurement noise is attenuated when the observer switches to the larger ε resulting in improved velocity tracking than the case with $\varepsilon = 0.0005$. We point out the importance of the delay T_d by noting that simulations with $T_d = 0$ resulted in repeated switching of ε between 0.01 and 0.0005.

5. CONCLUSIONS

This paper has considered the problem of output feedback control for a class of nonlinear systems using high-gain observers in the presence of measurement noise. We have derived a relationship on the state estimation error that exhibits the tradeoff inherent in the choice of observer gain. This tradeoff balances state reconstruction speed along with robustness to modeling uncertainty against the immunity to measurement noise. By studying the closed-loop output feedback system we have been able to argue boundedness and ultimate boundedness. Further, we have quantified the impact of the noise magnitude on

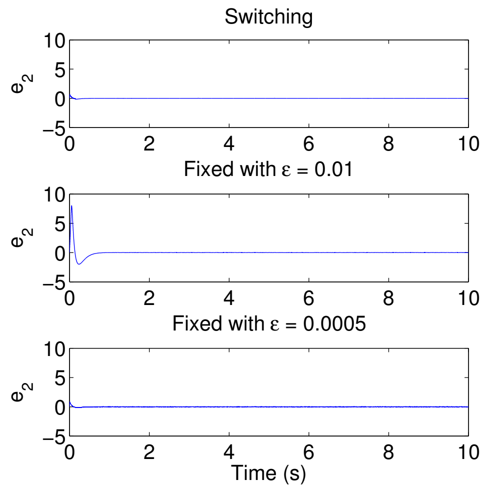

Fig. 2. Velocity tracking error ($e_2 = x_2 - \dot{r}$) for the switched-gain observer (top), the observer with $\varepsilon_2 = 0.01$ (middle), and the observer with $\varepsilon_1 = 5 \times 10^{-4}$ (bottom).

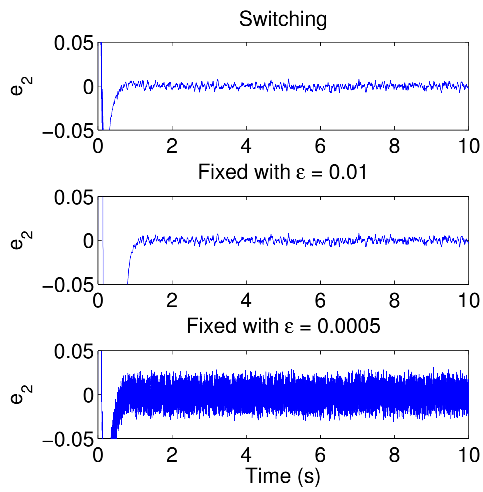

Fig. 3. Steady-State velocity tracking error ($e_2 = x_2 - \dot{r}$) for the switched-gain observer (top), the observer with $\varepsilon_2 = 0.01$ (middle), and the observer with $\varepsilon_1 = 5 \times 10^{-4}$ (bottom).

the recovery of the performance of the closed-loop output feedback system to the performance given by a globally bounded partial state feedback control. We have seen that we cannot recover the state feedback performance to an arbitrarily small degree.

Based on the forgoing we have designed a switched-gain version of the high-gain observer in an attempt to relax these tradeoffs. The idea uses high gain when the estimation error is large for fast state reconstruction at the expense of larger measurement noise error. When the output error becomes small we switch to a smaller gain to balance the error due to model uncertainty and measurement noise. To handle the peaking in the estimates we have included a switching delay timer in our scheme. Again, we are able to argue boundedness and ultimate boundedness of the closed-loop switched-gain output feedback system as well as closeness of trajectories to that of a globally bounded partial state feedback control.

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