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Conservatism-free Robust Stability Check of Fractional-order Interval Linear Systems ⋆

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Abstract: This paper addresses a necessary and sufficient robust stability condition of fractional-order interval linear time invariant systems. The state matrix A is considered as a parametric interval uncertain matrix and fractional commensurate order is considered belonging to $1 \leq \alpha < 2$. Using the existence condition of Hermitian matrix $P = P^*$ for a complex Lyapunov inequality, we show that a fractional-order interval linear system is robust stable if and only if there exist Hermitian matrices $P = P^*$ such that complex Lyapunov inequalities are satisfied for all vertex matrices, which is a set of selected matrices. Two numerical examples are presented to verify the validity of the proposed approach.

1. INTRODUCTION

Recently, the fractional order linear time invariant (FO-LTI) systems have attracted lots of attention in control systems society (Lurie, 1994; Podlubny, 1999b; Oustaloup et al., 1995, 1996; Raynaud and Zergaïnoh, 2000) even though fractional-order control problems were investigated as early as 1960's (Manabe, 1960, 1961). The fractional order calculus plays an important role in thermodynamics, mechatronics systems, chemical mixing, and biological system as well. It is recommended to refer to (Oustaloup, 1981; Axtell and Bise, 1990; Vinagre and Chen, 2002; Xue and Chen, 2002; Machado, 2002; Ortigueira and Machado, 2003) for the further engineering applications of FO-LTI systems. In the field of fractional-order control systems, there are many challenging and unsolved problems such as robust stability, input-output stability, internal stability, robust controllability, frequency domain analysis, robust observability, etc. (Rugh, 1993; Vidyasagar, 1971; Skaar et al., 1988; Matignon, 1996, 1998c,a,b; Bonnet and Partington, 2000; Matignon and d'Andréa Novel, 1996; Moze and Sabatier, 2005). In the fractional order controller, the fractional order integration or derivative of the output error is used for the current control force calculation. For the robust stability analysis of the fractional-order systems, model uncertainty, disturbance, and stochastic noises have been considered. Recently, parametric interval concept has been utilized to take account of the parameter variation in fractional-order uncertain dynamic systems (Petráš et al., 2004, 2005; Chen et al., 2005b,a; Ahn et al., 2007). Noticeably, matrix perturbation theory was used in (Chen et al., 2005a) to find the ranges of interval eigenvalues and Lyapunov inequality was used in (Ahn et al., 2007) to reduce the conservatism in the robust stability test of interval uncertain FO-LTI systems. However, (Chen et al., 2005a; Ahn et al., 2007) do not provide exact robust stability condition; instead the methods proposed in (Chen et al., 2005a; Ahn et al., 2007) estimate the robust stability property under some restrictive conditions. This paper is an extension of (Ahn et al., 2007); specifically this paper addresses a necessary and sufficient condition for the robust stability of fractional-order linear interval systems with fractional commensurate order of $1 \leq \alpha < 2$.

In the following section, we provide some backgrounds of FO linear interval systems. In Section 3, main results of the paper are presented. In Section 4, two examples are provided to validate the results. Conclusion will be given in Section 5.

2. ROBUST STABILITY OF FRACTIONAL-ORDER LINEAR INTERVAL SYSTEMS

Let us consider the FO-LTI systems governed by the following state-space form:

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t) \tag{1}$$

where $\alpha \in \mathbf{R}$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and α is the fractional commensurate order. The fractional-order interval linear time invariant systems (FO-ILTI) are defined as the FO-LTI systems whose "A" matrix is interval uncertain in parameter-wise. That is, when "A" matrix is defined as $A \in A^I = [a^I_{ij}]$ where a^I_{ij} is lower and upper bounded such as $a^I_{ij} := [\underline{a_{ij}}, \overline{a_{ij}}]$, we call the system (1) fractional-order interval linear time invariant systems (FO-ILTI). Note that A^I can be also defined as $[\underline{A}, \overline{A}]$

⋆ Corresponding author: Hyo-Sung Ahn. Tel: 82-062-970-2398 where $\underline{A} = [\underline{a_{ij}}]$ and $\overline{A} = [\overline{a_{ij}}]$. We call $A \in A^I$ interval matrix; $\underline{A}$ lower boundary matrix; and $\overline{A}$ upper boundary matrix. Moreover, we define vertex matrices of A^I such as $A^v = \{A = [a_{ij}] : \forall a_{ij} \in \{\underline{a_{ij}}, \overline{a_{ij}}\}\}$. Thus, the FO-ILTI system have parametric interval uncertainties in elements of A matrix. The robust stability problem of $0 < \alpha < 1$ was studied in (Chen et al., 2005a); so this paper focuses on the robust stability of $1 < \alpha < 2$, which was also studied in (Ahn et al., 2007) with some restrictions. Let us use Caputo definition for fractional derivative of order α of any function $f(t)$, which allows utilization of initial values of classical integer-order derivatives with known physical interpretations (Caputo, 1967; Podlubny, 1999a):

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \qquad (2)$$

where n is an integer satisfying $n - 1 < \alpha \le n$. As commented in (Moze and Sabatier, 2005), with $1 \le \alpha < 2$, when A matrix is deterministic without uncertainty, the stability condition for $\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t)$ is clearly

$$\min_i |\arg(\lambda_i(A))| > \alpha\pi/2, \quad i = 1, 2, \cdots, N. \qquad (3)$$

Thus, the robust stability condition of FO-ILTI systems is derived as follows:

$$\min_i |\arg(\lambda_i(A))| > \alpha\pi/2, \quad i = 1, 2, \cdots, N; \quad \forall A \in A^I. (4)$$

For more detailed introduction to the robust stability of FO-ILTI systems, the interested readers are referred to (Ahn et al., 2007). Now we state our main result in the following section. The result of the following section is a comprehensive version of Section 2 of (Ahn and Chen, 2007).

3. MAIN RESULTS

Based on (Molinari, 1975) and (Henrion and Meinsma, 2001), it is easily proved that the FO-ILTI system is robust stable if and only if there exist positive definite Hermitian matrices $P = P^* > 0$ and $Q = Q^* > 0$ such that the following equality holds:

$$\beta PA + \beta^* A^T P = -Q, \ \forall\ A \in A^I \qquad (5)$$

where $\beta = \eta + j\zeta$, and η and ζ are defined from $\tan(\pi/2 - \theta) = \eta/\zeta$ with $\theta = \frac{(\alpha-1)\pi}{\pi}$ (see Fig. 1 of (Ahn et al., 2007)). In (Ahn et al., 2007), a sufficient condition, which considers $P = I$, was developed. The condition given in (5) is equivalent to $\beta PA + \beta^* A^T P < 0,\ P = P^* > 0,\ \forall\ A \in A^I$, which means that eigenvalues of $\beta PA + \beta^* A^T P$ are negative. Therefore, we know that equality (5) holds if and only if the maximum eigenvalue of $\beta PA + \beta^* A^T P$ is negative (i.e., $\overline{\lambda}(\beta PA + \beta^* A^T P) < 0$). Let us summarize the above argument in the following lemma:

Lemma 1. Interval fractional order LTI system is robust stable if and only if there exists a positive definite Hermitian matrix $P = P^*$ such that $\overline{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^I$.

However it is impossible to check the condition of the above lemma because there are infinite number of matrices A such that $A \in A^I$. In what follows, we present that a set of finite matrices can be used for checking the condition of Lemma 1.

Let us first notice that since $\beta PA + \beta^* A^T P$ is a Hermitian matrix for any $A \in A^I$, the maximum eigenvalue is calculated as

$$\overline{\lambda} = \max_{A \in A^I} \left(\max_{\|x\|=1} x^*(\beta PA + \beta^* A^T P)x \right) \qquad (6)$$

where x is a length n column vector, $x = [x_1, x_2, \ldots, x_n]^T = [u_1 + jv_1, u_2 + jv_2, \ldots, u_n + jv_n]^T$. Here note that since the vector x can be normalized, we can enforce $v_1 = 0$. Let us expand (6) like (10). In (10), $\mathrm{Re}(\cdot)$ means the real part of $(\cdot)$; $\mathrm{Im}(\cdot)$ means the imaginary part of $(\cdot)$; $u = [u_1, u_2, \ldots, u_n]^T$ and $v = [v_1, v_2, \ldots, v_n]^T$. If we denote $P = C + jD$, then we can rewrite the right-hand side of (10) like below:

$$\begin{aligned} &2\eta u^T CAu + 2\eta v^T DAu - 2\eta u^T DAv + 2\eta v^T CAv \\ &-2\zeta u^T DAu - 2\zeta u^T CAv + 2\zeta v^T CAu - 2\zeta v^T DAv \end{aligned} \qquad (11)$$

Using $(CA)_{ij} = \sum_{k=1}^n c_{ik} a_{kj}$ and $(DA)_{ij} = \sum_{k=1}^n d_{ik} a_{kj}$, we rewrite (11) like (12) (note that (12) is on the next page).

Now defining

$$\begin{aligned} \alpha(k) = {} & \eta u_1 c_{1k} u_1 - \zeta u_1 d_{1k} u_1 + \sum_{i=2}^n (\eta u_i c_{ik} u_1 - \zeta u_i d_{ik} u_1) \\ & + \sum_{i=2}^n (\eta v_i d_{ik} u_1 + \zeta v_i c_{ik} u_1) \end{aligned} \qquad (13)$$

$$\begin{aligned} \beta(k, j) = {} & u_1(\eta c_{1k} - \zeta d_{1k}) u_j \\ & + \sum_{i=2}^n u_i(\eta c_{ik} - \zeta d_{ik}) u_j + \sum_{i=2}^n v_i(\eta c_{ik} - \zeta d_{ik}) v_j \\ & - u_1(\eta d_{1k} + \zeta c_{1k}) v_j - \sum_{i=2}^n u_i(\eta d_{ik} + \zeta c_{ik}) v_j \\ & + \sum_{i=2}^n v_i(\eta d_{ik} + \zeta c_{ik}) u_j, \end{aligned} \qquad (14)$$

we can simplify the right-hand side of (12) as

$$x^*(\beta PA + \beta^* A^T P)x = 2\sum_{k=1}^n \alpha(k) a_{k1} + 2\sum_{k=1}^n \sum_{j=2}^n \beta(k, j) a_{kj} \qquad (15)$$

It is required to maximize the right-hand side of (15) to find the maximum eigenvalue $(\overline{\lambda})$ of $\beta PA + \beta^* A^T P$ considering all possible interval uncertainties in $a_{ij} \in a_{ij}^I$. Here, we observe that $\overline{\lambda}$ depends on the signs of $\alpha(k)$ and $\beta(k, j)$. That is, if $\alpha(k) \ge 0$, then $\overline{\lambda}$ occurs at $\overline{a_{k1}}$; otherwise $\overline{\lambda}$ occurs at $\underline{a_{k1}}$. In the same way, if $\beta(k, j) \ge 0$, then $\overline{\lambda}$ occurs at $\overline{a_{kj}}$; otherwise $\overline{\lambda}$ occurs at $\underline{a_{kj}}$. We summarize this observation in the following lemma:

Lemma 2. For a positive definite Hermitian $P = P^*$, the maximum of the quadratic form $x^*(\beta PA + \beta^* A^T P)x$ given in (6) occurs as one of the vertex matrices of $A \in A^I$.

$$
\begin{aligned}
x^*(\beta PA+\beta^* A^T P)x &= [u^T - jv^T][(\eta+j\zeta)PA+(\eta-j\zeta)A^TP][u+jv] && (7)\\
&= \eta u^TPAu + j\eta u^TPAv - j\eta v^TPAu + \eta v^TPAv + \eta u^TA^TPu + j\eta u^TA^TPv - j\eta v^TA^TPu\\
&\quad +\eta v^TA^TPv + j\zeta u^TPAu - \zeta u^TPAv + \zeta v^TPAu + j\zeta v^TPAv - j\zeta u^TA^TPu + \zeta u^TA^TPv\\
&\quad -\zeta v^TA^TPu - j\zeta v^TA^TPv && (8)\\
&= \eta u^TPAu + j\eta u^TPAv - j\eta v^TPAu + \eta v^TPAv + \eta(u^TPAu)^* + j\eta(v^TPAu)^* - j\eta(u^TPAv)^*\\
&\quad +\eta(v^TPAv)^* + j\zeta u^TPAu - \zeta u^TPAv + \zeta v^TPAu + j\zeta v^TPAv - j\zeta(u^TPAu)^* + \zeta(v^TPAu)^*\\
&\quad -\zeta(u^TPAv)^* - j\zeta(v^TPAv)^* && (9)\\
&= 2\eta\mathrm{Re}[u^TPAu] + 2\eta\mathrm{Im}[v^TPAu] - 2\eta\mathrm{Im}[u^TPAv] + 2\eta\mathrm{Re}[v^TPAv] - 2\zeta\mathrm{Im}[u^TPAu]\\
&\quad -2\zeta\mathrm{Re}[u^TPAv] + 2\zeta\mathrm{Re}[v^TPAu] - 2\zeta\mathrm{Im}[v^TPAv] && (10)
\end{aligned}
$$

$$
\begin{aligned}
x^*(\beta PA+\beta^* A^T P)x &= 2\sum_{j=1}^n\sum_{i=1}^n u_i\left(\sum_{k=1}^n \eta c_{ik}a_{kj} - \zeta d_{ik}a_{kj}\right)u_j + 2\sum_{j=2}^n\sum_{i=2}^n v_i\left(\sum_{k=1}^n \eta c_{ik}a_{kj} - \zeta d_{ik}a_{kj}\right)v_j\\
&\quad -2\sum_{j=2}^n\sum_{i=1}^n u_i\left(\sum_{k=1}^n \eta d_{ik}a_{kj} + \zeta c_{ik}a_{kj}\right)v_j + 2\sum_{j=1}^n\sum_{i=2}^n v_i\left(\sum_{k=1}^n \eta d_{ik}a_{kj} + \zeta c_{ik}a_{kj}\right)u_j\\
&= 2u_1\left(\sum_{k=1}^n \eta c_{1k}a_{k1} - \zeta d_{1k}a_{k1}\right)u_1 + 2\sum_{i=2}^n u_i\left(\sum_{k=1}^n \eta c_{ik}a_{k1} - \zeta d_{ik}a_{k1}\right)u_1\\
&\quad +2\sum_{j=2}^n u_1\left(\sum_{k=1}^n \eta c_{1k}a_{kj} - \zeta d_{1k}a_{kj}\right)u_j + 2\sum_{j=2}^n\sum_{i=2}^n u_i\left(\sum_{k=1}^n \eta c_{ik}a_{kj} - \zeta d_{ik}a_{kj}\right)u_j\\
&\quad +2\sum_{j=2}^n\sum_{i=2}^n v_i\left(\sum_{k=1}^n \eta c_{ik}a_{kj} - \zeta d_{ik}a_{kj}\right)v_j - 2\sum_{j=2}^n u_1\left(\sum_{k=1}^n \eta d_{1k}a_{kj} + \zeta c_{1k}a_{kj}\right)v_j\\
&\quad -2\sum_{j=2}^n\sum_{i=2}^n u_i\left(\sum_{k=1}^n \eta d_{ik}a_{kj} + \zeta c_{ik}a_{kj}\right)v_j + 2\sum_{i=2}^n v_i\left(\sum_{k=1}^n \eta d_{ik}a_{k1} + \zeta c_{ik}a_{k1}\right)u_1\\
&\quad +2\sum_{j=2}^n\sum_{i=2}^n v_i\left(\sum_{k=1}^n \eta d_{ik}a_{kj} + \zeta c_{ik}a_{kj}\right)u_j && (12)
\end{aligned}
$$

Proof. We need to maximize the following summation:

$$\sum_{k=1}^n \alpha(k)a_{k1} + \sum_{k=1}^n\sum_{j=2}^n \beta(k,j)a_{kj}$$

considering all $x = [u + jv]$, which satisfies $\|x\| = 1$, and all $a_{ij} \in a_{ij}^I = [\underline{a_{ij}}, \overline{a_{ij}}]$. Noticing that $\alpha(k)$ and $\beta(k,j)$ depend on $x = [u+jv]$, let us select a particular $x^\dagger$, $\|x^\dagger\| = 1$, which determines $\alpha(1) = \alpha(1)^\dagger$, $\ldots$, $\alpha(n) = \alpha(n)^\dagger$ and $\beta(1,1) = \beta(1,1)^\dagger$, $\ldots$, $\beta(n,n) = \beta(n,n)^\dagger$. Then, for the particular $x^\dagger$ we obtain:

$$
\begin{aligned}
&\max_{a_{ij}\in a_{ij}^I}\left(\sum_{k=1}^n \alpha(k)a_{k1} + \sum_{k=1}^n\sum_{j=2}^n \beta(k,j)a_{kj}\right)\\
&= \sum_{k=1}^n \alpha(k)\overrightarrow{a_{k1}(S_{\alpha(k)})} + \sum_{k=1}^n\sum_{j=2}^n \beta(k,j)\overrightarrow{a_{kj}(S_{\beta(k,j)})} \quad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
\overrightarrow{a_{k1}(S_{\alpha(k)})} &= \begin{cases} \overline{a_{k1}}, & \text{if } \alpha(k) \geq 0;\\ \underline{a_{k1}}, & \text{if } \alpha(k) < 0 \end{cases};\\
\overrightarrow{a_{kj}(S_{\beta(k,j)})} &= \begin{cases} \overline{a_{kj}}, & \text{if } \beta(k,j) \geq 0;\\ \underline{a_{kj}}, & \text{if } \beta(k,j) < 0 \end{cases}. \quad (17)
\end{aligned}
$$

Therefore, since at any arbitrary selection x, the maximum of $\sum_{k=1}^n \alpha(k)a_{k1} + \sum_{k=1}^n\sum_{j=2}^n \beta(k,j)a_{kj}$ occurs at one of vertex matrices of $A \in A^I$, the maximum of $x^*(\beta PA + \beta^* A^T P)x$ occurs at one of vertex matrices of $A \in A^I$.

Now based on Lemma 1 and Lemma 2, we state the following theorem:

Theorem 3. Interval fractional order LTI system is robust stable if and only if there exists a positive definite Hermitian matrix $P = P^*$ such that $\overline{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$.

Proof. (Sufficient) Based on Lemma 2, since $\overline{\lambda}(\beta PA + \beta^* A^T P)$ occurs at one of vertex matrices of A^I, $\lambda_i(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^I$ if $\lambda_i(\beta PA + \beta^* A^T P) \leq \overline{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$. Therefore if there exists a positive definite Hermitian matrix $P = P^*$ such that $\overline{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$, then interval FO-LTI system is robust stable by Lemma 1.

(Necessary) Since there should exist $P = P^*$ such that $\overline{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^I$, it is necessary to ensure the existence of $P = P^*$ such that $\overline{\lambda}(\beta PA + \beta^* A^T P) < 0$ for all $A \in A^v$ because $A^v \subseteq A^I$.

The superiority of Theorem 3 over Lemma 1 is highlighted in the following remark.

Remark 4. Lemma 1 states that we need to check infinity number of matrices $A \in A^I$ to verify the existence of $P = P^*$ such that (5) holds. However Theorem 3 shows that a set of selected finite vertex matrices can be enough for checking the existence of $P = P^*$. Therefore a selected finite vertex matrices can be used for checking the robust stability of FO-LTI interval systems.

4. ILLUSTRATIVE EXAMPLES

4.1 Example-1

Consider the following fractional-order linear interval system, which was studied in (Ahn et al., 2007):

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax,\ A \in A^I \tag{18}$$

where $\alpha = 1.5$, which makes $\beta = \eta + j\zeta = 1 + j$, and $A \in A^I = [\underline{A}, \overline{A}]$ with

$$\underline{A} = \begin{pmatrix} -1.8 & 0.4 & 0.8 \\ -1.2 & -3.6 & 0.8 \\ -0.6 & -1.8 & -3.0 \end{pmatrix};\ \overline{A} = \begin{pmatrix} -1.2 & 0.6 & 1.2 \\ -0.8 & -2.4 & 1.2 \\ -0.4 & -1.2 & -2.0 \end{pmatrix}$$

From Theorem 3, we need to check all vertex matrices and for individual vertex matrix $A \in A^v$, there should exist $P = P^* > 0$ such that $\beta PA + \beta^* A^T P < 0$. The existence of $P = P^* > 0$ can be checked by LMI formulation. However, the system considered in this paper is complex system; thus the standard LMI approach should be reformulated based on the following fact [1]:

Fact 5. A complex Hermitian H is $H < 0$ if and only if $\begin{pmatrix} \mathrm{Re}(H) & \mathrm{Im}(H) \\ -\mathrm{Im}(H) & \mathrm{Re}(H) \end{pmatrix} < 0.$

Therefore, if there exists $P = P^* > 0$ such that the following holds

$$PB + B^* P < 0 \tag{19}$$

where $B = \begin{pmatrix} \mathrm{Re}(A) & \mathrm{Im}(A) \\ -\mathrm{Im}(A) & \mathrm{Re}(A) \end{pmatrix}$, $\forall A \in A^v$, then we can conclude that the FO-interval LTI system is robust stable. The above condition can be easily checked using MATLAB LMI commands `setlmis, lmivar, lmiterm, getlmis, feasp, dec2mat` . Using the algorithm given in Fig. 1, we find that there exists $P = P^*$ such that inequality (19) hold for all $A \in A^v$. For example, when $A = \underline{A}$, we obtain the following symmetric matrix:

$$\begin{pmatrix} 0.6224 & 0.0264 & 0.0439 & 0.0000 & 0.0900 & 0.1144 \\ 0.0264 & 0.3861 & -0.0525 & -0.0900 & 0.0000 & 0.1573 \\ 0.0439 & -0.0525 & 0.3978 & -0.1144 & -0.1573 & 0.0000 \\ 0.0000 & -0.0900 & -0.1144 & 0.6224 & 0.0264 & 0.0439 \\ 0.0900 & 0.0000 & -0.1573 & 0.0264 & 0.3861 & -0.0525 \\ 0.1144 & 0.1573 & 0.0000 & 0.0439 & -0.0525 & 0.3978 \end{pmatrix}$$

whose eigenvalues are 0.1868, 0.1868, 0.5128, 0.5128, 0.7068, 0.7068, and when $A = \overline{A}$, we obtain the following symmetric matrix:

$$\begin{pmatrix} 0.8575 & 0.1313 & 0.1613 & -0.0000 & 0.1332 & 0.3652 \\ 0.1313 & 0.7062 & -0.0051 & -0.1332 & 0.0000 & 0.5039 \\ 0.1613 & -0.0051 & 1.0618 & -0.3652 & -0.5039 & -0.0000 \\ -0.0000 & -0.1332 & -0.3652 & 0.8575 & 0.1313 & 0.1613 \\ 0.1332 & 0.0000 & -0.5039 & 0.1313 & 0.7062 & -0.0051 \\ 0.3652 & 0.5039 & -0.0000 & 0.1613 & -0.0051 & 1.0618 \end{pmatrix}$$

whose eigenvalues are 0.2437, 0.2437, 0.7653, 0.7653, 1.6165, 1.6165.

4.2 Example-2

Suppose we are given

$$\underline{A} = \begin{pmatrix} -1.8 & 0.4 & 0.8 \\ -1.2 & -3.6 & 0.8 \\ -0.6 & -1.8 & -3.0 \end{pmatrix};\ \overline{A} = \begin{pmatrix} 1.2 & 0.6 & 1.2 \\ -0.8 & -2.4 & 1.2 \\ -0.4 & -1.2 & -2.0 \end{pmatrix}$$

Using the same algorithm given in Fig. 1, however we find that there does not exist positive definite matrix P when

$$A = \begin{pmatrix} 1.2 & 0.4 & 0.8 \\ -1.2 & -3.6 & 0.8 \\ -0.6 & -1.8 & -3.0 \end{pmatrix} \in A^v$$

Therefore, the system is not robustly stable.

5. CONCLUSIONS

This paper presented an exact robust stability condition of fractional-order interval linear systems without conservatism. The motivation of this paper is to remove conservatism of our existing result (Ahn et al., 2007). Using the existence condition of Hermitian matrix $P = P^*$ for a complex Lyapunov inequality, we showed that a fractional-order interval linear system is robustly stable if and only if there exist Hermitian matrices $P = P^*$ such that complex Lyapunov inequalities are satisfied for all vertex matrices. The existence of $P = P^* > 0$ was checked by LMI formulation. However, the LMI systems considered in this paper were complex systems; thus the standard LMI approach was reformulated. Two numerical examples were presented to verify the validity of the proposed approach.

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[1] See MATLAB LMI toolbox

```
clear all

zerobin = dec2bin(0)
beta = 1+j;
Al=[-1.8, 0.4, 0.8; -1.2, -3.6, 0.8; -0.6, -1.8, -3.0];
Au=[-1.2, 0.6, 1.2; -0.8, -2.4, 1.2; -0.4, -1.2, -2.0];
Ar = (Au - Al) ;

for ii=0:1:(2^9-1)
    setlmis([])
    tt=dec2bin(ii,9)
    pp=0;
    for jj=1:1:3
        for kk=1:1:3
            pp=pp+1;
            if tt(pp)==zerobin
                Aadded(jj,kk) = 0;
            else
                Aadded(jj,kk) = Ar(jj,kk);
            end
        end
    end

    AAA= Al+Aadded;
     AA = beta*AAA ;

    A = [real(AA),imag(AA) ; -imag(AA), real(AA)]

    X = lmivar(1,[6 1]) ;
    lmiterm([1 1 1 X], 1, A);
    lmiterm([1 1 1 X], A', 1);
    lmiterm([1 1 1 0],0);
    lmiterm([-2 1 1 X], 1, 1);
    lmis = getlmis;
    [tmin, xfeas] = feasp(lmis);

    X = dec2mat(lmis,xfeas,X) ;
    if min(eig(X))<0
        disp('Not stable')
        break
    end

    clear X
end
```

Fig. 1. LMI formulation for robust stability test of fractional-order interval linear time invariant systems (FO-ILTI).

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