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A Continuous-time Fixed-lag Smoother Converging in Finite Time

Bo Kyu Kwon * **Wook Hyun Kwon** *

* *The School of Electrical Engr. & Computer Science, Seoul National University, Seoul, Korea. (Tel: 82-2-873-4704; e-mail:{bkkwon,whkwon}@cisl.snu.ac.kr).*

Abstract: In this paper, we propose a new fixed-lag smoother that estimates the fixed-delayed state for a continuous-time stochastic system. The estimation error variance of the proposed smoother is minimized under the constraint that the estimated state converges to the real state exactly in finite time after noises or uncertainties disappear. For numerical computing, the proposed smoother is represented in a differential form. In order to achieve the convergence in finite time, any additional processes such as batch processing and sampling data through discrete-time techniques are not required. A numerical example is presented to illustrate the finite time convergence of the proposed smoother in comparison with the asymptotic convergence of the conventional Kalman smoothers.

1. INTRODUCTION

Estimation problems for finding out unknown variables have been widely dealt with in science and engineering areas. Among estimation problems, it has been considered important to find out a state in a dynamic system when only the partial information on the state is available Barnett [1975], Franklin [1980], Kailath [1985]. In case that a small delay is tolerable, a smoother, as one of state estimators, has been commonly used to estimate the state at the delayed time by using outputs measured up to the current time. These smoothers hold its decision for a more correct estimation so that it has the better performance than the filter without delay or lag. In this paper, we propose a new smoother that has the finite time convergence as an attractive feature.

When temporary uncertainties or abrupt large noises break out and disappear in a little time, the large estimation error happens transiently and can converge to zero asymptotically with an exponential rate by suitably choosing the estimator eigenvalues Kailath [1985]. Furthermore, in case of discrete-time systems, the convergence can be carried out in finite time by choosing the zero eigenvalues of an estimator, *i.e.*, called a deadbeat response. It is known that the guaranteed convergence time is n times the sampling time and can be adjusted by choosing the estimator eigenvalues Franklin [1980], Isermann [1981]. However, the deadbeat response for a continuous-time system does not arise naturally without some special processing. For a deadbeat estimator, batch processing was employed with least square and least mean square criteria Medvedev [1992], Han [2001]. In Urikura [1987], the sample and hold scheme was applied to achieve the deadbeat response. By using some special structures, deadbeat controls and lag-free filters were proposed without batch processing and sampling data Engel [2002], Nobuyama [1991]. To the best of the author's knowledge, there is no result on the continuous-time fixed-lag smoother which converges in finite time without additional process such as batch processing and sampling data.

In this paper, as a general version of the existing deadbeat filter, we propose a new fixed-lag smoother that guarantees the convergence in finite time. The proposed smoother is obtained in such an optimal fashion that the estimation error variance is minimized while keeping the deadbeat property. If the lag size is set to zero, the smoother in this paper reduces to the deadbeat filter that is an improved version of Engel [2002] since external noises are concerned and the optimization based design is employed.

This paper is organized as follows: In Section 2, an optimal fixed-lag smoother which converges in finite time is proposed for a continuous state space model. In Section 3, via numerical example, the performance of the proposed smoother with the finite time convergence is compared with that of the Kalman smoother with the asymptotical convergence. Finally, the conclusion is presented in Section 4.

2. A CONTINUOUS-TIME FIXED-LAG SMOOTHER CONVERGING IN FINITE TIME

Let us consider the following linear continuous-time state space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t), \tag{1}$$

$$y(t) = Cx(t) + v(t), \tag{2}$$

where $x(t)$, $y(t)$, $u(t)$, $v(t)$, and $w(t)$ are the state, the measurement, the input, the system noise, the measurement noise, respectively. The covariances of $w(t)$ and $v(t)$ are denoted by Q and R, respectively. The pair (A, C) of the system (1)–(2) is assumed to be observable so that stabilized observers can be constructed.

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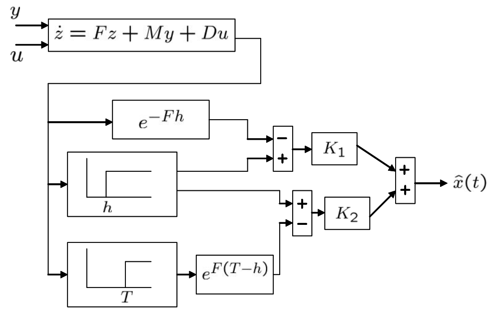

Fig. 1. The structure of the proposed smoother

If we choose two matrices M_i $(i = 1, 2)$ such that $A - M_iC$ are Hurwitz and $\{A - M_iC, M_i\}$ are controllable, two standard identity estimators for the system (1)-(2) are obtained as

$$\dot{z}_i(t) = F_i z_i(t) + M_i y(t) + Bu(t), \quad i = 1, 2, \tag{3}$$

where $F_i = A - M_iC$ Chen [1998]. Note that it is known to guarantee that $z(t)$ goes to $x(t)$ as $t \to \infty$.

Augmenting parameters of two observers as

$$F \triangleq \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \; M \triangleq \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},$$
$$D \triangleq \begin{bmatrix} B \\ B \end{bmatrix}, \; L \triangleq \begin{bmatrix} I \\ I \end{bmatrix}, \; z(t) \triangleq \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \tag{4}$$

we have

$$\dot{z}(t) = Fz(t) + My(t) + Du(t). \tag{5}$$

Here, the state $x(t-h)$ delayed from the current time t by h is estimated from $z(\cdot)$ generated from (5). For this purpose, the smoother is given as the following form:

$$\begin{aligned} \hat{x}(t-h|t) = \; & K_1[z(t-h) - e^{-Fh}z(t)] \\ & + \; K_2[z(t-h) - e^{F(T-h)}z(t-T)], \end{aligned} \tag{6}$$

where T is a positive number greater than h, and K_1 and K_2 are the gain matrices to be determined later on. It is noted that we use $z(\cdot)$ at three time points, t, $t-h$, and $t-T$ for estimating the state $x(t-h)$ at the current time t. The structure of the smoother (6) is depicted in Fig. 1.

By using the relationship $F_i = A - M_iC$, the following dynamic equation can be obtained

$$\begin{aligned} & \frac{d}{dt}\{z(t) - Lx(t)\} \\ & = Fz(t) + M\{Cx(t) + v(t)\} + Du(t) \\ & - L\{Ax(t) + Bu(t) + Gw(t)\} \\ & = F\{z(t) - Lx(t)\} + Mv(t) - LGw(t), \end{aligned} \tag{7}$$

From (7), we can represent $z(t) - Lx(t)$ in terms of $z(t-h) - Lx(t-h)$ and noises on the horizon $[t-h, t]$,

$$z(t) - Lx(t) = e^{Fh}\{z(t-h) - Lx(t-h)\} + \Xi(t), \tag{8}$$

where the noise term $\Xi(t)$ is given as

$$\Xi(t) = \int_0^h e^{F(h-\tau)}\eta(t-h+\tau)d\tau \tag{9}$$

with $\eta(t) = Mv(t) - LGw(t)$. Solving for $z(t-h) - Lx(t-h)$ yields

$$z(t-h) - Lx(t-h) = e^{-Fh}\{z(t) - Lx(t) - \Xi(t)\}. \tag{10}$$

Taking similar steps, we also have the following relationship.

$$\begin{aligned} & z(t-h) - Lx(t-h) \\ & = e^{F(T-h)}\{z(t-T) - Lx(t-T)\} + \Gamma(t) \end{aligned} \tag{11}$$

where $\Gamma(t)$ is given as

$$\Gamma(t) = \int_0^{T-h} e^{F(T-h-\tau)}\eta(t-T+\tau)d\tau. \tag{12}$$

By using (10) and (11), the estimated state (6) can be rewritten as

$$\begin{aligned} \hat{x}(t-h) = & [K_1L + K_2L]x(t-h) - K_1e^{-Fh}Lx(t) \\ & - K_2e^{F(T-h)}Lx(t-T) - K_1e^{-Fh}\Xi(t) \\ & + K_2\Gamma(t). \end{aligned} \tag{13}$$

In order to make the estimated state $\hat{x}(t-h)$ track down the real state $x(t-h)$ exactly when noises or disturbances disappear on the horizon $[t-T, \; t]$, $K_1L + K_2L$, K_1e^{-Fh}, and $K_2e^{F(T-h)}$ are set to an identity matrix, a zero matrix, and a zero matrix, respectively. By incorporating the constraints on K_1 and K_2, the estimation error $\hat{x}(t-h) - x(t-h)$ is represented only in terms of noises,

$$\begin{aligned} e(t-h|t) & \triangleq \hat{x}(t-h|t) - x(t-h) \\ & = -K_1e^{-Fh}\Xi(t) + K_2\Gamma(t). \end{aligned} \tag{14}$$

Then, the covariance matrix of the estimation error is obtained as

$$\mathbf{E}[e(t-h)e^T(t-h)] = K_1W_1K_1^T + K_2W_2K_2^T,$$

where W_1 and W_2 are given as

$$W_1 = \int_0^h e^{-F\tau}\Omega e^{-F^T\tau}d\tau,$$
$$W_2 = \int_0^{T-h} e^{F(T-h-\tau)}\Omega e^{F^T(T-h-\tau)}d\tau,$$

with $\Omega = MRM^T + LGQG^TL^T$.

The object is now to determine gain matrices K_1 and K_2 that minimize $\mathbf{E}[e(t-h)e^T(t-h)]$ while meeting the constraints on K_1 and K_2. The optimization problem to solve can be stated as follows:

$$\min_{K_1, K_2} K_1W_1K_1^T + K_2W_2K_2^T. \tag{15}$$

subject to $K_1L + K_2L = I$, $K_1e^{-Fh}L = 0$, and $K_2e^{F(T-h)}L = 0$.

Let α and β be K_1L and K_2L, respectively. By using the constraints, K_1 and K_2 can be expressed as

$$K_1 = [\,\alpha \;\; 0\,]\,[\,L \;\; e^{-Fh}L\,]^{-1}, \tag{16}$$
$$K_2 = [\,\beta \;\; 0\,]\,[\,L \;\; e^{F(T-h)}L\,]^{-1}, \tag{17}$$

where the inverses of matrix blocks are checked later on. If we define two vectors $\tilde{W}_1$ and $\tilde{W}_2$ as

$$\tilde{W}_1 = N_1 W_1 N_1^T,$$
$$\tilde{W}_2 = N_2 W_2 N_2^T,$$

with matrices N_1 and N_2 given by

$$N_1 = \left[\, e^{-F_1 h} e^{F_2 h} \quad -e^{-F_1 h} e^{F_2 h} + I \,\right],$$
$$N_2 = \left[\, e^{-F_1 (T-h)} e^{F_2 (T-h)} \quad -e^{-F_1 (T-h)} e^{F_2 (T-h)} + I \,\right],$$

the optimization problem (15) can be represented in terms of α and β

$$\min_{\alpha,\beta} \alpha \tilde{W}_1 \alpha^T + \beta \tilde{W}_2, \tag{18}$$

subject to $\alpha + \beta = I$.

Using a technique of completing the square, we can obtain optimal α and β as

$$\alpha = (\tilde{W}_1 + \tilde{W}_2)^{-1} (\tilde{W}_2 + \tilde{W}_2^T), \tag{19}$$
$$\beta = I - (\tilde{W}_1 + \tilde{W}_2)^{-1} (\tilde{W}_2 + \tilde{W}_2^T). \tag{20}$$

Substituting the above α and β into (16) and (17), we have optimal K_1 and K_2.

From now on, we check the inversions of $\left[\, L \ e^{-Fh} L \,\right]$ and $\left[\, L \ e^{F(T-h)} L \,\right]$ in (16) and (17). To begin with, we compute the determinants of two block matrices:

$$\det(\left[\, L \ e^{-Fh} L \,\right]) = (-1)^n \det(I - e^{-F_1 h} e^{F_2 h}) \times \det(e^{-F_2 h}), \tag{21}$$
$$\det(\left[\, L \ e^{F(T-h)} L \,\right]) = (-1)^n \det(e^{F_1 (T-h)}) \times \det(I - e^{-F_1 (T-h)} e^{F_2 (T-h)}). \tag{22}$$

It was shown in Engel [2002] that if M is chosen such that

$$Re\ \lambda_j(F_2) < \sigma < Re\ \lambda_j(F_2),\ j = 1, 2, \cdots, n \tag{23}$$

for some $\sigma < 0$, then $\left[\, L \ e^{Fr} L \,\right]$ for almost all positive r. According to this result, the inverses of block matrices in (17) are guaranteed.

We can see that, from (5) and (6), the proposed smoother has the following batch form

$$\hat{x}(t-h|t) = -K_1 \int_0^h e^{-F\tau} \mathcal{Y}(t-h+\tau) d\tau + K_2 \int_0^{T-h} e^{T-h-\tau} \mathcal{Y}(t-T+\tau) d\tau, \tag{24}$$

where K_1 and K_2 are given in (16) and (17), and $\mathcal{Y}(\tau) = My(\tau) + Du(\tau)$.

If the fixed-lag size h in this paper is set to zero, the proposed smoother reduces to the finite convergence filter to estimate the current state $x(t)$. This filter would be an improved version of Engel [2002] since external noises are concerned and the optimization based design is employed.

3. NUMERICAL EXAMPLE

In this section, we give a numerical example to illustrate the finite time convergence of the proposed smoother.

Consider a state space model:

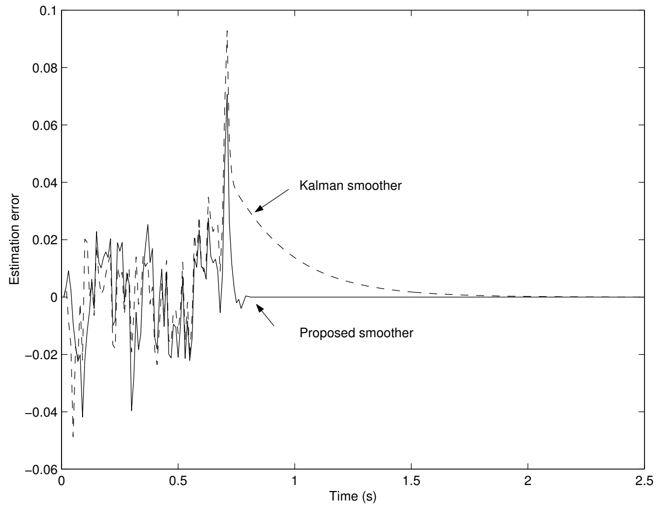

Fig. 2. Estimation errors of proposed smoother and Kalman smoother

$$\dot{x}(t) = \begin{bmatrix} -8.68 + \delta(t) & 0.37 \\ -0.27 & -0.80 + \delta(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.27 \\ 5 \end{bmatrix} w(t), \tag{25}$$
$$y(t) = [\, 1 + 0.01\delta(t) \ \ 1 + 0.01\delta(t) \,]\, x(t) + v(t), \tag{26}$$

where $\delta(t)$ is an uncertain model parameter. The system noise covariance Q is 0.02 and the measurement noise covariance R is 0.02. T and the fixed-lag size h are taken as $T = 0.1\ sec$ and$h = 0.03 sec$, respectively. $\delta(t)$ for temporary uncertainties is given as

$$\delta(t) = \begin{cases} 0.1, & 0.5 \le t < 0.7, \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

The proposed smoother is compared with a general Kalman smoother and we will check their convergence property when noises and disturbances do not exist. The estimation errors of the proposed smoother and fixed-lag Kalman smoother are compared in Fig. 2. We can see that the estimation error of the proposed smoother converge to zero in finite time while that of the fixed-lag Kalman smoother converges to zero asymptotically. As designed in advance, the proposed smoother has the finite time convergence when noises and disturbances do not exist.

4. CONCLUSION

In this paper, a new smoother was proposed to guarantee the finite time convergence for a continuous-time state space model. The proposed smoother was obtained so as to achieve the minimum variance of the estimation error under the constraint of the finite time convergence. We employed a new structure of a smoother instead of batch process with high computation load and sampling data through discretization in order to obtain the finite time convergence. In addition, since stochastic systems with noises are considered, the smoother in the paper could be more widely utilized than existing deadbeat estimators for deterministic systems without noises.

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