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Approximate observer error linearization for nonlinear systems with input

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Abstract: This paper presents an approach to the design of nonlinear observers by approximate error linearization. It extends the results in Deutscher and Bäuml (2006) to systems with input applying Lyapunov's Auxiliary Theorem. By using a Galerkin approach on the basis of multivariable Legendre polynomials the L_2-norm of the remaining nonlinearity in the resulting error dynamics can be made small on a specified multivariable interval in the state space. Linear matrix equations are derived for determining the corresponding change of coordinates and output injections. Consequently, the proposed design procedure can easily be implemented in a numerical software package. A dc motor with a boost converter as actuator demonstrates the properties of the proposed numerical observer design.

Keywords: nonlinear observers; approximate error linearization; Galerkin method; multivariable Legendre polynomials; L_2-approximation.

1. INTRODUCTION AND PROBLEM FORMULATION

Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

with state $x \in \Omega \subseteq \mathbb{R}^n$, where Ω is an open set such that $0 \in \Omega$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}^m$, $m < n$. In the sequel, f and h are supposed to be real analytic functions in Ω, i.e. $f \in C^\omega(\Omega)$ and $h \in C^\omega(\Omega)$. The function g is assumed to be smooth in Ω, i.e. $g \in C^\infty(\Omega)$. Furthermore, let without loss of generality $x = 0$ be an equilibrium point of (1) for $u = 0$, i.e. $f(0) = 0$ and assume that $g(0) \neq 0$ and $h(0) = 0$. In this paper the problem of constructing a nonlinear observer for (1)–(2) is considered that estimates the state x using the measurements y and the input u. To his end, it is assumed that system (1)–(2) is *linearly observable*, i.e. its Jacobian linearization

$$A = \frac{\partial f}{\partial x}(0), \quad b = g(0), \quad C = \frac{\partial h}{\partial x}(0) \tag{3}$$

is observable. Although the observer design for systems with one input is discussed, the extension to systems with multiple inputs is straightforward and is omitted for the clarity of the presentation.

The design of observer for nonlinear systems, especially without input, has been studied extensively in the last two decades (see, e.g. the surveys Walcott et al. (1987) and Krener (2003)). One technique for construction nonlinear observers is to linearize the error dynamics by using a change of coordinates and an output injection which was investigated for single-output (see Krener and Isidori (1983)) and multi-output systems (see Krener and Respondek (1985), Xia and Gao (1989)). However, it turned out that the conditions for the linearizability of the error dynamics are quite stringent. A new approach for the design of nonlinear observers was presented in Kazantzis and Kravaris (1998) and Krener and Xiao (2002) (see also Krener and Xiao (2004)), that can be applied to a wider class of systems without input. In this paper this approach is extended to systems with input. The essential part of the design is the computation of a nonlinear change of coordinates

$$z = \bar{\phi}(x), \tag{4}$$

which transforms (1) into the normal form

$$\dot{z} = (A - LC)\, z - \beta(y) - \alpha(y)u \tag{5}$$

where

$$L = -\frac{\partial \beta}{\partial y}(0) \tag{6}$$

is chosen such that $\mathrm{Re}\lambda_i(A - LC) < 0$, $\forall i = 1, 2, \ldots, n$ and $\lambda_i(\cdot)$ denoting the ith *eigenvalue* of a matrix. This L exists since (C, A) is observable. The required change of coordinates can be achieved by solving the initial value problem

$$\frac{\partial \bar{\phi}(x)}{\partial x} f(x) = (A - LC)\, \bar{\phi}(x) - \beta(h(x)) \tag{7}$$

$$\bar{\phi}(0) = 0 \tag{8}$$

with an additional condition

$$\frac{\partial \bar{\phi}(x)}{\partial x} g(x) = -\alpha(h(x)) \tag{9}$$

which arises from considering a system (1) with input. This problem formulation consists of two linear PDEs with $\beta \in C^\omega(h(\Omega))$, $\beta(0) = 0$ and $\alpha \in C^\omega(h(\Omega))$, $\alpha(0) \neq 0$. An observer for (5) is given by

$$\dot{\hat{z}} = (A - LC)\hat{z} - \beta(y) - \alpha(y)u \tag{10}$$

which obviously has the linear error dynamics

$$\dot{z} - \dot{\hat{z}} = (A - LC)\,(z - \hat{z}) \tag{11}$$

In order to circumvent the inversion of $\bar{\phi}$ to determine $\hat{x}$ from $\hat{z}$ the observer (10) can be represented in the x-coordinates by

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u \tag{12}$$
$$- \left(\frac{\partial\bar{\phi}(\hat{x})}{\partial\hat{x}}\right)^{-1} (\beta(y) - \beta(h(\hat{x})) + (\alpha(y) - \alpha(h(\hat{x})))\,u)$$

which has the error dynamics (11). In comparison to the observer for systems without input (see Deutscher and Bäuml (2006)) a second PDE (9) has to be solved in addition to the original initial value problem (7)–(8). In this paper the following procedure is proposed to solve the PDEs (7) and (9). First a solution of the initial value problem (7)–(8) is determined. Existence and uniqueness of this solution can be checked by using Lyapunov's Auxiliary Theorem:

Theorem 1. Consider the initial value problem (7)–(8) and let the following assumption be satisfied:

1. The Jacobian matrix A in (3) has all its eigenvalues either in the open left or in the open right complex half plane.
2. The eigenvalues $\lambda_i(A)$, $i = 1, 2, \ldots, n$ of the matrix A (see (3)) and the eigenvalues $\lambda_i(A - LC)$, $i = 1, 2, \ldots, n$ of $A - LC$ (see (5)) satisfy
$$\lambda_i(A - LC) \neq p_1\lambda_1(A) + \cdots + p_n\lambda_n(A) \tag{13}$$
$\forall i = 1, 2, \ldots, n$, and $\forall$ non-negative integers p_j such that $p_1 + \cdots + p_n \geq 2$.

Then, in a neighborhood of $x = 0$ there exists a unique analytic solution
$$\bar{\phi}(x) = x + \bar{\phi}_{nl}(x) \tag{14}$$
with $\bar{\phi}_{nl}(0) = 0$ and $\frac{\partial\bar{\phi}_{nl}}{\partial x}(0) = 0$, solving the initial value problem (7)–(8).

Proof. See Kazantzis and Kravaris (1998).

Since (C, A) is supposed to be observable there always exists an L such that the eigenvalues of $A - LC$ meet (13). Hence, by Theorem 1 one can always compute a solution (14) of (7) provided that the first assumption holds. If only the second assumption is satisfied then at least a unique formal series solution of (7) exists. Since the nonlinear part of β in (7) can be chosen freely there remain degrees of freedom in the solution of the initial value problem (7)–(8) which can be used together with a suitable α such that $\bar{\phi}$ also satisfies (9).

Based on Theorem 1 a Taylor series approximation is proposed in Kazantzis and Kravaris (1998) and Krener and Xiao (2002) to solve the initial value problem (7). The drawback of this approach is that in many cases the local character of this series solution method leads to a small region of attraction for the approximately linear error dynamics. Furthermore, the observer performance may deteriorate if the initial conditions of the system and of the observer are far away from their respective origins. The Galerkin approach presented in Deutscher and Bäuml (2006) for determining a solution of the initial value problem (7) circumvents the drawbacks of the Taylor series solution method. The Galerkin method approximates the solution of (7) by a finite series of orthogonal basis functions up to degree N. In order to determine the coefficients of this approximation the equation error resulting from substituting the approximate solution in the PDE (7) is computed. Then, the free parameters in the approximate solution are chosen such that the coefficients of the first N terms of the series expansion of the equation error vanish. By compensating the first terms of this series the L_2-norm of the equation error will presumably become very small. Thereby the interval in the state space where the L_2-norm of the equation error is small can be specified beforehand. Thus, the domain where the observer dynamics becomes nearly linear can be assigned in the design. Since the PDEs (7) and (9) to be solved are linear, linear algebraic equations can be derived for determining an approximate solution.

In this paper multivariable Legendre polynomials are used in the Galerkin approach as basis function. These polynomials are presented in the next section, where also the Galerkin approach for the nonlinear observer design is investigated. A practical application demonstrates the results of the presented approach in Section 3.

2. GALERKIN APPROACH FOR NONLINEAR OBSERVER DESIGN

2.1 Multivariable Legendre polynomials

For the approximate solution of the initial value problem with an additional condition (7)–(9) a finite series of orthonormal basis functions is used to represent the solution. A natural choice for the basis functions are polynomials. In general the approximate solution will consist of all state variables x, hence multivariable polynomials are needed, which are orthonormal on a given state interval. A simple method to generate such multivariable polynomials is a product of one-dimensional polynomials
$$\varphi_{k_1\cdots k_n}(x) = \varphi_{k_1}(x_1)\cdot\ldots\cdot\varphi_{k_n}(x_n) \tag{15}$$
Each $\varphi_{k_\nu}(x_\nu)$ in (15) represents a one-dimensional Legendre polynomial of degree k_ν, which is defined by
$$\varphi_{k_\nu}(x_\nu) = \sqrt{\frac{2k+1}{2}}\frac{1}{2^k k!}\frac{d^k}{dx_\nu^k}(x_\nu^2 - 1)^k, \quad k \in \mathbb{N}_0 \tag{16}$$
(for details concerning Legendre polynomials see e.g. Courant and Hilbert (1989)). As a result the multivariable Legendre polynomials $\varphi_{k_1\cdots k_n}(x)$ of degree $k = k_1 + \ldots + k_n$ are obtained. The advantage of the Legendre polynomials (16) is, that they are *orthonormal* with the constant weighting function 1 on the interval $I = [-1, 1]$, i.e. the *scalar product* is given by
$$\langle\varphi_i, \varphi_j\rangle = \int_{-1}^{1} \varphi_i(x_\nu)\varphi_j(x_\nu)dx_\nu = \delta_{ij}, \quad \forall i, j \in \mathbb{N}_0 \tag{17}$$
where
$$\delta_{ij} = \begin{cases} 0 : i \neq j \\ 1 : i = j \end{cases} \tag{18}$$
denotes the *Kronecker delta function.* The set of one-dimensional Legendre polynomials is complete, this means that every function $g(x_\nu)$ with finite L_2-norm $\|g\|_2 = (\langle g, g\rangle)^{\frac{1}{2}}$ can be approximated by $\hat{g}(x_\nu) = \sum_{i=0}^{N} c_i\varphi_i(x_\nu)$ with arbitrary small error $e = (\langle g - \hat{g}, g - \hat{g}\rangle)^{\frac{1}{2}}$, if the degree of the approximation N is chosen large enough. In Deutscher and Bäuml (2006) it is shown, that the properties of the one-dimensional Legendre polynomials (16) are transferred to the multivariable Legendre polynomials (15). Hence they represent an orthonormal and complete set of functions to the n-dimensional interval

$I = [-1, 1]^n$. The L_2-approximation of arbitrary intervals can be achieved by using an affine linear transformation, which maps a chosen approximation interval $\bar{I} = [x_{1,min}, x_{1,max}] \times \cdots \times [x_{n,min}, x_{n,max}]$ on the interval $I = [-1, 1]^n$ (see Deutscher and Bäuml (2006)).

2.2 Derivation of linear matrix equations for the observer design

Consider the approximate solution of the initial value problem (7)–(8) in the form

$$\bar{\phi}_N(x) = T_\Phi \Phi(x) \tag{19}$$

where $\Phi(x)$ is the vector of multivariable Legendre polynomials up to degree N and T_Φ is the corresponding matrix of coefficients. The output injections in (5) are arranged in the form

$$\beta(y) = -Ly - L_{\Phi_y}\Phi_y(y) \tag{20}$$

$$\alpha(y) = -b - M_{\Phi_y}\Phi_y(y) \tag{21}$$

with the vector $\Phi_y(y)$ of Legendre polynomials in y_1, y_2, …, y_m up to degree N. In general (19) will be an approximate solution of the PDEs (7) and (9) for given output injections (20)–(21), thus substituting (19) in the PDEs yields the equation errors

$$r_N(x) = \frac{\partial \bar{\phi}_N(x)}{\partial x} f(x) - (A - LC)\bar{\phi}_N(x) + \beta(h(x)) \tag{22}$$

$$\bar{r}_N(x) = \frac{\partial \bar{\phi}_N(x)}{\partial x} g(x) + \alpha(h(x)) \tag{23}$$

The basic idea of the Galerkin approach is to expand the equation errors (22)–(23) into multivariable Legendre polynomials with minimal L_2-error norm in a chosen interval. This yields the L_2-approximations

$$r_N \approx \hat{r}_N(x) = R_N \Phi(x) \tag{24}$$

$$\bar{r}_N \approx \hat{\bar{r}}_N(x) = \bar{R}_N \Phi(x) \tag{25}$$

where the matrices R_N and $\bar{R}_N$ are given by

$$R_N = \left\langle r_N, \Phi^T \right\rangle = \begin{bmatrix} \langle r_{1,N}, \Phi_1\rangle & \cdots & \langle r_{1,N}, \Phi_{n_\Phi}\rangle \\ \vdots & & \vdots \\ \langle r_{n,N}, \Phi_1\rangle & \cdots & \langle r_{n,N}, \Phi_{n_\Phi}\rangle \end{bmatrix} \tag{26}$$

$$\bar{R}_N = \left\langle \bar{r}_N, \Phi^T \right\rangle = \begin{bmatrix} \langle \bar{r}_{1,N}, \Phi_1\rangle & \cdots & \langle \bar{r}_{1,N}, \Phi_{n_\Phi}\rangle \\ \vdots & & \vdots \\ \langle \bar{r}_{n,N}, \Phi_1\rangle & \cdots & \langle \bar{r}_{n,N}, \Phi_{n_\Phi}\rangle \end{bmatrix} \tag{27}$$

(for details see Deutscher and Bäuml (2006)). Note that $\hat{r}_N$ and $\hat{\bar{r}}_N$ exist since f, β, $\alpha \in C^\omega(\Omega)$ and $g \in C^\infty(\Omega)$. In (26)–(27) $r_{i,N}$ and $\bar{r}_{i,N}$, respectively, denote the ith component of r_N and $\bar{r}_N$, respectively, and Φ_i the ith component of Φ with n_Φ =dimΦ. The choice of the coefficients in (26)–(27) ensures that $\hat{r}_N$ and $\hat{\bar{r}}_N$ approximate the equation errors with minimal L_2-error norm, i.e.

$$\|r_N - \hat{r}_N\|_2 = \left(\int_I (r_N - \hat{r}_N)^T (r_N - \hat{r}_N)\, dx \right)^{\frac{1}{2}} = \min \tag{28}$$

$$\left\|\bar{r}_N - \hat{\bar{r}}_N\right\|_2 = \left(\int_I \left(\bar{r}_N - \hat{\bar{r}}_N\right)^T \left(\bar{r}_N - \hat{\bar{r}}_N\right) dx \right)^{\frac{1}{2}} = \min \tag{29}$$

Because the functions in (22)–(23) are smooth, small approximation errors (28)–(29) can be achieved with relatively low approximation degrees (see Deutscher and Bäuml (2006)). If the change of coordinates (19) and the output injections (20)–(21) are determined such that the coefficient matrices satisfy

$$R_N = \left\langle r_N, \Phi^T \right\rangle \stackrel{!}{=} 0 \tag{30}$$

$$\bar{R}_N = \left\langle \bar{r}_N, \Phi^T \right\rangle \stackrel{!}{=} 0 \tag{31}$$

then the equation errors will be small in the least square sense in the approximation interval for sufficiently large N. By substituting (22)–(23) in (26)–(27) the coefficient matrices R_N and $\bar{R}_N$ of the approximation of the equation errors read

$$R_N = T_\Phi A_\Phi - (A - LC)T_\Phi - LH_\Phi - L_{\Phi_y}C_\Phi \stackrel{!}{=} 0 \tag{32}$$

$$\bar{R}_N = T_\Phi N_\Phi - B_\Phi - M_{\Phi_y}C_\Phi \stackrel{!}{=} 0 \tag{33}$$

in view of (19)–(21) and by using $B_\Phi = \left[\sqrt{2}^n b \;\; 0\right]$, $A_\Phi = \left\langle (\partial\Phi(x)/\partial x) f, \Phi^T \right\rangle$, $N_\Phi = \left\langle (\partial\Phi(x)/\partial x) g, \Phi^T \right\rangle$, $H_\Phi = \left\langle h, \Phi^T \right\rangle$ and $C_\Phi = \left\langle \Phi_y \circ h, \Phi^T \right\rangle$ (see Deutscher and Bäuml (2006)). However, the solution of (32)–(33) has to meet the auxiliary conditions

$$\bar{\phi}_N(0) = 0, \quad \frac{\partial \bar{\phi}_N}{\partial x}(0) = I \tag{34}$$

$$\beta(0) = 0, \quad \frac{\partial \beta}{\partial y}(0) = -L, \quad \alpha(0) = b \tag{35}$$

to determine a valid change of coordinates and output injections. The first condition (34) ensures that the approximate solution (19) satisfies the initial value of the initial value problem (7)–(8) and is a change of coordinates in the neighborhood of $x = 0$. The second condition (35) is necessary to preserve the form of the output injections (20)–(21). It is shown in Deutscher and Bäuml (2006) that these conditions yield additional linear equations

$$T_\Phi A_{\bar{\phi}} = B_{\bar{\phi}}, \quad L_{\Phi_y} A_\beta = 0, \quad M_{\Phi_y}\Phi_y(0) = 0 \tag{36}$$

for the coefficient matrices T_Φ, L_{Φ_y} and M_{Φ_y}. With the application of the operational matrices of differentiation D_ν and $D_{y\mu}$

$$\frac{\partial \Phi(x)}{\partial x_\nu} = D_\nu \Phi(x), \quad \nu = 1, \ldots, n \tag{37}$$

$$\frac{\partial \Phi_y(y)}{\partial y_\mu} = D_{y\mu}\Phi_y(y), \quad \mu = 1, \ldots, m \tag{38}$$

the matrices in (36) are given by

$$A_{\bar{\phi}} = [\Phi(0) \;\; D], \quad B_{\bar{\phi}} = [0 \;\; I_n] \tag{39}$$

$$A_\beta = [\Phi_y(0) \;\; D_y C] \tag{40}$$

with

$$D = [D_1\Phi(0) \; \ldots \; D_n\Phi(0)] \tag{41}$$

$$D_y = [D_{y1}\Phi_y(0) \; \ldots \; D_{ym}\Phi_y(0)] \tag{42}$$

The set of equations (32)–(33) and (36) must be solved for the design of the observer. It can be shown that this set of equation is in general overdetermined.

2.3 Solution of the matrix equations

Before solving the set of equations (32)–(33) and (36) the general solutions of (36) are computed since (34)–(35) have to be satisfied in any case. The general solution of $T_\Phi A_{\bar{\phi}} = B_{\bar{\phi}}$ (see (36)) exists (see Deutscher and Bäuml (2006)) and reads

$$T_\Phi = B_{\bar{\phi}} A_{\bar{\phi}}^{+} + \bar{T}_\Phi A_{\bar{\phi}}^{\perp} \tag{43}$$

where $\bar{T}_\Phi$ represents an arbitrary coefficient matrix and $A_{\bar{\phi}}^{+}$ denotes the *Moore-Penrose inverse* (see e.g. Ben-Israel and Greville (2003)) and $A_{\bar{\phi}}^{\perp}$ the *left annihilator* of $A_{\bar{\phi}}$, i.e.

$$A_{\bar{\phi}}^{\perp} A_{\bar{\phi}} = 0 \tag{44}$$

with $A_{\bar{\phi}}^{\perp}$ having full row rank. The general solution of $L_{\Phi_y} A_\beta = 0$ (see (36)) is given by

$$L_{\Phi_y} = \bar{L}_{\Phi_y} A_\beta^{\perp} \tag{45}$$

where $\bar{L}_{\Phi_y}$ is an arbitrary coefficient matrix. If the condition $\binom{m+N}{m} > n+1$ is fulfilled which can be achieved by an approximation degree chosen large enough (45) is a solution of $L_{\Phi_y} A_\beta = 0$ (see Deutscher and Bäuml (2006)). The general solution of $M_{\Phi_y}\Phi_y(0) = 0$ (see (36)) is given by

$$M_{\Phi_y} = \bar{M}_{\Phi_y} \Phi_y^{\perp}(0) \tag{46}$$

where $\bar{M}_{\Phi_y}$ is an arbitrary coefficient matrix. Equation (46) is a nonvanishing solution of $M_{\Phi_y}\Phi_y(0) = 0$ if the vector $\Phi_y(0)$ has more than one row which is true if the approximation degree satisfies $N \geq 1$. The arbitrary coefficient matrices $\bar{T}_\Phi$, $\bar{L}_{\Phi_y}$ and $\bar{M}_{\Phi_y}$ in (43), (45) and (46) parametrize all changes of coordinates (19) and output injections (20)–(21) that fulfil the conditions (34)–(35). Substituting the general solutions (43), (45) and (46) in (32)–(33) yield the design equations

$$\begin{aligned} F\bar{T}_\Phi A_{\bar{\phi}}^{\perp} - \bar{T}_\Phi A_{\bar{\phi}}^{\perp} A_\Phi =& B_{\bar{\phi}} A_{\bar{\phi}}^{+} A_\Phi - F B_{\bar{\phi}} A_{\bar{\phi}}^{+} \\ & - LH_\Phi - \bar{L}_\Phi A_\beta^{\perp} C_{\Phi_y} \end{aligned} \tag{47}$$

$$\bar{T}_\Phi A_{\bar{\phi}}^{\perp} N_\Phi - \bar{M}_{\Phi_y} \Phi_y^{\perp}(0) C_{\Phi_y} = B_\Phi - B_{\bar{\phi}} A_{\bar{\phi}}^{+} N_\Phi \tag{48}$$

for the coefficient matrices $\bar{T}_\Phi$, $\bar{L}_{\Phi_y}$ and $\bar{M}_{\Phi_y}$, with $F = A - LC$. In order to solve the matrix equations (47)–(48) introduce the *vec operator* $\text{vec}(M)$ of a matrix $M = [m_1 \ m_2 \ \ldots \ m_q]$ (see, e.g. Shi and Steeb (1997)) that yields the column vector $\text{vec}(M)=\text{col}(m_1, \ldots, m_q)$ by stacking the columns m_i, $i = 1, 2, \ldots, q$, of M in one vector. The property $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$ of the vec operator can be used to convert a linear equation for matrices into a linear equation for a column vector (see Shi and Steeb (1997)). At first an approximate solution of the initial value problem is determined by applying this property to the matrix equation (47) which gives

$$\begin{aligned} \text{vec}\left(\bar{T}_\Phi\right) =& P^{+}\Big(-\left(\left(A_\beta^{\perp} C_{\Phi_y}\right)^T \otimes I_n\right)\text{vec}\left(\bar{L}_{\Phi_y}\right) \\ & + \text{vec}\left(B_{\bar{\phi}} A_{\bar{\phi}}^{+} A_\Phi - F B_{\bar{\phi}} A_{\bar{\phi}}^{+} - LH_\Phi\right)\Big) \end{aligned} \tag{49}$$

with $P = (A_{\bar{\phi}}^{\perp})^T \otimes F - (A_{\bar{\phi}}^{\perp} A_\Phi)^T \otimes I_n$ and P^{+} denoting the Moore-Penrose inverse of P. In many cases (49) is only an approximate solution of (47). This is due to the fact that degrees of freedom in T_Φ and L_{Φ_y} are used to satisfy (36). Then, (49) yields an approximate solution with a minimal Euclidean norm of the equation error, i.e.

$$\begin{aligned} \|e\| = \Big\| & P\text{vec}\left(\bar{T}_\Phi\right) + \left(\left(A_\beta^{\perp} C_{\Phi_y}\right)^T \otimes I_n\right)\text{vec}\left(\bar{L}_{\Phi_y}\right) \\ & - \text{vec}\left(B_{\bar{\phi}} A_{\bar{\phi}}^{+} A_\Phi - F B_{\bar{\phi}} A_{\bar{\phi}}^{+} - LH_\Phi\right)\Big\| = \min \end{aligned} \tag{50}$$

The degrees of freedom contained in the nonlinear part of β (i.e. in $\bar{L}_{\Phi_y}$) and in α (i.e. in $\bar{M}_{\Phi_y}$) can now be used to satisfy (9). By applying the vec operator to (48) and substituting (49) in (48) one obtains

$$L_N v_N = w_N \tag{51}$$

with the matrix

$$L_N^T = \begin{bmatrix} \left(\left(\left(\left(A_{\bar{\phi}}^{\perp} N_\Phi\right)^T \otimes I_n\right) P^{+}\left(-\left(A_\beta^{\perp} C_{\Phi_y}\right)^T \otimes I_n\right)\right)^T \\ -\left(\left(\Phi_y^{\perp}(0) C_{\Phi_y}\right)^T \otimes I_n\right)^T \end{bmatrix} \tag{52}$$

and

$$v_N = \begin{bmatrix} \text{vec}\left(\bar{L}_{\Phi_y}\right) \\ \text{vec}\left(\bar{M}_{\Phi_y}\right) \end{bmatrix} \tag{53}$$

$$\begin{aligned} w_N =& \text{vec}\left(B_\Phi - B_{\bar{\phi}} A_{\bar{\phi}}^{+} N_\Phi\right) \\ & - \left(\left(A_{\bar{\phi}}^{\perp} N_\Phi\right)^T \otimes I_n\right) P^{+}\text{vec}\left(B_{\bar{\phi}} A_{\bar{\phi}}^{+} A_\Phi - F B_{\bar{\phi}} A_{\bar{\phi}}^{+} - LH_\Phi\right) \end{aligned} \tag{54}$$

If (51) is solvable, i.e.

$$\text{rank}\, L_N = \text{rank}\,[L_N \ \ w_N] \tag{55}$$

then a solution is given by

$$v_N = L_N^{+} w_N \tag{56}$$

which yields the coefficient matrices $\bar{L}_{\Phi_y}$ and $\bar{M}_{\Phi_y}$. Once again (51) may not be solvable but the Euclidean norm of the equation error $\|L_N v_N - w_N\|$ is minimal because of the Moore-Penrose inverse in (56). Using this results in (49) the coefficient matrix $\bar{T}_\Phi$ can be computed. Finally, applying these matrices to the general solutions (43), (45) and (46) the change of coordinates (19) and the output injections (20)–(21) can be calculated.

2.4 Resulting error dynamics

The error dynamics of the observer (12) in the z-coordinates is given by

$$\begin{aligned} \dot{z} - \dot{\hat{z}} =& (A - LC)(z - \hat{z}) \\ & + \rho_N(z) - \rho_N(\hat{z}) + (\bar{\rho}_N(z) - \bar{\rho}_N(\hat{z}))u \end{aligned} \tag{57}$$

in view of (22)–(23) with

$$\rho_N(z) = r_N\left(\bar{\phi}_N^{-1}(z)\right), \quad \rho_N(\hat{z}) = r_N\left(\bar{\phi}_N^{-1}(\hat{z})\right) \tag{58}$$

$$\bar{\rho}_N(z) = \bar{r}_N\left(\bar{\phi}_N^{-1}(z)\right), \quad \bar{\rho}_N(\hat{z}) = \bar{r}_N\left(\bar{\phi}_N^{-1}(\hat{z})\right) \tag{59}$$

It is shown in Deutscher and Bäuml (2006) that the transformed equation error ρ_N has the property

$$\rho_N(0) = 0, \quad \frac{\partial \rho_N}{\partial z}(0) = 0 \tag{60}$$

when taking the conditions (34)–(35) into account. With the condition $\alpha(0) = b$ in (35) the property of the equation error $\bar{\rho}_N$ is given by

$$\bar{\rho}_N(0) = 0 \tag{61}$$

Hence the convergence of the error dynamics in a sufficiently small neighborhood of $z = \hat{z} = 0$ can be assured if all eigenvalues $\bar{\lambda}_i$ of the matrix $A - LC$ meet $\text{Re}\lambda_i(A - LC) < 0$. The substitution of $x = \bar{\phi}_N^{-1}(z)$ and $\hat{x} = \bar{\phi}_N^{-1}(\hat{z})$ respectively in r_N and $\bar{r}_N$ (see (22)–(23)) yield the nonlinearities ρ_N and $\bar{\rho}_N$. The values of r_N and $\bar{r}_N$ are small on the approximation interval I because of the decrease of the L_2-norm $\|r_N\|_2$ and $\|\bar{r}_N\|_2$ in a least square sense. Thus the resulting error dynamics has small nonlinearities in a least square sense on a free chosen interval in the state space.

3. EXAMPLE

In this section the results of the presented observer design is implemented in a practical application. Considered is a dc motor with boost converter as actuator. This system has the state space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{L}(E - R_L x_1 - x_2) \\ \frac{1}{C}x_1 - \frac{1}{CR}x_2 - \frac{1}{R}x_3 \\ \frac{1}{L_M}(x_2 - R_M x_3 - K_E x_4) \\ \frac{1}{J}(-\tau + K_M x_3 - B x_4) \end{bmatrix} + \begin{bmatrix} \frac{1}{L}x_2 \\ -\frac{1}{C}x_1 \\ 0 \\ 0 \end{bmatrix} u \quad (62)$$

$$y = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \quad (63)$$

where x_1 is the converter current, x_2 the capacitor voltage, x_3 the motor current and x_4 the revolutions per second (rps) of the motor shaft. E in (62) is the constant converter supply voltage and τ is a constant friction torque and B is the coefficient of the dynamic friction torque of the motor. The system parameters are $C = 470\mu F$, $L = 1.335mH$, $R = 10k\Omega$, $R_M = 6\Omega$, $L_M = 8.9mH$, $K_E = 28.6 \cdot 10^{-2}Vsec$, $K_M = 7.24 \cdot 10^{-3}\frac{Nm}{A}$, $J = 7.95 \cdot 10^{-6}kgm^2$, $\tau = 8.2 \cdot 10^{-4}Nm$, $B = 4 \cdot 10^{-6}Nmsec$ and $E = 9V$. The input is constrained to be in the interval $u \in [0,\ 1]$ as u is the duty cycle of the PWM input signal to the converter. In order to show the application of the proposed observer design in a closed loop system a linear state feedback controller is designed for the desired equilibrium point at $x_d = [0.27A\ \ 16.585V\ \ 0.144A\ \ 55rps]^T$ for $u = 0.462$. The eigenvalues $\tilde{\lambda}_i$ of the closed loop system are assigned to $\tilde{\lambda}_i = -600$, $i = 1, \ldots, 4$. By a simple change of coordinates this equilibrium point is transformed to the origin, so that the results of the paper are applicable. Furthermore, the Jacobian linearization (3) of this equilibrium point is observable and satisfies the first condition of Theorem 1. By choosing the eigenvalues $\bar{\lambda}_1 = -898.7$, $\bar{\lambda}_2 = -892.3$, $\bar{\lambda}_3 = -885.9$ and $\bar{\lambda}_4 = -879.5$ for the observer the second condition is also met. Therefore a solution of the of the initial value problem exists. The interval

$$\bar{I} = [-5, 5] \times [-20, 20] \times [-4, 4] \times [-40, 40] \quad (64)$$

is chosen for the observer design using multivariable Legendre polynomials. The boundaries are picked larger than the expected values to avoid singularities in the inverse matrix in (12) which especially appear at the edges of the chosen interval. Such singularities lead to instability of the observer in most cases so its important that the interval is not to small. On the other hand if the interval is too large the linearization effect of the error dynamics becomes worse. To show the properties of the proposed observer (12) it is compared to a nonlinear constant gain observer

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + L(y - h(\hat{x})) \quad (65)$$

with the same eigenvalue assignment. In order to investigate the effect of the linearization the absolute error between the resulting error dynamics and the linear error dynamics

$$\dot{\xi}_{lin} = (A - LC)\xi_{lin} \quad (66)$$

is plotted. This means that the error in question for the presented observer reads $e_{i,leg} = |z_i - \hat{z}_i - \xi_{i,lin}|$ and for the constant gain observer $e_{i,cg} = |x_i - \hat{x}_i - \xi_{i,lin}|$. At first, both observers are simulated with the initial condition $x(0) = [0.241A\ \ 15.139V\ \ 0.141A\ \ 50rps]^T$ for the system (62) and the initial condition $\hat{x}(0) = x_d$ in the original coordinates of the system. Figure 1 shows the linearization errors for a linearization degree $N = 2$ for the Legendre-based observer and the constant gain observer. Although the matrix equations (47)–(48) cannot be solved

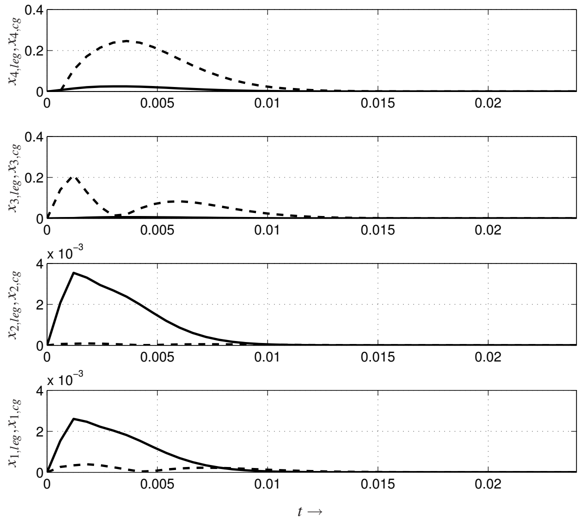

Fig. 1. Comparison of the simulation of the error linearization of the constant gain observer (----) and the Legendre-based observer for linearization degree 2 (—)

the Moore-Penrose inverse in (56) yields a least square solution which obviously yields a good linearization for the Legendre-based observer.

For the implementation of the controller and observer a real time hardware is used which is certainly more extensive for the Legendre-based observer but still possible. The same configuration for the system and the observers in practical application is shown in Figure 2. The error

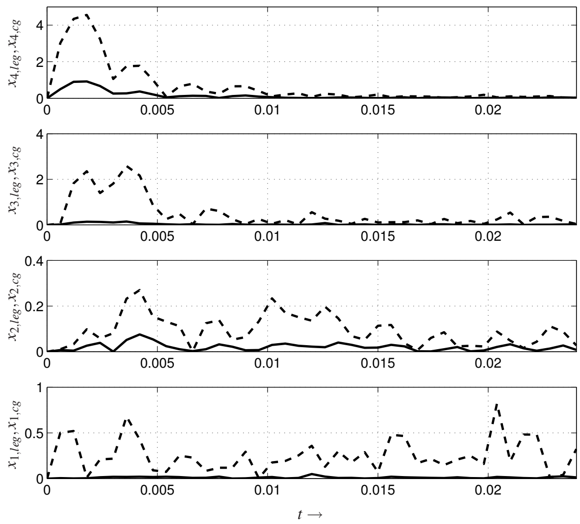

Fig. 2. Comparison of the measurement of the error linearization of the constant gain observer (----) and the Legendre-based observer for linearization degree 2 (—)

linearization in Figure 2 does not tend to zero because of the noise in the measurements. It is obvious that the linearization effect of the Legendre-based observer compared to the constant gain observer is still better in the practical application. Figure 3 shows the trajectories of the system and the Legendre-based observer for the considered initial conditions. The trajectories of the system and

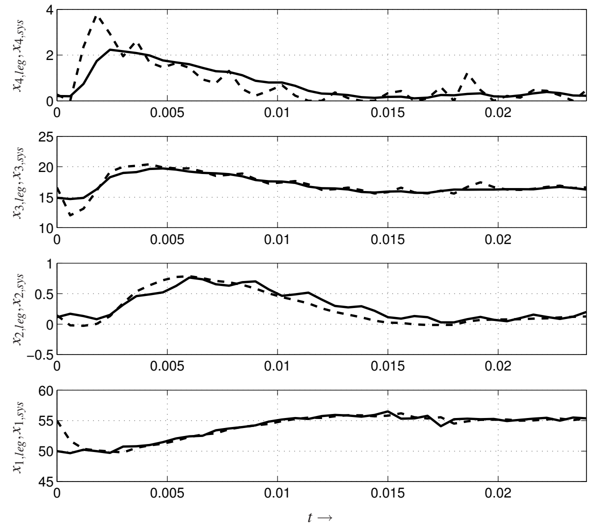

Fig. 3. Trajectories of the Legendre-based observer for linearization degree 2 (----) and the system (—) with the initial condition $x(0) = [0.241A \;\; 15.139V \;\; 0.141A \;\; 50rps]^T$

the Legendre-based observer for initial conditions $x(0) = [0.166A \;\; 11.086V \;\; 0.133A \;\; 36rps]^T$ far away from the operating point x_d of the observer are shown in Figure 4. Even with an initial condition far away from the operating

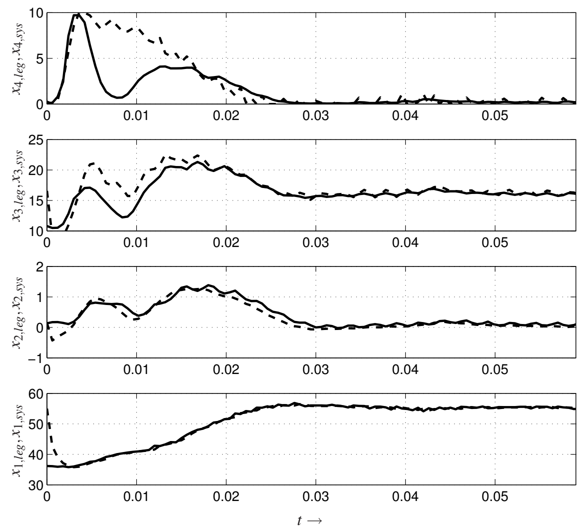

Fig. 4. Trajectories of the Legendre-based observer for linearization degree 2 (----) and the system (—) with the initial condition $x(0) = [0.166A \;\; 11.086V \;\; 0.133A \;\; 36rps]^T$

point of the observer a good closed loop behavior is given by the system with the linear state feedback controller and Legendre-based observer. Although the linearization is only effective in the new coordinates z the observation error in the original coordinates will tend to zero in the same time as the approximately linear error dynamics (57). This feature is preserved by the change of coordinates in view of $\bar{\phi}_N(0) = 0$.

4. CONCLUSION

This paper extends the observer design procedure of Deutscher and Bäuml (2006) to systems with input in order to use the observer in the closed loop system. By computing an approximate solution of the corresponding initial value problem with an additional condition using the Galerkin approach the error dynamics can be approximately linearized uniformly on a specified interval in the state space. Thus, the proposed approach circumvents the drawbacks of the solution method using a Taylor series approach. The corresponding design problem can be solved by computing the solution of linear matrix equations which can easily be calculated by a numerical software package. As multivariable orthonormal polynomials products of Legendre polynomials in one variable are employed for implementing the Galerkin approach. Consequently, existing algorithms and results for Legendre polynomials in one variable can be used for computing the multivariable Legendre polynomials as well as the operational matrices for differentiation of polynomials enabling an efficient numerical observer design. The linearization effects are shown for the laboratory setup of a dc motor with boost converter. The presented results demonstrate that the Legendre-based observer can be used in practical applications.

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