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Analysis and Control of Time Delayed Systems via the Lambert W Function ⋆

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Abstract: This paper summarizes recent research results by the authors for the analytical solution to systems of delay differential equations using the matrix Lambert W function, and its applications to analysis and control of time-delay systems. The solution has the form of an infinite series of modes written in terms of the matrix Lambert W function. This solution is analytical in terms of the parameters, coefficients and delay time, of the system, and each eigenvalue in the infinite eigenspectrum is distinguished in terms of the branches of the Lambert W function. This enables extension of methods for systems of ordinary differential equations to systems of delay differential equations. These include stability analysis, controllability and observability, as well as methods for eigenvalue assignment.

1. INTRODUCTION

Time-delay systems (TDS) arise from an inherent time delay in the components of the system or a deliberate introduction of time delay into the system for control purposes. Time delays occur often in electrical, mechanical, biological, metallurgical, and chemical systems. Such time-delay systems can be represented by delay differential equations (DDEs), which belong to the class of functional differential equations, and have been extensively studied over the past decades [Richard (2003)].

The principal difficulty in studying DDEs is that such equations always lead to an infinite spectrum of eigenvalues. The determination of this spectrum requires a corresponding determination of roots of certain analytic, but infinite-dimensional, equations. Therefore, typically one needs graphical, numerical, and/or approximate methods. Widely used approximation methods are the rational approximations (e.g., Padé approximation), which treat an infinite-dimensional system like a finite-dimensional one [Richard (2003)]. However, such approximations, and other methods, have limits in analyzing or controlling time-delay systems (see, e.g., Richard (2003), Silva et al. (2001), and Yi et al. (2008b))

An analytic approach to solving systems of DDEs based on the concept of the Lambert W function, which has been known to be useful to analyze DDEs [Corless et al. (1996)], was developed by Asl and Ulsoy (2003). Unlike other existing methods, the solution has an analytical form expressed in terms of the parameters of the DDE. One can explicitly determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is distinguished in terms of the branch of the Lambert W function. However, their approach was only correct in the scalar case and in the special case where certain matrices in the systems of DDEs commute. In Yi and Ulsoy (2006), the analytical approach was extended to general systems of DDEs and to non-homogeneous DDEs. The method has been validated, for stability [Yi et al. (2007a)], and for free and forced responses, by comparison to numerical integration [Yi and Ulsoy (2006)]. As shown in Table 1, the approach using the Lambert W function provides a solution form for DDEs similar to that to the free and forced solution of linear time-invariant ordinary differential equations (ODEs) in terms of the state transition matrix. This analogy enables extensions of the methods for systems of ODEs to systems of DDEs as shown by our recent results.

This paper summarizes recent results by the authors on the analytical solution of systems of linear time-invariant (LTI) delay differential equations with a single delay using the matrix Lambert W function. All the results presented in this paper in summary form have previously been presented in detail in Yi et al. (2007b), Yi et al. (2007a), Yi et al. (2008a), Yi et al. (2008b), and Yi et al. (2008c). For a given system of DDEs, the analytical solution is derived in terms of the matrix Lambert W function Yi et al. (2007b). From the solution form, the stability of the system is determined [Yi et al. (2007a)] and controllability and observability is analyzed [Yi et al. (2008a)]. For a point-wise controllable system, a linear feedback controller is designed via eigenvalue assignment [Yi et al. (2008b)], which can be extended to robust controllers and time-domain specifications [Yi et al. (2008c)] (see Figure 1). We present here a comprehensive discussion and perspective on these related topics.

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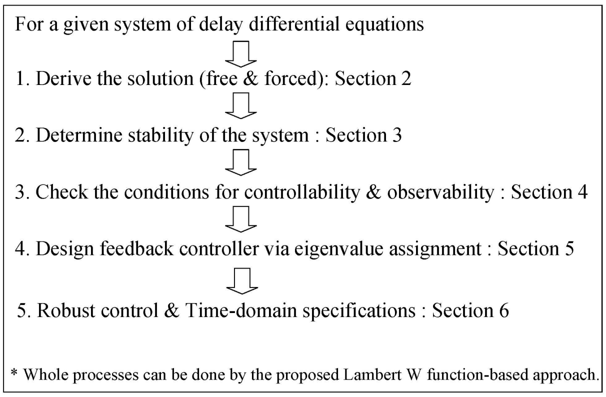

Fig. 1. The matrix Lambert W function-based approach: using the Lambert W function, these whole processes, which are standard for systems of ODEs, are tractable.

2. SOLUTION USING THE MATRIX LAMBERT W FUNCTION

Consider a linear time-invariant system of delay differential equations with a single constant delay, h,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t), & t > 0 \\ \mathbf{x}(t) &= \mathbf{g}(t), & t \in [-h, 0) \\ \mathbf{x}(t) &= \mathbf{x}_0, & t = 0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{A_d}$ are $n \times n$ coefficient matrices, and $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{B}$ is an $n \times r$ matrix, $\mathbf{u}(t)$, an $r \times 1$ vector, is a function representing the external excitation, and $\mathbf{g}(t)$ and $\mathbf{x}_0$ are a specified preshape function and an initial point, respectively. The output matrix $\mathbf{C}$ is $p \times n$ and $\mathbf{y}(t)$ is a $p \times 1$ measured output vector. The existence and uniqueness of the solution for the system of linear DDEs in (1) was studied by Hale and Lunel (1993).

First we assume a free solution form, i.e., $\mathbf{u}(t) = \mathbf{0}$, as

$$\mathbf{x}(t) = e^{\mathbf{S}t}\mathbf{x}_0 \tag{2}$$

where $\mathbf{S}$ is $n \times n$ matrix. In the usual case, the characteristic equation for (1) is obtained from the equation by looking for nontrivial solution of the form $e^{st}\mathbf{C}$ where 's' is a scalar variable and $\mathbf{C}$ is constant (e.g., Bellman and Cooke (1963) and Hale and Lunel (1993)). However, such an approach does not lead to an interesting result, nor does it help in deriving a solution to systems of DDEs in (1). Alternatively, one could assume the form of (2) to derive the solution to systems of DDEs in (1) using the matrix Lambert W function. Substitution of (2) into (1) enables one to obtain a homogeneous solution given by

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I \tag{3}$$

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k) + \mathbf{A} \tag{4}$$

The constant matrices $\mathbf{C}_k^I$ in (3) are computed from a given preshape function $\mathbf{g}(t)$, and an initial point, $\mathbf{x}_0$. The matrix, $\mathbf{Q}_k$, is obtained from the following condition, that can be used to solve for the unknown matrix, $\mathbf{Q}_k$,

$$\mathbf{W}_k(\mathbf{A}_d h\mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)+\mathbf{A}h} = \mathbf{A_d}h \tag{5}$$

Our result, in numerous examples studied to date, always yields a unique solution, $\mathbf{Q}_k$, from the numerical solution to (5) for each k, the branch of the Lambert W function. However, conditions for existence and uniqueness of such a solution are still required. Note that $\mathbf{W}_k$ in (4) denotes the matrix Lambert W function which satisfies the definition, [Asl and Ulsoy (2003)]

$$\mathbf{W}_k(\mathbf{H}_k)e^{\mathbf{W}_k(\mathbf{H}_k)} = \mathbf{H}_k \tag{6}$$

The matrix Lambert W function, $\mathbf{W}_k(\mathbf{H}_k)$, is complex valued, with a complex argument, $\mathbf{H}_k$, and has an infinite number of branches, $\mathbf{W}_k(\mathbf{H}_k)$, where $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$ [Corless et al. (1996)]. Corresponding to each branch, k, of the matrix Lambert W function, $\mathbf{W}_k$, there is a solution $\mathbf{Q}_k$ from (5), and for $\mathbf{H}_k = \mathbf{A_d}h\mathbf{Q}_k$, we compute the Jordan canonical form $\mathbf{J}_k$ from $\mathbf{H}_k = \mathbf{Z}_k\mathbf{J}_k\mathbf{Z}_k^{-1}$. $\mathbf{J}_k = \text{diag}(J_{k1}(\hat{\lambda}_1), J_{k2}(\hat{\lambda}_2), \ldots, J_{kp}(\hat{\lambda}_p))$, where $J_{ki}(\hat{\lambda}_i)$ is $m \times m$ Jordan block and m is multiplicity of the eigenvalue, $\hat{\lambda}_i$. Then, the matrix Lambert W function can be computed as

$$\begin{aligned} &\mathbf{W}_k(\mathbf{H}_k) = \\ &\mathbf{Z}_k \left\{ \text{diag}\left(\mathbf{W}_k(J_{k1}(\hat{\lambda}_1)), \ldots, \mathbf{W}_k(J_{kp}(\hat{\lambda}_p)) \right) \right\} \mathbf{Z}_k^{-1} \end{aligned} \tag{7}$$

where

$$\mathbf{W}_k(J_{ki}(\hat{\lambda}_i) = \begin{bmatrix} W_k(\hat{\lambda}_i) & W_k'(\hat{\lambda}_i) & \cdots & \frac{1}{(m-1)!}W_k^{(m-1)}(\hat{\lambda}_i) \\ 0 & W_k(\hat{\lambda}_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k(\hat{\lambda}_i) \end{bmatrix} \tag{8}$$

The principal and other branches of the Lambert W function in (8) can be calculated analytically using a series expansion [Corless et al. (1996)], or alternatively, using commands already embedded in the various commercial software packages, such as Matlab, Maple, and Mathematica.

When $\mathbf{u}(t) \neq \mathbf{0}$ in (1), the solution in (3) can be extended to the form, [Yi et al. (2007b)]

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^N\mathbf{B}\mathbf{u}(\xi)d\xi \tag{9}$$

The coefficient $\mathbf{C}_k^I$ in (9) is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and the preshape function, $\mathbf{g}(t)$, and the initial point, $\mathbf{x}_0$, while $\mathbf{C}_k^N$ is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and does not depend on $\mathbf{g}(t)$ or $\mathbf{x}_0$. The numerical and analytical methods for computing $\mathbf{C}_k^I$ and $\mathbf{C}_k^N$ were developed respectively in Asl and Ulsoy (2003) and Yi et al. (2006). Conditions for convergence of the infinite series in (9) have been studied in Banks and Manitius (1975), Bellman and Cooke (1963), Hale and Lunel (1993), and Lunel (1989). For example, for a bounded external excitation, $\mathbf{u}(t)$, if the coefficient matrix, $\mathbf{A_d}$, is nonsingular, the infinite series converges to the solution.

Table 1. Comparison of the solutions to ODEs and DDEs. The solution to DDEs in terms of the Lambert W function shows a formal semblance to that of ODEs [Yi et al. (2007b)].

ODEs	DDEs
Scalar Case	
$\dot{x}(t) = ax(t) + bu(t), \quad t > 0$	$\dot{x}(t) = ax(t) + a_d x(t-h) + bu(t), \quad t > 0$
$x(t) = x_0, \quad t = 0$	$x(t) = g(t), \quad t \in [-h, 0)\ ;\ x(t) = x_0, \quad t = 0$
$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\xi)} bu(\xi) d\xi$	$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\xi)} C_k^N bu(\xi) d\xi$ where, $S_k = \frac{1}{h} W_k(a_d h e^{-ah}) + a$
Matrix-Vector Case	
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad t > 0$	$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t), \quad t > 0$
$\mathbf{x}(t) = \mathbf{x}_0, \quad t = 0$	$\mathbf{x}(t) = \mathbf{g}(t), \quad t \in [-h, 0)\ ;\ \mathbf{x}(t) = \mathbf{x}_0, \quad t = 0$
$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\xi)} \mathbf{B}\mathbf{u}(\xi) d\xi$	$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t} \mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)} \mathbf{C}_k^N \mathbf{B}\mathbf{u}(\xi) d\xi$ where, $\mathbf{S}_k = \frac{1}{h} \mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) + \mathbf{A}$

From the Laplace transform of the system (1), the solution to (1) in the Laplace domain is

$$\mathbf{X}(s) = \underbrace{\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1} \left\{\mathbf{x}_0 + \mathbf{A_d}\mathbf{G}(s)e^{-sh}\right\}}_{\text{free}} + \underbrace{\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1} \left\{\mathbf{B}\mathbf{U}(s)\right\}}_{\text{forced}} \tag{10}$$

Comparing the solution in the Laplace domain in (10) with the solution in the time domain in terms of the matrix Lambert W function in (9) yields,

$$\mathfrak{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1}\right\} = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t} \mathbf{C}_k^N \tag{11}$$

Note that, compared with results by other existing methods for the series solutions to DDEs in Banks and Manitius (1975), Bellman and Cooke (1963), Lunel (1989), where eigenvalues are obtained from exhaustive numerical computation, the solution in terms of the Lambert W function has an analytical form expressed in terms of the parameters, $\mathbf{A}$, $\mathbf{A_d}$ and h, of the DDE in (1). Hence, one can determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is distinguished by k, which indicates the branch of the Lambert W function. The solution to DDEs in terms of the Lambert W function, is analogous to that of ODEs in terms of the state transition matrix as summarized in Table 1.

3. STABILITY OF TIME-DELAY SYSTEMS

The solution form in (9) with (4) reveals that the stability condition of the systems of (1) depend on the eigenvalues of the matrix $\mathbf{S}_k$, and, thus, also on the matrix $e^{\mathbf{S}_k}$. A time delayed system characterized by (9) is *asymptotically stable* if and only if all the eigenvalue of $\mathbf{S}_k$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, have negative real parts or, equivalently in the sense of Lyapunov, all the eigenvalue of $e^{\mathbf{S}_k}$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, lie within the unit circle. However, computing the matrix $\mathbf{S}_k$ or $e^{\mathbf{S}_k}$ for an infinite number of branches is not practical. We have observed, in numerous examples, that if coefficient matrix $\mathbf{A_d}$ does not have repeated zero eigenvalues, then, the eigenvalues of $\mathbf{S}_k$ obtained using the principal branch ($k = 0$) are closest to the imaginary axis and, thus, determine stability of the system. That is, [Yi et al. (2007a)]

$$\max\{\Re\{\text{eigenvalues for } k = 0\}\} \geq \Re\{\text{all other eigenvalues}\} \tag{12}$$

For the scalar DDE case, it has been proven that the root obtained with the principal branch always determines stability [Shinozaki and Mori (2006)], using monotonicity of the Lambert W function with resect to the branch of the Lambert W function. Such a proof can readily be extended to systems of DDEs where $\mathbf{A}$ and $\mathbf{A_d}$ are simultaneously triangularizable and, thus, commute with each other. Even though such a proof is not available in the case of general matrix-vector DDEs, we have observed such behavior in all the examples we have considered.

With this useful observation, the approach based on the matrix Lambert W function is applied to solve a problem in a machining process.

Example - Regenerative chatter in the turning process [Yi et al. (2007a)]: The linearized chatter equation can be expressed in state space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t - T) \tag{13}$$

where $\mathbf{x} = \{x \quad \dot{x}\}^T$ and ^{T} indicates transpose, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\left(1 + \frac{k_c}{k_m}\right)\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad \mathbf{A_d} = \begin{bmatrix} 0 & 0 \\ \frac{k_c}{k_m}\omega_n^2 & 0 \end{bmatrix}. \tag{14}$$

Here, $\mathbf{A}$ and $\mathbf{A_d}$ are the linearized coefficient matrices of the process model and are functions of the machine-tool and workpiece structural parameters such as natural

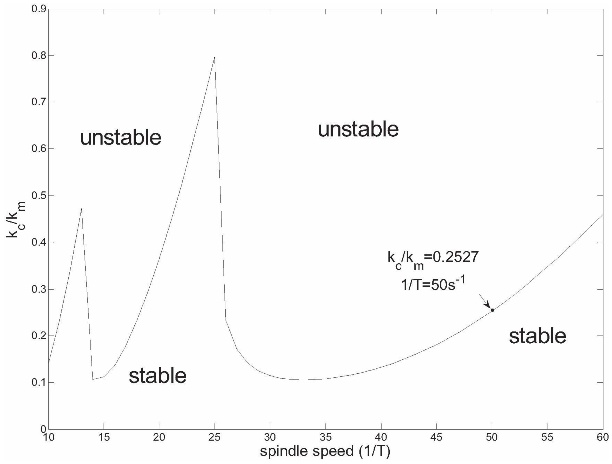

Fig. 2. Stability lobes for the chatter equation [Yi et al. (2007a)]

frequency (ω_n), and damping ratio (ζ). Time delay, T, which is equivalent to h in (1), is the inverse of the spindle speed of the workpiece. The ratio k_c/k_m depends on the depth-of-cut, as well as the workpiece material properties. If we observe the roots obtained using the principal branch, we can find the critical point when the roots cross the imaginary axis. For example, when spindle speed $(1/T) = 50(\sec^{-1})$, $\omega_n = 150(\sec^{-2})$ and $\zeta = 0.05$, the critical ratio of gains (k_c/k_m) is 0.2527 (see Figure 2). This value agrees with the result obtained by the Lyapunov method [Malek-Zavarei and Jamshidi (1987)], the Nyquist criterion and the computational method of [Chen et al. (1997)]. The stability lobes by this method are depicted in Figure 2 for given parameters of the system with respect to $1/T$, i.e., the spindle speed (*rps*, revolution per second). With these lobes, one can find the safe operating spindle speed, which does not leave any chatter marks on the surface of the workpiece.

In obtaining the result for stability of the system shown in the Figure 2, we note that the roots obtained using the principal branch always determine stability. One of the advantages of using the matrix Lambert W function over other methods appears to be the observation that the stability of the system can be obtained from only the principal branch among an infinite number of roots. The eigenvalues are expressed analytically in terms of the parameters of the systems. Therefore, one can know how each parameter affects the rightmost eigenvalues and stability.

4. CONTROLLABILITY AND OBSERVABILITY

Controllability and observability of linear time delay systems has been studied, and various definitions and criteria have been presented since the 1960s. For a detailed review, refer to Malek-Zavarei and Jamshidi (1987), Richard (2003), and Yi et al. (2008a). However, the lack of an analytical solution approach has limited the applicability of the existing theory. Using the solution form in terms of the matrix Lambert W function, algebraic conditions and Gramians for controllability and observability of DDEs were derived by Yi et al. (2008a) in a manner analogous to the well-known controllability and observability results for the ODE case.

The system (1) is *point-wise controllable* (or equivalently, defined as *fixed-time completely controllable* or $\mathbb{R}^n$-*controllable to the origin* in other literature) if, for any given initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, there exists a time t_1, $0 < t_1 < \infty$, and an admissible (i.e., measurable and bounded on a finite time interval) control segment $\mathbf{u}(t)$ for $t \in [0, t_1]$ such that $\mathbf{x}(t_1; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}(t)) = \mathbf{0}$.

If a system (1) is point-wise complete, there exist a control which results in *point-wise controllability* in finite time of the solution of (1) for any initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, if and only if the controllability Gramian, $\mathcal{C}$, computed with the kernel has a full rank. Using the result in (11), the rank condition can be expressed as

$$\begin{aligned}\text{rank}\Bigg[\mathcal{C}(0,t_1) \equiv \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \\ \times\mathbf{B}\mathbf{B}^T\left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi\Bigg] = n\end{aligned} \quad (15)$$

Similarly, a rank criteria for observability was developed. The system of (1) is *point-wise observable*, (or equivalently, *observable* in other literature) in $[0, t_1]$ if the initial point, $\mathbf{x}_0$, can be uniquely determined from the knowledge of $\mathbf{u}(t)$, $\mathbf{g}(t)$, and $\mathbf{y}(t)$. Then, if and only if the observability Gramian $\mathcal{O}(0, t_1)$ computed with the kernel defined in (11) satisfies the condition, i.e.,

$$\begin{aligned}\text{rank}\Bigg[\mathcal{O}(0,t_1) \equiv \int_0^{t_1} \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T \\ \times\mathbf{C}^T\mathbf{C} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi\Bigg] = n\end{aligned} \quad (16)$$

the system of (1) is *point-wise observable*. These conditions were applied to determine whether a time-delay system is controllable/observable with examples, and to derive other algebraic conditions for point-wise controllability and observability.

The results presented agree with those obtained using previously existing algebraic methods. However, using the method of Gramians developed in [Yi et al. (2008a)], one can acquire more information. The controllability and observability Gramians in (15)-(16) indicate how controllable and observable the corresponding states are [Holford and Agathoklis (1996)], while algebraic conditions tell only whether a system is controllable/observable or not. With the condition using Gramian concepts, one can determine how the change in some specific parameters of the system or the delay time, h, affect the controllability and observability of the system via the changes in the Gramians. Furthermore, for systems of ODEs, a balanced realization in which the controllability Gramian and observability Gramian of a system are equal and diagonal was introduced in Moore (1981) and its existence was investigated in Verriest and Kailath (1983). By balancing a realization we mean that we symmetrize a certain input property (controllability) with a certain output property (observability) through a suitable choice of basis [Verriest and Kailath (1983)]. The significance of the method has been established because of its desirable properties such as good error bounds, computational simplicity, stability, and its close connection to robust multi-variable control

[Lu et al. (1987)]. However, for systems of DDEs, results on balanced realizations have been lacking. Using the Gramians defined in (15) and (16), the concept of the balanced realization has been extended to systems of DDEs for the first time [Yi et al. (2008a)].

5. EIGENVALUE ASSIGNMENT

Consider the system in (1) and a generalized feedback containing current and delayed state,

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) + \mathbf{K_d}\mathbf{x}(t-h) \tag{17}$$

Then, the closed-loop system becomes

$$\dot{\mathbf{x}}(t) = \{\mathbf{A} + \mathbf{BK}\}\mathbf{x}(t) + \{\mathbf{A_d} + \mathbf{BK_d}\}\mathbf{x}(t-h) \tag{18}$$

The controllability of such system, using the solution form of (9) was studied in Yi et al. (2008b). In the case of LTI systems of ODEs, if it is completely controllable, then the eigenvalues can be assigned by choosing the feedback gains for DDE's. The gains, $\mathbf{K}$ and $\mathbf{K_d}$ are determined as follows. First, select desired eigenvalues, $\lambda_{i,desired}$ for $i = 1, \cdots, n$, and set an equation so that the selected eigenvalues become those of the matrix $\mathbf{S}_0$ as

$$\lambda_i(\mathbf{S}_0) = \lambda_{i,desired} \tag{19}$$

for $i = 1, \cdots, n$, where, $\lambda_i(\mathbf{S}_0)$ is i^{th} eigenvalue of the matrix $\mathbf{S}_0$. Second apply the new two coefficient matrices $\mathbf{A} \equiv \mathbf{A} + \mathbf{BK}$ and $\mathbf{A_d} \equiv \mathbf{A_d} + \mathbf{BK_d}$, as (18) to (5) and solve numerically to obtain the matrix $\mathbf{Q}_0$ for the principal branch ($k = 0$). Note that $\mathbf{K}$ and $\mathbf{K_d}$ are unknown matrices with all unknown elements, and the matrix $\mathbf{Q}_0$ is a function of the unknown $\mathbf{K}$ and $\mathbf{K_d}$. For the third step, substitute the matrix $\mathbf{Q}_0$ from (5) into (4) to obtain $\mathbf{S}_0$ and its eigenvalues as the function of the unknown matrix $\mathbf{K}$ and $\mathbf{K_d}$. Finally, equation (19) with the matrix, $\mathbf{S}_0$, is solved for the unknown $\mathbf{K}$ and $\mathbf{K_d}$ using numerical methods, such as *fsolve* in MATLAB. Depending on the structure or parameters of given system, there exists limitation of the rightmost eigenvalues and some values are not proper for the rightmost eigenvalues. In that case, the above approach does not yield any solution for $\mathbf{K}$ and $\mathbf{K_d}$. To resolve the problem, one may try again with fewer desired eigenvalues, or different values of the desired rightmost eigenvalues. Then, the solution, $\mathbf{K}$ and $\mathbf{K_d}$, is obtained numerically for a variety of initial conditions by an empirical trial and error procedure.

Example - Eigenvalues assignment [Yi et al. (2008b)]: Consider the system in (1) with parameters,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A_d} = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad h = 0.1 \tag{20}$$

with $\mathbf{B} = [\,0\ 1\,]^T$. Before applying feedback, the eigenvalues for $k = 0$ are 0.1098 and −1.1183, and the system is unstable (see Fig. 3). The system is point-wise controllable by the criterion in Section 4, then, using the eigenvalue assignment method, the gains, $\mathbf{K}$ and $\mathbf{K_d}$, can be chosen to locate the eigenvalues at desired positions in the complex plane. For example, when the desired eigenvalues are −1.0000 and −6.0000, the computed gains are $\mathbf{K} = [-0.1391\ -1.8982]$ and $\mathbf{K_d} = [-0.1236\ -1.8128]$, or $\mathbf{K} = [-0.1687\ -3.6111]$ and $\mathbf{K_d} = [1.6231\ -0.9291]$ for −2.0000 and −4.0000. By applying the obtained feedback gains, the system is stabilized and the eigenvalues placed at a desired position (*) in the complex plane (see Fig. 3).

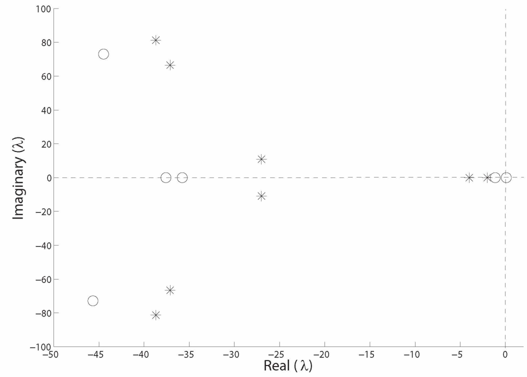

Fig. 3. Movement of eigenvalues after applying feedback (without feedback: o, with feedback:*). The rightmost eigenvalues are located at the exact desired location −2.0000 and −4.0000 with the computed feedback gains $\mathbf{K} = [-0.1687\ -3.6111]$ and $\mathbf{K_d} = [1.6231\ -0.9291]$ [Yi et al. (2008b)].

Many researchers have sought to establish the connection between point-wise controllability presented in Section 4 and eigenvalue assignment by linear feedback (not predictive) control for systems of DDEs as for ODEs [Tsoi (1978)]. Even though some partial results have been presented for functional controllability (see e.g., Vandevenne (1972) and the references therein) several theoretical results for point-wise controllability are not yet available. We have presented some examples for such a connection in Yi et al. (2008b). In that study, it has been shown by examples that if the system of DDEs is point-wise controllable, it is possible to design the linear feedback controllers via rightmost eigenvalue assignment for systems of DDEs as in (1); otherwise, it is not.

6. CONCLUDING REMARKS AND FUTURE WORK

Recent results by the authors on the solution of delay differential equation using the matrix Lambert W function and its applications are summarized in this paper. The main advantage of this method is that the solution in terms of the Lambert W function has an analytical form expressed in terms of the parameters, $\mathbf{A}$, $\mathbf{A_d}$ and h, of the DDE in (1). Hence, one can determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is distinguished by k, which indicates the branch of the Lambert W function. The method has been validated, for stability, and for free and forced responses, by comparison to numerical integration. The solution to DDEs in terms of the Lambert W function, is analogous to that of ODEs in terms of the state transition matrix. This suggests that some analyses used for systems of ODEs, based upon the concept of the state transition matrix, can potentially be extended to systems of DDEs. These include controllability and observability, methods for eigenvalues assignment for linear feedback controller design. Also, their extension to robust stability and time-domain specifications are tractable and are being studied by the authors [Yi et al. (2008c)]. Stability of time invariant linear DDEs can be determined using the approach and extension to stability of time-varying DDEs is also currently under investigation. Even though, for nonlinear systems of DDEs, the presented appraoch cannot be applied directly, via linearization (e.g., Piecewise Linear Approach in Sontag (1981)), it can be helpful to analyze such systems.

In this survey paper we have also introduced several outstanding research problems associated with the solution of systems of DDEs using the matrix Lambert W function. First, conditions for existence and uniqueness of a solution $\mathbf{Q}_k$ to (5) are needed. Second, a general proof that the stability of the systems in (1) can be determined by the principal ($k = 0$) branch is lacking. Third, the connection between point-wise controllability and eigenvalue assignment by linear feedback (not predictive) control for systems of DDEs is also another open problem. We hope that researchers in the DDE community will be interested in those problems.

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