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Robust Stability and Synthesis of Nonlinear Discrete Control Systems under Uncertainty

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Abstract: With the use of Lyapunov functions chosen as the norm of state vector, we obtain the robust stability sufficient conditions for a wide class of nonlinear, and generally nonstationary, discrete-time control systems with the given set-valued parameter estimates. For a strictly monotone nonlinear function, validation of these conditions is equivalent to solution of a series of combinatorial problem in the state space.
Synthesis of robustly stable control systems in a domain is performed on the basis of the obtained sufficient conditions of robust stability.

Keywords: Uncertainty, robust stability, set-valued estimates, stability domains, nonlinear control systems, control synthesis.

1. INTRODUCTION

The problem of stability for nonlinear discrete-time systems has gained a nearly 50-year history written by the well-known researchers like Ya. Z. Tsypkin, V. A. Yakubovich, B. T. Polyak, E. Juri, R. Kalman, A. Khalanai and others. These authors contributed essentially to solution of the problem, however the problem remains actual and far from being solved.

The apparatus of Lyapunov functions has been (and still remains) the major tool of stability analysis for nonlinear discrete systems. This tool is used once again for obtaining the robust stability sufficient conditions presented below. The stability analysis and control synthesis is considered in the present paper for a domain (a given bounded set in the state space) specific for the considered class of systems.

Hereinafter, the following class of nonlinear, generally nonstationary, systems is considered,

$$X_{n+1} = F(X_n, L_n); \quad X_0 = \overset{\circ}{X}; \quad n = 0, 1, \dots , \tag{1}$$

where $X_n \in \mathbb{R}^m$ is a state vector, $F(\cdot)$ is a nonlinear continuous single-valued m-dimensional function which satisfies the condition $F(0, L_n) = 0 \ \forall n \in [0; \infty)$, L_n is a vector of (generally) time-dependent parameters. We assume that $F(\cdot)$ is linear in parameters L_n.

Consider obtaining the conditions of asymptotic stability in a convex set $\mathbf{X}$, $0 \in \mathbf{X}$, for the system (1) with the use of Lyapunov function

$$v_n = \|X_n\|, \tag{2}$$

where the norm is not fixed yet. In view of (1), the first difference of (2) is calculated as

$$v_{n+1} - v_n = \|F(X_n, L_n)\| - \|X_n\|.$$

Fulfilment of the following inequality provides the robust stability of the system (1),

$$\max_{X_n \in \mathbf{X}} \{\|F(X_n, L_n)\| - \|X_n\|\} < 0 \quad \forall n \in [0; \infty). \tag{3}$$

In order of obtaining verifiable sufficient stability conditions from the inequality (3), one needs to present a nonlinear function $F(X_n, L_n)$ in the form of quasi-linear parameterized function, where the parameters minimize the left-hand side of the inequality.

2. DOMAIN ROBUST STABILITY SUFFICIENT CONDITIONS

Consider set $\mathbf{X}$ in (3) is given as

$$\mathbf{X} = \{X : \|X\| \leq \rho = \text{const}\}.$$

Assume at the beginning that all components $f_i(\cdot)$ of the vector function $F(\cdot)$ are strictly monotone functions in X.

Since $\|X_n\|$ is a convex function, maximum in (3) is reached at the boundary of $\mathbf{X}$. In view of this, choose the vector norm in (2) in the form of either

$$\|X\|_\infty = \max_{j=1,\dots,m} |x_j| \quad \text{or} \quad \|X\|_1 = \sum_{j=1}^{m} |x_j|$$

and represent $\mathbf{X}$ as a convex hull of its vertices,

$$\mathbf{X} = \operatorname*{conv}_{k=1,\dots,2^m} \{X^k\}, \tag{4}$$

where X^k is the kth vertex of either an m-dimensional cube with a side of the length 2ρ or an m-dimensional octahedron depending respectively on the norm chosen. Since (regardless to the choice of either 1-norm or ∞-norm)

$$\max_{X_n \in \mathbf{X}} \|X\| = \max_{k=1,\dots,2^m} \|X^k\| = \rho,$$

the inequality (3) takes the form of

$$\max_{k=1,\dots,2^m}\{\|F(X^k,L_n)\|\}<\rho \quad \forall n\in[0;\infty). \tag{5}$$

Assume additionally that a set-valued estimate is given for a time-variant parameter vector L_n,

$$L_n\in\mathbf{L}=\underset{s=1,\dots,S}{\operatorname{conv}} L^s \quad \forall n\in[0;\infty), \tag{6}$$

where L^s is the sth vertex of a set $\mathbf{L}$ and S is the number of vertices.

Taking into account that $F(\cdot)$ is linear in L, maximum of $F(L)$ is reached at a vertex L^s of $\mathbf{L}$. In other words, the inequality (5) in view of the assumption (6) can be rewritten as follows,

$$\max_{k=1,\dots,2^m;s=1,\dots,S}\{\|F(X^k,L^s)\|\}<\rho. \tag{7}$$

Consider now a stationary subclass of the class (1), for which a parameter vector is a time-independent uncertain vector $\overset{\circ}{L}$ with the given set-valued estimate

$$\overset{\circ}{L}\in\overset{\circ}{\mathbf{L}}=\underset{s=1,\dots,S}{\operatorname{conv}} \overset{\circ}{L}{}^s,$$

where $\overset{\circ}{L}{}^s$ is the sth vertex of a polytope $\overset{\circ}{\mathbf{L}}$. In the case, the domain sufficient robust stability condition is identical to (7) to the extent of notations.

3. DOMAIN SUFFICIENT ROBUST STABILITY CONDITIONS FOR NONLINEAR SYSTEMS WITH A LINEAR PART

The following widely considered subclass of the class (1) is worth of independent research,

$$X_{n+1}=\boldsymbol{A}_nX_n+\Phi(X_n);\quad X_0=\overset{\circ}{X};\quad n=0,1,\dots\ , \tag{8}$$

where $\boldsymbol{A}_n$ is an $m\times m$-dimensional matrix with uncertain coefficients bounded with the given set-valued (particularly, interval) estimates, $\Phi(\cdot)$ is a nonlinear continuous single-valued m-dimensional vector function, $\Phi(0)=0$. This function is assumed to have the following presentation,

$$\Phi(X)=\boldsymbol{P}\overset{\circ}{\Phi}(X), \tag{9}$$

where $\overset{\circ}{\Phi}(\cdot)$ is a given function and

$$\boldsymbol{P}=\operatorname{diag}\{p_j\}_{j=1}^m,\quad \underline{p}_j\le p_j\le\overline{p}_j,$$

and the components $\overset{\circ}{\phi}_j(X)$ of function $\overset{\circ}{\Phi}(X)$ are strictly monotone in X.

Introduce the polyhedral estimate for the jth row of matrix $\boldsymbol{A}$ as follows,

$$A_{jn}^T\in\mathbf{A}_j=\underset{v=1,\dots,V_j}{\operatorname{conv}}\{A_j^v\},\quad j=1,\dots,m,$$

where A_j^v is the vth vertex of polytope $\mathbf{A}_j$ and V_j is the number of its vertices.

A sufficient condition of robust stability for the class of systems (8),(9) in the domain $\mathbf{X}$ (similarly to (7)) takes the form

$$\max_{\substack{j=1,\dots,m\\k=1,\dots,2^m}}\ \max_{\substack{v=1,\dots,V^j\\p_j=\underline{p}_j,\overline{p}_j}}\left|(A_j^v)^TX^k+p_j\overset{\circ}{\phi}_j(X^k)\right|<\rho. \tag{10}$$

Since

$$\left|(A_j^v)^TX^k\right|\le\|A_j^v\|\cdot\|X^k\|$$

and $\|X^k\|=\rho$, inequality (10) can be transformed into the following one,

$$\max_{\substack{j=1,\dots,m\\k=1,\dots,2^m}}\ \max_{\substack{v=1,\dots,V^j\\p_j=\underline{p}_j,\overline{p}_j}}\left(\|A_j^v\|+\rho^{-1}p_j\left\|\overset{\circ}{\phi}_j(X^k)\right\|\right)<1. \tag{11}$$

If, in particular, $\Phi(X_n)=0$, the inequality (11) degenerates into the known sufficient robust stability condition for linear nonstationary systems,

$$\|\ \boldsymbol{A}_n\ \|<1\quad \forall n\in[0;\infty).$$

Consider in details the subclass of systems (8) under the condition that $\Phi(X)=\phi(X)B$ is a scalar function and $\phi(0)=0$. Here B is given constant vector of the respective dimension. This particular case is widely met in applications. Thus, we shall be considering the system

$$X_{n+1}=\boldsymbol{A}_nX_n+\phi(X_n)B,\quad X_0=\overset{\circ}{X},\quad n=0,1,\dots\ . \tag{12}$$

Here $\boldsymbol{A}_n$ is an $m\times m$ Frobenius matrix with the mth row A_{mn} a priori estimated by

$$A_{mn}^T\in\mathbf{A}=\underset{v=1,\dots,V}{\operatorname{conv}}\{A^v\},\quad \forall n=1,\dots, \tag{13}$$

where A^v is the vth vertex of polytope $\mathbf{A}$ and V is the number of its vertices.

Assume function $\phi(X)$ is given in the form

$$\phi(X)=p\,\overset{\circ}{\phi}(X),$$

where $\overset{\circ}{\phi}(\cdot)$ is a known function and the unknown parameter p is a priori estimated with the interval,

$$\underline{p}\le p\le\overline{p}.$$

It is easily seen that the inequality (11) cannot be fulfilled for the system (12), because $\|\boldsymbol{A}_n\|=1\ \forall n\in[0;\infty)$. On the other hand, the considered system can be robustly stable in domain X. This paradox in robust stability analysis for linear discrete systems with a Frobenius matrix was mentioned by Polyak and Scherbakov [2002b, 2005, 2002a] and Kuntsevich [2007] and resolved by Kuntsevich [2006a,b, 2007]. Here, we generalize the method of robust stability analysis, presented by Kuntsevich [2007, 2006a], to the considered class of nonlinear systems. With this purpose in view, following Barbashin [1978], represent function $\overset{\circ}{\phi}(\cdot)$ as follows,

$$\overset{\circ}{\phi}(X_n)=\overset{\circ}{\Psi}{}^T(X_n,L)X_n, \tag{14}$$

where

$$X_n=\{x_{jn}\}_{j=1}^m\,;\ L=\{l_j\}_{j=1}^m\,;$$
$$\overset{\circ}{\Psi}{}^T(X_n,L)=\left\{l_j\overset{\circ}{\psi}(X_n)\right\}_{j=1}^m. \tag{15}$$

Here L is a vector of unknown parameters to be calculated.

Rewrite the system (12) in the quasi-linear form:

$$X_{n+1}=\boldsymbol{H}(X_n,\boldsymbol{A}_n,L)X_n, \tag{16}$$

where

$$\boldsymbol{H}(\cdot)=\boldsymbol{A}_n+P\overset{\circ}{\Psi}{}^T(X_n,L)$$

is a Frobenius matrix with the mth row H_m given by the equality

$$H_m^T(X_n, A_{mn}, L) = A_{mn}^T + P\overset{\circ}{\Psi}^T(X_n, L).$$

Assume $\overset{\circ}{\psi}_j(X) = x_j^{-1}\overset{\circ}{\phi}(X),\ j = 1\ldots,m$. Hence, from (14,15), one obtains

$$\sum_{j=1}^{m} l_j = 1. \qquad (17)$$

We shall prove that the following inequality is a sufficient robust stability condition in domain $\mathbf{X}$ for the class of systems (12,13),

$$\max_{\substack{X_n\in\mathbf{X}\\ A_{mn}^T\in\mathbf{A}\\ p=\underline{p},\overline{p}}} \left\{ \sum_{j=1}^{m} \left| a_{mj,n} + p\, l_j \overset{\circ}{\psi}_j(X_n) \right| \right\} \le q < 1, \qquad (18)$$

where $a_{mj,n}$ are coefficients of the mth row of matrix $\boldsymbol{A}_n$ and q is a constant. The following statement is a generalization of the one given by Kuntsevich [2006b, 2007].

Lemma 1. For a matrix

$$\boldsymbol{H} = \boldsymbol{H}(S_m)\,\boldsymbol{H}\,(S_{m-1})\cdots\boldsymbol{H}\,(S_1),$$

which is the product of m Frobenius matrices of the dimension $m \times m$ depending respectively on parameter vectors $S_i,\ \ i = 1,\ldots,m$, the following inequality is fulfilled,

$$\|\,\boldsymbol{H}\,\| \le q < 1,$$

if the mth rows $H_m(S_i),\ i = 1,\ldots,m$, of the respective matrices satisfy the condition

$$\|H_m(S_i)\| = \sum_{j=1}^{m} |h_{mj}(S_i)| \le q < 1.$$

Note that the parameter vectors P_i can be state vectors and/or discrete time, etc.

Theorem 1. The class of nonlinear stationary systems (12,13) with a Frobenius matrix $\boldsymbol{H}(X_n, A_{m,n}, L)$ is stable in the set $\mathbf{X}$ if the inequality (18) is fulfilled for the mth row H_m^T of the matrix $\boldsymbol{H}$.

See the proof in Appendix.

The function $\xi(L)$ in (**??**) is defined to the extent of the parameter vector L. Further, we need to calculate this vector.

Consider first a particular case, when $A_{m,n}^T$ is a vector of constants, meaning $\mathbf{A}$ is a point-wise set which contains the only vector $\overset{\circ}{A}{}_m^T$. It is desired to find L which minimizes $\xi(L)$ at $L \in \mathbf{L}$, where $\mathbf{L}$ is given by the equality (17) and the condition $l_j \ge 0,\ j = 1,\ldots,m$. However, this minimization problem has no analytical solution, and finding a numeric solution to the problem is rather complicated. Instead of solving the problem directly, make use of the inequality

$$\max_{X_n\in\mathbf{X}} \left\{ \sum_{j=1}^{m} |h_{mj}(X_n, L)| \right\} \le \sum_{j=1}^{m} \left\{ \max_{X_n\in\mathbf{X}} |h_{mj}(X_n, L)| \right\}$$

and strengthen the inequality (**??**):

$$\overline{\xi}(L) = \sum_{j=1}^{m} \max_{X_n\in\mathbf{X}} |h_{mj}(X_n, L)| \le q < 1. \qquad (19)$$

Find L as solution to the problem

$$\min_{L\in\mathbf{L}} \left\{ \overline{\xi}(L) = \sum_{j=1}^{m} \max_{X_n\in\mathbf{X}} |h_{mj}(X_n, L)| \right\} \le q < 1. \qquad (20)$$

Assume $\overset{\circ}{\psi}_j(X)$ are symmetric functions, $\overset{\circ}{\psi}_j(-X) = -\overset{\circ}{\psi}_j(X),\ j = 1,\ldots,m$. This assumption is not fundamental and it is made for simplification reasons only.

Next, find a solution to the following optimization problem,

$$\overset{\star}{\psi}_j = \max_{X\in\mathbf{X}} \left\{ \overset{\circ}{\psi}_j(X) = x_j^{-1}\overset{\circ}{\phi}(X) \right\}, \quad j = 1,\ldots,m,$$

either analytically (if possible) or by application of the routine by Kappel and Kuntsevich [2000]. In particular, if $\overset{\circ}{\psi}_j(X)$ are monotone functions in X, a solution $\overset{\star}{\psi}_j$ is found at a vertex of the set $\mathbf{X}$.

Substitute the obtained solutions $\overset{\star}{\psi}_j(X),\ j = 1,\ldots,m$, into (20) and find the required vector L_{opt} as a solution to the problem

$$\min_{L\in\mathbf{L}} \left\{ \sum_{j=1}^{m} \left| h_{mj}(\overset{\star}{\psi}_j, L) \right| \right\}. \qquad (21)$$

Note that $|h_{mj}(L)|$ are convex functions and $\mathbf{L}$ is a convex set, hence the problem (21) is a local minimization problem which can be efficiently solved particularly with SolvOpt (see Kappel and Kuntsevich [2000]).

Consider now a more general case, when uncertain parameter vectors $A_{m,n}^T$ are estimated by (13). Instead of fulfilment of the inequality (19), we require fulfilment of the following condition,

$$\overline{\xi}(L) = \sum_{j=1}^{m} \max_{\substack{X_n\in\mathbf{X}\\ A_{m,n}^T\in\mathbf{A}\\ p=\underline{p},\overline{p}}} |h_{mj}(X_n, A_{m,n}, L, p)| \le q < 1.$$

In this case, we obtain the desired vector L_{opt} as solution to the problem

$$\min_{L\in\mathbf{L}} \left\{ \sum_{j=1}^{m} \max_{\substack{X_n\in\mathbf{X}\\ A_{m,n}^T\in\mathbf{A}\\ p=\underline{p},\overline{p}}} \left| h_{mj}(\cdot) = a_{mj,n} + p\, l_j \overset{\circ}{\psi}_j(X_n) \right| \right\}. \qquad (22)$$

Assume $\mathbf{A}$ is an interval set,

$$\mathbf{A} = \mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_m,$$

where

$$\mathbf{a}_j = \{a_{mj} : \underline{a}_{mj} \le a_{mj} \le \overline{a}_{mj}\}, \quad j = 1,\ldots,m,$$

and the numerical bounds $\underline{a}_{mj}$ and $\overline{a}_{mj}$ are known.

If the functions $\overset{\circ}{\psi}_j(X_n)$ are strictly monotone, maximum in (22) is reached at the boundary of set $\mathbf{X}$, which is defined in (4) as a hyper-box, and therefore, the problem (21) is reduced to the following one,

$$\min_{L\in\mathbf{L}} \left\{ \sum_{j=1}^{m} \max_{\substack{k=1,\ldots,2^m \\ a_{mj,n}=\underline{a}_{mj},\overline{a}_{mj} \\ p=\underline{p},\overline{p}}} \left| a_{mj,n} + p\, l_j \overset{\circ}{\psi}_j(X^k) \right| \right\}. \qquad (23)$$

Finding maximum in (23) does not require essential computational efforts with $m \sim 10$, and therefore, the minimization problem (23) can be efficiently solved again with SolvOpt.

4. SYNTHESIS OF ROBUST STABILIZING SYSTEMS WITH SCALAR CONTROLS

Consider a widely applicable description of discrete-time control systems given by the difference equation

$$X_{n+1} = \Phi(X_n, u_n, L_n), \qquad (24)$$

where X_n is a state vector as above, u_n is a scalar control at a discrete time n, $\Phi(\cdot)$ is an m-dimensional nonlinear function, $\Phi(0, L_n) = 0$, and L_n is a vector of generally time-varying uncertain parameters with the given set-valued estimates $L_n \in \mathbf{L}$.

Assume that X_n is measurable exactly at any n.

Our objective is calculation of controls $u_n = u(X_n)$ providing the robust stability of the closed-loop system,

$$X_{n+1} = \Phi(X_n, u_n, L_n),$$

in the given domain $\mathbf{X}$, $X_0 \in \mathbf{X}$, and, if possible, with the given parameter set-valued estimate $\mathbf{L}$.

For the Lyapunov function (2) and the equation (24), find the first difference as follows,

$$\Delta v_n = \|\Phi(X_n, u_n, L_n)\| - \|X_n\|,$$

and calculate the required control u_n at a discrete time n as minimizer for the first difference Δv_n (see Kuntsevich and Lychak [1977]),

$$\min_{u_n} \|\Phi(X_n, u_n, L_n)\|.$$

Consider a particular subclass of systems (24), widely met in applications,

$$X_{n+1} = \boldsymbol{A}_n X_n + \psi(X_n) B_n + u_n C_n, \qquad (25)$$

where $\psi(X)$ is a scalar nonlinear function, $\psi(0) = 0$, $\boldsymbol{A}_n$ is a Frobenius matrix, and vectors B_n and C_n are of the standard form,

$$B_n^T = b_n(0;\ldots;0;1), \quad C_n^T = c_n(0;\ldots;0;1).$$

Assume the following set-valued estimates are given for the mth row, $A_{m,n}$, of matrix $\boldsymbol{A}_n$ and scalars b_n and c_n at $n = 0, 1, \ldots$,

$$A_{m,n}^T \in \mathbf{A} = \operatorname*{conv}_{k=1,\ldots,K}\{A^k\}, \qquad (26)$$

$$b_n \in \mathbf{b} = \{b : \underline{b} \le b \le \overline{b}\}, \quad c_n \in \mathbf{c} = \{c : \underline{c} \le c \le \overline{c}\}. \; (27)$$

Here A^k, $k = 1, ..., K$, are vertices of the polytope $\mathbf{A}$. Assume also $\underline{b} > 0$ and $\underline{c} > 0$ without loosing a generality.

The following inequality provides a sufficient robust stability condition for the systems (25-27) in domain $\mathbf{X}$,

$$\max_{\substack{X_n\in\mathbf{X} \\ A_{m,n}^T\in\mathbf{A} \\ b\in\mathbf{b} \\ c\in\mathbf{c}}} \|\, \boldsymbol{A}\,(A_{m,n})X_n + \psi(X_n)B(b_n) + u_n C(c_n)\| - $$
$$-\|X_n\| < 0.$$

Due to the structural features of matrices $\boldsymbol{A}_n$ and vectors B_n and C_n, the only mth coefficient $x_{m,n+1}$ of vector X_{n+1} depends on control u_n. Hence the optimal control at a time n should be found as solution to the problem

$$\min_{u_n} \max_{\substack{X_n\in\mathbf{X} \\ A_{m,n}^T\in\mathbf{A} \\ b\in\mathbf{b} \\ c\in\mathbf{c}}} |A_{m,n}^T X_n + b_n\psi(X_n) + c_n u_n|. \qquad (28)$$

In the boundary case, when $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ are point-wise sets consisting respectively of the only points $\overset{\circ}{A}{}_m^T$, $\overset{\circ}{b}$ and $\overset{\circ}{c}$, solution of the problem (28),

$$\min_{u_n} \max_{X_n\in\mathbf{X}} |\overset{\circ}{A}{}_{m,n}^T X_n + \overset{\circ}{b}\psi(X_n) + \overset{\circ}{c}u(X_n)|,$$

is trivial,

$$\overset{\star}{u}_n = -\overset{\circ}{c}{}^{-1}\left(\overset{\circ}{A}{}_m^T X_n + \overset{\circ}{b}\psi(X_n)\right). \qquad (29)$$

Substitute (29) into (25) and obtain the equation of a linear stationary system,

$$X_{n+1} = \overline{\boldsymbol{A}} X_n,$$

where $\overline{\boldsymbol{A}}$ is a singular nilpotent Frobenius matrix.

If we account the upper bound restriction on controls u_n, which is inevitably present in practical applications,

$$u_n \in \mathbf{u} = \{u : \; |u| \le \sigma\}, \qquad (30)$$

then the optimal control (29) can be implemented only in the domain

$$\Omega = \left\{ X : \left| -\overset{\circ}{c}{}^{-1}\left(\overset{\circ}{A}{}_m^T X + \overset{\circ}{b}\psi(X)\right) \right| \le \sigma \right\}.$$

If the set $\mathbf{X}$, which is determined by the constant ρ, is such that $\mathbf{X} \subset \Omega$, then the condition (30) is of no concern and does not influence the control synthesis procedure. If the sets $\mathbf{X}$ and Ω have a common subset, the optimal control (29) "linearizes" the closed-loop system and, consequently, provides its asymptotic stability only in the domain $\mathbf{X} \cap \Omega$.

The control synthesis procedure differs insignificantly from the given above in a more general case, when set-valued estimates $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ contain more than a single point. In general, minimax problems cannot be solved analytically, but fortunately the problem (28) is an exception. More precisely, analytical solution of the minimax problem (28) requires presentation of the given set-valued estimates in the centralized form,

$$\mathbf{A} = \overset{\circ}{A} + \delta\mathbf{A}, \quad \delta\mathbf{A} = \operatorname*{conv}_{s=1,\ldots,S}\{\delta A^s = A^s - \overset{\circ}{A}\}. \qquad (31)$$

Here $\overset{\circ}{A}$ is the center of the upper-bound ellipsoid for the polytope $\mathbf{A}$. An efficient numerical algorithm for calculation of the lower and upper bound ellipsoids for a given set of points (particularly, the set of vertices of a polytope) is presented by Shor and Berezovski [1992].

Similarly, the sets $\mathbf{b}$ and $\mathbf{c}$ will be also represented in the centralized form,

$$\mathbf{b} = \overset{\circ}{b} + \delta\mathbf{b}, \quad \overset{\circ}{b} = 0.5(\underline{b} + \bar{b}),$$
$$\delta\mathbf{b} = \mathrm{conv}\{\delta b_1 = \underline{b} - \overset{\circ}{b}; \delta b_2 = \bar{b} - \overset{\circ}{b}\}, \tag{32}$$

$$\mathbf{c} = \overset{\circ}{c} + \delta\mathbf{c}, \quad \overset{\circ}{c} = 0.5(\underline{c} + \bar{c}),$$
$$\delta\mathbf{c} = \mathrm{conv}\{\delta c_1 = \underline{c} - \overset{\circ}{c}; \delta c_2 = \bar{c} - \overset{\circ}{c}\}, \tag{33}$$

With the introduced notations, the problem (28) can be rewritten as follows,

$$\min_{u_n} \max_{\substack{X_n \in \mathbf{X} \\ \delta A_m^T \in \delta\mathbf{A} \\ \delta b \in \delta\mathbf{b} \\ \delta c \in \delta\mathbf{c}}} |(\overset{\circ}{A}_m + \delta A_m)^T X_n + (\overset{\circ}{b} + \delta b)\psi(X_n) + (\overset{\circ}{c} + \delta c)u_n|. \tag{34}$$

The problem (34) is identical to the one solved by Kuntsevich and Kuntsevich [1999] to the extent of notations. The analytically calculated minimizer is given by the following equality

$$\overset{\star}{u}_n = -\overset{\circ}{c}^{-1}\left(\overset{\circ}{A}{}^T_m X_n + \overset{\circ}{b}\psi(X_n)\right), \tag{35}$$

and it is the same as the minimizer (29) for the case of point-wise set estimates.

In general, with arbitrary set-valued estimates $\delta\mathbf{A}$, $\delta\mathbf{b}$ and $\delta\mathbf{c}$, the control (35) cannot guarantee the robust stability of systems (25-27),(35). Therefore, we need to verify the stability conditions for the closed-loop control system. Aiming this, substitute the obtained solution (35) into the equation (25) and obtain the equation which describes the closed-loop control system,

$$X_{n+1} = F(X_n), \tag{36}$$

where $F(X_n)$ is a vector function, $F(X_n) = \{f_j(X_n)\}_{j=1}^m$, which is calculated as follows,

$$F(X_n) = \left\| \begin{matrix} 0 & \\ \vdots & \boldsymbol{I} \\ 0 & \\ \cdots\cdots & \cdots\cdots \\ & -\delta c\, \delta A^T \end{matrix} \right\| X_n + \left\| \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ (\delta b - \overset{\circ}{b}\,\delta c) & & \psi(X_n) \end{matrix} \right\|. \tag{37}$$

Next, the mth component of $F(X_n)$,

$$f_m(X_n) = -\delta c\, \delta A^T X_n + (\delta b - \overset{\circ}{b}\,\delta c)\psi(X_n), \tag{38}$$

has to be represented as above in quasi-linear form. With this purpose in view, we introduce the notations

$$\phi_j(X_n) = x_j^{-1}\psi(X_n)$$

and

$$\psi(X_n) = \Phi^T(X_n, \tilde{L})X_n, \tag{39}$$

where

$$\Phi(X_n, \tilde{L}) = \{\tilde{l}_j\, \phi_j(X_n)\}_{j=1}^m.$$

Make use of the notations (39) and rewrite (38) as follows,

$$f_m(\cdot) = \left[-\delta c\, \delta\overset{\circ}{A}{}^T_m + (\delta b - \overset{\circ}{b}\,\delta c)\Phi^T(X_n, \tilde{L})\right] X_n.$$

It was shown above that the inequality

$$\max_{\substack{X_n \in \mathbf{X} \\ \delta A_m^T \in \delta\mathbf{A} \\ \delta b \in \delta\mathbf{b} \\ \delta c \in \delta\mathbf{c}}} |-\delta c\, \delta\overset{\circ}{A}{}^T_m X_n + (\delta b - \overset{\circ}{b}\,\delta c)\Phi^T(X_n, \tilde{L})X_n| \le q < 1 \tag{40}$$

is a sufficient robust stability condition for the class of systems (36,37) with the set-valued estimates (31,32,33) in the domain $\mathbf{X}$.

The above made remark on accounting the bounds on control absolute values and, as a consequence, on the fulfilment of the sufficient robust stability condition (40) in domain $\mathbf{X} \cap \Omega$ remains actual in the considered case as well.

The inequality (40) contains the only unknown vector $\tilde{L}$ of parameters of the decomposition (39). We described above the method of calculating this vector as minimizer to the left side of the inequality (40).

If $\psi(X_n)$ is a strictly monotone function in the domain $\mathbf{X}$, verification of the condition (40) is simplified considerably. In this case, maximum of the function $|\psi(X_n)|$ is reached at the boundary (particularly, at a vertex) of the set $\mathbf{X}$. Hence the condition (40) takes the form

$$\max_{\substack{k=1,\dots,2^m \\ s=1,\dots,S \\ \iota=1,2 \\ \kappa=1,2}} |-\delta c_\kappa\, (\delta A^s)^T X^k + (\delta b_\iota - \overset{\circ}{b}\,\delta c_\kappa)\, \Phi^T(X^k, \tilde{L})X_n| \le q < 1 \tag{41}$$

Since the dimension of the combinatorial problem (41) is small, a solution can be found by searching among all $4 \times 2^m \times S$ candidates.

5. CONCLUSIONS

The obtained results can be easily generalized to multi-dimensional nonlinear (generally, nonstationary) dynamic plants described by the equation

$$X_{n+1} = \boldsymbol{A}_n X_n + F(X_n) + \boldsymbol{B}U_n,$$

where U_n is a vector of controls and $\boldsymbol{B}$ is a matrix of the respective dimension. The detailed description of this generalization cannot be given here due to the limitations put on the paper size. Let us note that B can be either square or rectangular non-singular matrix. In the first case, the optimal control is calculated with the use of B^{-1}. In the second case, a pseudo-inverse matrix B^{-1} is used.

We have considered above a constructive method for solution of the control synthesis problem providing the robust stability of a wide class of nonlinear (generally, nonstationary) systems.

The robust stability of discrete-time systems cannot be guaranteed with arbitrary set-valued estimates for uncertain system parameters. Therefore, the final step of a synthesis procedure necessarily has to include verification of sufficient robust stability conditions. If none of the applicable conditions is satisfied, the a priori data has to be refined. Possibly, one can either reduce a given domain $\mathbf{X}$ or improve set-valued estimates of uncertain values. If all of the improvements do not provide the system robust stabilizability, it is still possible to obtain the desired result by application of adaptive control procedures aiming reduction of uncertainty.

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APPENDIX

Here we present the proof for the theorem.

We shall prove first that the system (16) is stable by Lyapunov if the inequality (18) is fulfilled.

Introduce the Lyapunov function

$$v_n = \|X_n\|. \tag{A.1}$$

Define the first difference of function (A.1) for the system (16) as

$$\begin{aligned} \Delta v_n &= v_{n+1} - v_n = \|X_{n+1}\| - \|X_n\| = \\ &= \| \boldsymbol{H}_n\, X_n\| - \|X_n\| \le (\| \boldsymbol{H}_n \| - 1)\|X_n\|. \end{aligned} \tag{A.2}$$

The fulfilment of the condition (18) provides the correctness of the equality $\| \boldsymbol{H}_n \| = 1$, hence $\Delta v_n = 0$ in (A.2) and the system (16) is stable by Lyapunov.

Next, we shall prove the asymptotic stability of the system (16) in the set $\mathbf{X}$. Aiming this, we shall select the following subsequence of the norms of state vectors,

$$\|X_n\|,\ \|X_{n+m}\|,\ \|X_{n+2m}\|,\ \ldots, \tag{A.3}$$

out of the sequence $\{\|X_{n+i}\| : i = 0, \ldots\}$. The dynamics of the subsequence (A.3) is described by the equation

$$X_{n+m} = \boldsymbol{H}_n^m X_n, \tag{A.4}$$

where

$$\boldsymbol{H}_n^m = \boldsymbol{H}_{n+m-1}\, \boldsymbol{H}_{n+m-2} \cdots \boldsymbol{H}_n\ .$$

Define the first extended difference of the function (A.1) for the system (A.4) as follows,

$$\begin{aligned} \Delta v_{n+m} &= v_{n+m} - v_n = \|X_{n+m}\| - \|X_n\| = \\ &= \| \boldsymbol{H}_n^m\, X_n\| - \|X_n\| \le (\| \boldsymbol{H}_n^m \| - 1)\|X_n\|. \end{aligned} \tag{A.5}$$

Due to the fulfilment of inequality (**??**) and the equality $\| \boldsymbol{H}_n \| = 1$, one obtains

$$\|X_{n+1}\| = \|X_n\|. \tag{A.6}$$

Hence the inequality analogous to (**??**) remains fulfilled for the step $(n+1)$ due to (A.6).

The aforesaid is correct also for each step $n+k$, where $1 \le k \le m-1$, because of the fulfilment of the equality

$$\|X_{n+k}\| = \|X_n\|, \quad 1 \le k \le m-1. \tag{A.7}$$

Since the conditions of lemma are fulfilled, the inequality

$$\| \boldsymbol{H}_n^m \| \le q < 1, \tag{A.8}$$

take place and one obtains from (A.5) and (A.8) the desired inequality

$$\Delta v_{n+m} < 0, \tag{A.9}$$

The inequality (A.9) provides the convergence of subsequence (A.4) to zero. This result together with the equality (A.7) provides the required convergence of state vectors in norm,

$$\lim_{n\to\infty} \|X_n\| = 0.$$

Remark. As it results from (A.7) and (A.9), a strictly monotone convergence of the sequence $\{\|X_n\|\}$ does not take place.