Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Improving sector-based results for systems with deadzone nonlinearities ⋆

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Abstract: The conservatism of sector-based results for systems consisting of an interconnection of a linear-time-invariant (LTI) system and a static, sector-bounded nonlinearity is well known. Despite this, they are widely used in control system analysis and synthesis. This paper shows how, when the sector-bounded nonlinearity is a deadzone, standard sector based results can be modified to allow the synthesis of a *nonlinear* controller which can deliver improved performance over standard results. The appealing feature of the method is that, as the main part of the design is done using standard sector results, the computation associated with the improved controller is essentially the same as with the standard controller. An override control example is used to illustrate the potential of the results.

1. INTRODUCTION

Sector based stability results form the cornerstone of the theory of analysis of feedback loops involving a stable linear-time-invariant (LTI) system and an isolated, memoryless nonlinearity. Indeed, sector-boundedness of the nonlinearity is vital for establishing both the popular Circle and Popov criteria (see Khalil (1996) for example). These sector-based absolute stability results have found many applications in control engineering due to their relative ease of application and their computational tractability (Boyd et al. (1994)) and are widely used. Their importance is particularly noteworthy in the control of constrained systems where it is often possible to pose the stability problem as that of ensuring the stability of a feedback interconnection of a (normally) LTI part and a deadzone nonlinearity, which is sector bounded. Thus sector-based results feature heavily in the anti-windup literature (Kothare et al. (1994); Mulder et al. (2001); Grimm et al. (2003); Pittet et al. (1997) for example) where researchers have exploited them to design (normally) linear anti-windup compensators. They have also been used in override control, where the aim is to design an "override" controller to prevent output constraints from being exceeded - see Glattfelder et al. (1983); Glattfelder and Schaufelberger (2003); Turner and Postlethwaite (2002a, 2004); Herrmann et al. (2007) for more details.

Unfortunately, sector-based results tend to be conservative because sector-bounding a nonlinearity effectively assigns it to be a member of a certain set, to which other nonlinearities also belong. Thus any results derived on the basis of sector-bounds hold not only for the nonlinearity under consideration but for the whole set of nonlinearities which inhabit the same sector. For example the Sector $[0, I]$ contains both the saturation and deadzone operators, which have very different small and large signal behaviours.

The conservatism of sector based results has been of particular concern in anti-windup control and many authors have sought to overcome these limitations by various methods. A number of researchers have developed generalised sector conditions (Tarbouriech et al. (2006); Hu et al. (2004)) which provide less conservative and more attractive analysis tools for designing AW compensators by providing improved regions of attraction. Other researchers (Nguyen and Jabbari (2000); Turner and Postlethwaite (2002c); Zaccarian and Teel (2004); Lin and Saberi (1995); Turner et al. (2005)) have devoted their attention to performance and have conducted analyses which consider local as well as global performance. While there is merit to all of these approaches, some of them have a rather intricate construction and have significant computational requirements which may prevent their application to many practical or complex problems.

In Turner and Postlethwaite (2002c), the conservatism of sector based approaches was examined from a different perspective, with particular attention to override control. Although the results presented in Turner and Postlethwaite (2002c) were preliminary, they indicated that by augmenting the override controller with a simple nonlinear static element of a particular form, performance improvements could be obtained. This additional nonlinear element was chosen so that the composition of this and the deadzone nonlinearity did not violate the Sector$[0, I]$ constraint which already existed on the deadzone. Thus, the standard sector based results used to derive the linear control (Turner and Postlethwaite (2002a, 2004)) were able to be duplicated in the design of the nonlinear override controller, thus making the computation and construction of the nonlinear override controller virtually identical to that of the linear controller.

This paper takes the same approach as Turner and Postlethwaite (2002c) but widens the results. The basic aim of the paper is to propose simple nonlinear modifications to "controllers" (which may mean anti-windup/override controllers) which enable them to perform better while still meeting standard sector stability requirements. Section 2 formulates the problem in a fairly general way. In particular, although most applications are likely to involve feedback interconnections of LTI systems and sector-bounded nonlinearities, more generality can be obtained by considering general nonlinear systems instead. This makes the results of some relevance to the work of Wu and Soto (2004) where anti-windup for LPV systems is considered. In this section, a crucial lemma gives conditions under which the deadzone nonlinearity can be replaced by a composite nonlinearity which preserves the sector bound. The following section then applies these results to linear cascade systems

⋆ This work was supported in part by the UK Engineering and Physical Sciences Research Council

which arise in, for example anti-windup and override control, and demonstrates how appropriate choices of the additional nonlinearity can "push" the graph of the deadzone closer to the upper sector bound and thus potentially allow one to obtain improved performance. A simple override example shows the effectiveness of the scheme and finally some conclusions are given.

Notation used in the paper is standard. For functions $f(.)$ and $g(.)$ we denote their composition as $f \circ g(.)$, i.e. $f \circ g = f(g(.))$. Extensive use is made of the signum function $\text{sign}(.) : \mathbb{R} \mapsto \mathbb{R}$ which we define as

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{1}$$

2. GENERIC RESULTS

2.1 Problem formulation

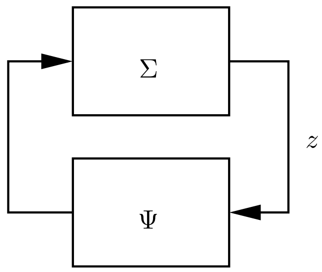

Fig. 1. System with sector-bounded nonlinearity

The problem is first posed in a fairly general way and is then specialised to certain cases of interest. Thus, consider the interconnection in Figure 1 which is described by the equations below:

$$\Sigma \sim \begin{cases} \dot{x} = f(x, v) \\ z = h(x, v) \end{cases} \tag{2}$$

$$v = \Psi(z) \tag{3}$$

where $x \in \mathbb{R}^n$ represents the system's state, $v \in \mathbb{R}^m$ its input and $z \in \mathbb{R}^m$ its output. In typical constrained control applications Σ would be linear time invariant, but this is not necessary. The static nonlinearity $\Psi(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is, for the moment, simply considered to be a memoryless decentralised nonlinearity such that

$$\Psi(z) = [\psi_1(z_1), \ldots, \psi_m(z_m)]' \tag{4}$$

where

$$\psi_i \in \text{Sector}[0, 1] \quad \forall i \in \{1, \ldots, m\} \tag{5}$$

Note that the above equation is sufficient to ensure that $\Psi \in \text{Sector}[0, I]$. Note also that equation (5) is equivalent (Khalil (1996)) to the following inequalities

$$\psi_i^2(z_i) \leq \psi_i(z_i) z_i \leq z_i^2 \quad \forall i \in \{1, \ldots, m\} \tag{6}$$

We assume that sector-based results have been used to prove that the system in Figure 1 is asymptotically stable; formally we make the following assumption.

Assumption 1. The origin of the interconnection of Σ and Ψ is globally asymptotically stable for all $\Psi \in \text{Sector}[0, I]$.

Such an assumption can normally be made, for example in the context of anti-windup compensation, if absolute stability results, such as the Circle or Popov Criteria, have been used to establish the stability of the system and the nonlinearity in question is $\Psi(z) = \text{Dz}(z)$, where $\text{Dz}(z)$ is defined as

$$\text{Dz}(z) = [\text{Dz}_1(z_1), \ldots, \text{Dz}_m(z_m)]' \tag{7}$$

where

$$\text{Dz}_i(z_i) = \begin{cases} 0 & |z_i| < \bar{z}_i \\ \text{sign}(z_i)(|z_i| - \bar{z}_i) & |z_i| \geq \bar{z}_i \end{cases} \tag{8}$$

and $\bar{z}_i > 0 \quad \forall\, i$. It is easy to see that $\text{Dz}_i \in \text{Sector}[0, 1] \quad \forall\, i$ and thus $\text{Dz} \in \text{Sector}[0, I]$. It is convenient to define the set

$$\mathcal{Z} = [-\bar{z}_1, \bar{z}_1] \times \ldots \times [-\bar{z}_m, \bar{z}_m] \tag{9}$$

and note that $\text{Dz}(z) = 0 \quad \forall z \in \mathcal{Z}$. However, also note that the deadzone nonlinearity is one of many nonlinearities inhabiting $\text{Sector}[0, I]$ and thus any results derived on this basis will be conservative. Many researchers Hindi and Boyd (1998); Pittet et al. (1997); Wu and Soto (2004) have noted that, for finite z, the sector which the deadzone inhabits can be narrowed to $\text{Dz} \in \text{Sector}[0, \epsilon I]$ for $\epsilon \in (0, 1)$. For $\epsilon < 1$ this means that the graph of the deadzone is closer to the "narrowed" sector boundary. Figure 2 illustrates this idea for the scalar deadzone case. In the context of anti-windup compensation, this often allows the synthesis of more aggressive anti-windup compensators which can deliver better performance (Zaccarian and Teel, 2004).

This sector narrowing approach ensures that the graph of the deadzone (for $z < \bar{z}^{(2)}$) is closer to its upper boundary and is actually on this upper boundary for $|z| = \bar{z}^{(2)}$ and therefore the hope is that this will reduce conservatism and allow better performance to be obtained. In this paper we shall attempt to "push" the graph of the deadzone closer to its upper boundary by adding another nonlinear function to increase the "gain" of the composition of this function and the deadzone, again in the hope that this will allow improved performance to be obtained.

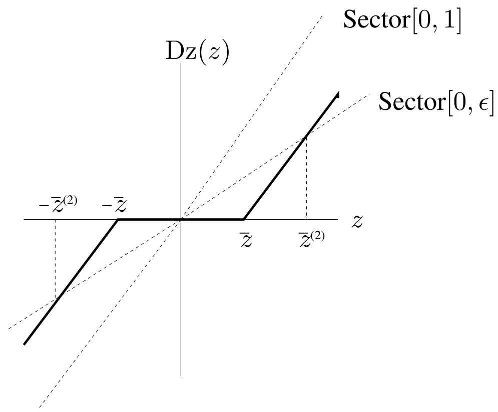

Fig. 2. Deadzone with local and global sector bounds

2.2 Main results

The main technical result of this paper shows how the deadzone nonlinearity can be augmented with another static nonlinear element, $\mathcal{Q}(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ in order to "push" the graph of the composite nonlinear element, $\mathcal{Q} \circ \text{Dz}(.)$ closer to the sector boundary to extract better performance from the system. Subsequent sections will then apply this general result to specific areas of interest. Thus, the crucial technical result of the paper is

Lemma 1. The decentralised nonlinearity

$$\mathcal{Q}(.) = [\mathcal{Q}_1(.), \ldots, \mathcal{Q}_m(.)]' \quad \mathcal{Q}_i(.) : \mathbb{R} \mapsto \mathbb{R} \tag{10}$$

is such that $\mathcal{Q} \circ \text{Dz}(.) \in \text{Sector}[0, I]$ and Lipschitz if and only if $\mathcal{Q}(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ satisfies the following properties.

(1) $\mathcal{Q}(.)$ is Lipschitz.
(2) $\mathcal{Q}(0) = 0$
(3) $\mathcal{Q}(.)$ is such that $\text{sign}(Q_i(x_i)) = \text{sign}(x_i)$ or $\text{sign}(Q_i(x_i)) = 0$ for all $i \in \{1, 2, \ldots, m\}$.

(4) $\mathcal{Q}(.)$ is such that $|\mathcal{Q}_i(x_i)| \leq |x_i| + \bar{z}_i$ for all $i \in \{1, 2, \ldots, m\}$.

Proof: As both $\mathcal{Q}(.)$ and $\mathrm{Dz}(.)$ are decentralised it suffices to prove items 1-4 for $\mathcal{Q}_i \circ \mathrm{Dz}_i(.)$.

ITEM 1

As $\mathrm{Dz}(.)$ is Lipschitz, then $\mathcal{Q} \circ \mathrm{Dz}(.)$ is Lipschitz if and only if $\mathcal{Q}(.)$ is Lipschitz.

ITEMS 2-4

These items pertain to the sector boundedness of $Q_i \circ \mathrm{Dz}_i(.)$ and by equation (5) require the following inequality to hold

$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]^2 \leq [\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]z_i \leq z_i^2 \tag{11}$$

We first prove that Items 2-4 are sufficient for this inequality to hold and then prove they are also necessary.

Sufficiency: $z \in \mathcal{Z}$. Note that if $z \in \mathcal{Z}$, then $\mathrm{Dz}_i(z_i) = 0 \ \ \forall i$. In this case, by Item 2, inequality (11) reduces to $0 \leq 0 \leq z_i^2$ which holds trivially.

Sufficiency: $z \notin \mathcal{Z}$. In this case $z_i > \bar{z}_i$ and thus $\mathrm{Dz}(z_i) = \mathrm{sign}(z_i)(|z_i| - \bar{z}_i)$. Thus we have

$$[\mathcal{Q}_i \circ \mathrm{Dz}(z_i)]z_i = \mathrm{sign}[\mathcal{Q}_i \circ \mathrm{Dz}(z_i)]|\mathcal{Q}_i \circ \mathrm{Dz}(z_i)| \times \mathrm{sign}(z_i)|z_i|$$

By item 3, we then have two options:

- First assume that $\mathrm{sign}(Q_i(x_i)) = \mathrm{sign}(x_i)$. Then we have
$$\begin{aligned}[\mathcal{Q}_i \circ \mathrm{Dz}(z_i)]z_i &= \mathrm{sign}(z_i)^2|\mathcal{Q}_i \circ \mathrm{Dz}(z_i)||z_i| \\ &= |\mathcal{Q}_i \circ \mathrm{Dz}(z_i)||z_i| \\ &\leq ||z_i| - \bar{z}_i + \bar{z}_i||z_i| = z_i^2 \end{aligned} \tag{12}$$
where the inequality is due to Item 4. This proves the right inequality in (11).
Also note that as $\mathrm{sign}[\mathrm{Dz}(z_i)] = \mathrm{sign}(z_i)$, by Item 3 we have
$$\begin{aligned}&[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]^2 \\ &= \mathrm{sign}(z_i)^2|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| \times |\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| \\ &\leq \mathrm{sign}(z_i)^2|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| \times (|\mathrm{Dz}_i(z_i)| + \bar{z}_i) \\ &= \mathrm{sign}(z_i)^2|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| \times (|z_i| - \bar{z}_i + \bar{z}_i) \\ &= \mathrm{sign}(z_i)^2|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| \times |z_i| \\ &= [\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]z_i \end{aligned} \tag{13}$$
which proves the left hand inequality in (11).
- Next assume that $\mathrm{sign}(\mathcal{Q}_i(x_i)) = 0$, which implies that $\mathcal{Q}_i(x_i) = 0$. In this case we have
$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]^2 = 0 \tag{14}$$
$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]\, z_i = 0 \tag{15}$$
Thus both inequalities in (11) are seen to hold trivially.

Necessity: $z \in \mathcal{Z}$. Assume that $\mathrm{Q}_i(0) \neq 0$ and instead $\mathrm{Q}_i(0) = \alpha_i$. Then for $z = 0 \in \mathcal{Z}$ it follows that

$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]^2 = \alpha_i^2 \tag{16}$$

$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]\, z_i = 0 \tag{17}$$

which obviously violates the sector condition (11). Thus Item 2 is necessary.

Necessity: $z \notin \mathcal{Z}$. Again consider

$$[\mathcal{Q}_i \circ \mathrm{Dz}(z_i)]z_i = \mathrm{sign}[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| \times \mathrm{sign}(z_i)|z_i|$$

If Item 3 is not satisfied, then this implies that

$$\mathrm{sign}[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)] = -\mathrm{sign}(z_i) \tag{18}$$

and thus also that

$$\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i) \neq 0 \tag{19}$$

Thus we have that

$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]z_i = -\mathrm{sign}(z_i)^2|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)||z_i| < 0 \tag{20}$$

Now as $[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]^2 \geq 0$, the sector condition (11) is violated. Thus Item 3 is necessary.

Next assume Item 3 is satisfied (as it is necessary) but that Item 4 is not satisfied; i.e. there exists a z_i such that $|\mathcal{Q}_i \circ \mathrm{Dz}_i(z)| > |\mathrm{Dz}_i(z_i)| + \bar{z}_i$. For this to be the case, it follows that $z_i \notin \mathcal{Z}$ (otherwise we would have $0 > |\mathrm{Dz}_i(z_i)z_i| + \bar{z}_i$ which can never hold). Again we have two options:

- First, when $\mathrm{sign}(\mathcal{Q}_i(x_i)) = \mathrm{sign}(x_i)$ we have $\mathrm{sign}[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)] = \mathrm{sign}(z_i)$ and thus we obtain
$$[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)]z_i = |\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)||z_i| \tag{21}$$
$$> (|z_i| - \bar{z}_i + \bar{z}_i)|z_i| \tag{22}$$
$$= z_i^2 \tag{23}$$
which contradicts the sector condition (11).
- Next, when $\mathrm{sign}(\mathcal{Q}_i(x_i)) = 0$. Thus we have $\mathrm{sign}[\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)] = 0$ which implies $\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i) = 0$. Thus if Item 4 is not satisfied we get
$$|\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i)| > |\mathrm{Dz}_i(z_i)| + \bar{z}_i > 0 \tag{24}$$
But as $\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i) = 0$ by assumption, we have a contradiction.

Thus we see that Item 4 is also necessary. □□

Remark 1: Item 3 is quite general and it is normally desirable to replace this with the assumption that $\mathcal{Q}(.)$ is an odd nonlinearity, which would be more useful in practice. Obviously this reduces the necessary and sufficient nature of Lemma 1 to being simply sufficient. □

Remark 2: One of the key conditions on the nonlinearity $\mathcal{Q}(.)$ is that $|\mathcal{Q}_i(x)| \leq |x_i| + \bar{z}_i$. This suggests that $\mathcal{Q}_i(x_i)$ would typically take the following form

$$\mathcal{Q}_i(x_i) = \eta_i(x_i)(|x_i| + \bar{z}_i), \quad \eta_i(x_i) \in [0, 1] \tag{25}$$

Note as $|x_i| + \bar{z}_i$ is even, it follows that $\eta_i(x_i)$ must be odd to ensure $\mathcal{Q}_i(x_i)$ is also odd. □

Remark 3: Lemma 1 is actually a special case of a more general result. A similar proof to that given above can be used to prove that a decentralised $\mathcal{Q} \circ \mathrm{Dz}(.) \in \mathrm{Sector}[0, \epsilon I]$ and Lipschitz if and only if $\mathcal{Q} : \mathbb{R}^m \mapsto \mathbb{R}^m$ satisfies the first 3 properties of Lemma 1 and the fourth property is replaced by

4. $\mathcal{Q}(.)$ is such that $|\mathcal{Q}_i(x_i)| \leq \epsilon|x_i| + \bar{z}_i$ for all $i \in \{1, 2, \ldots, m\}$.

This allows the results here to be adapted to the cases when global asymptotic stability is not achievable and one has to consider local sector bounds ($\epsilon \in (0, 1)$) to obtain local stability results. See Hindi and Boyd (1998) and Pittet et al. (1997) for more discussion. □

Remark 4: Lemma 1 guarantees that the composition $\mathcal{Q} \circ \mathrm{Dz}(.)$ is Lipschitz continuous and normally this is desirable. However, the class of $\mathcal{Q}(.)$ preserving sector boundedness can be widened if one drops the Lipschitz requirement. □

The results of Lemma 1 and Assumption 1 can now be assembled to form the main Theorem in the paper.

Theorem 2. Let Assumption 1 be satisfied. Then the origin of the interconnection of Σ and Ψ is globally asymptotically stable when $\Psi(.) \equiv \mathcal{Q} \circ \mathrm{Dz}(.)$ where $\mathcal{Q}$ satisfies the conditions of Lemma 1.

Proof: Assumption 1 implies that the origin of Σ is globally asymptotically stable when $\Psi \in \mathrm{Sector}[0, I]$. Thus when Ψ is chosen as $\Psi \equiv \mathcal{Q} \circ \mathrm{Dz}(.)$ where $\mathcal{Q}$ satisfies the conditions of Lemma 1, it follows that $\Psi \in \mathrm{Sector}[0, I]$ and thus the system is globally exponentially stable. □□

3. APPLICATIONS

3.1 Linear systems with deadzone nonlinearities

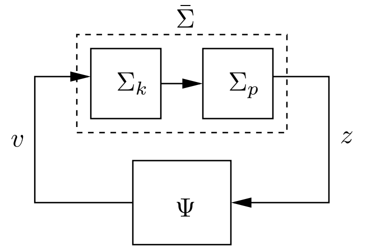

Fig. 3. Cascade system with sector-bounded nonlinearity

The obvious and most straightforward place to apply these results is to the control of linear systems which contain saturation and/or deadzone nonlinearities. It is well known that the saturation and deadzone nonlinearities satisfy the following identity

$$\mathrm{sat}(u) = u - \mathrm{Dz}(u) \tag{26}$$

where $\mathrm{sat}(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is the standard decentralised saturation nonlinearity. This allows many saturation problems to be equivalently posed as deadzone problems. This immediately makes the results of Section 2 applicable to many constrained control problems of practical interest. In fact in such situations we are often faced with cascade systems of the form

$$\bar{\Sigma} \sim \begin{cases} \left.\begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_p y_k \\ z &=& C_p x_p + D_p y_k \end{array}\right\} \sim \Sigma_p \\ \left.\begin{array}{rcl} \dot{x}_k &=& A_k x_k + B_k v \\ y_k &=& C_k x_k + D_k v \end{array}\right\} \sim \Sigma_k \end{cases} \tag{27}$$

$$v = \Psi(z) \tag{28}$$

as shown in Figure 3 where $\Psi \equiv \mathrm{Dz}$. In this Figure, $\bar{\Sigma}$ is a cascade of a "plant" Σ_p, which is given, and a "controller" Σ_k which is to be designed. Note the designation "plant" and "controller" may be something of a misnomer and in, for example anti-windup and override control, the "plant" may actually represent the physical system plus the linear controller and the "controller" represents the anti-windup compensator or override controller to be designed. In most cases, this problem is then solved by sector bounding the deadzone and using absolute stability results to obtain a *linear* controller. Using the results of the previous section we can easily obtain simple nonlinear controllers which provide the same stability guarantees as their linear counterparts, but with potentially improved performance.

Corollary 3. Assume there exists a linear Σ_k, as described by equation (27), such that the system consisting of the interconnection of Σ_k, Σ_p (27) and $v = \Psi(z)$ where $\Psi \in \mathrm{Sector}[0, I]$ is globally asymptotically stable. Define $\tilde{\Sigma}_k$ as

$$\tilde{\Sigma}_k \sim \begin{cases} \dot{x}_k &= A_k x_k + B_k v \\ y_k &= C_k x_k + D_k v \\ v &= \mathcal{Q}(w) \end{cases} \tag{29}$$

where $\mathcal{Q}(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ satisfies the assumptions of Lemma 1. Then the interconnection of $\tilde{\Sigma}_k$, Σ_p and $w = \mathrm{Dz}(z)$ is globally asymptotically stable.

Proof: The proof is straightforward: simply note that the cascade of Σ_k and Σ_p can be put into the form of Σ as defined in equation (2); also, as $\mathcal{Q}(.)$ satisfies the assumptions of Lemma 1, it follows that the composition $\mathcal{Q} \circ \mathrm{Dz}(.) \in \mathrm{Sector}[0, I]$. Application of Theorem 1 thus completes the proof. □□

3.2 Performance enhancement

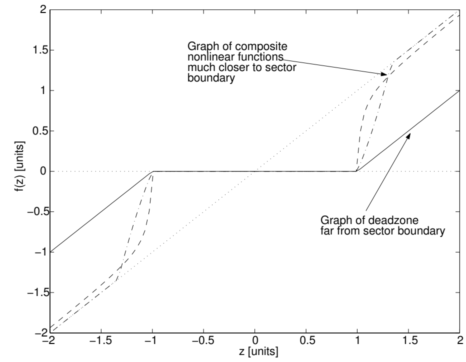

Fig. 4. Sector$[0, 1]$ and graphs of $\mathrm{Dz}_i(.)$ (solid), $Q_{i,frac} \circ \mathrm{Dz}_i(.)$ (dashed) and $Q_{i,sat} \circ \mathrm{Dz}_i(.)$ (dash-dotted)

The main reason to use the nonlinear $\tilde{\Sigma}_k$ instead of the linear Σ_k is to improve performance. Note that in the situation of linear systems with deadzone nonlinearities, both linear and nonlinear "controllers" are driven by the deadzone, the graph of which is depicted in Figure 4. For the linear case, observe that if z strays outside $\mathcal{Z}$ (in the case of Figure 4, $\mathcal{Z} = [-1, 1]$) by only a small amount, then the value of $\mathrm{Dz}(z)$ will only be small, which, by linearity, will only correspond to a small "reaction" by the controller Σ_k. The natural objective of the nonlinear "controller" Σ_k would be to increase the "gain" of the override controller when z_i is small, and thus far from the sector boundary, and make the graph of $\mathcal{Q} \circ \mathrm{Dz}(.)$ lie closer to the sector boundary.

There is obviously a continuum of suitable choices for $\mathcal{Q}(.)$ but we will choose functions which have the following properties, in addition to those stipulated by Lemma 1: $\mathcal{Q}$ is a monotonically non-decreasing function and $\lim_{z \to \infty} \{\mathcal{Q} \circ \mathrm{Dz}(z) - z\} = 0$. One such function is

$$\mathcal{Q}_{i,frac}(x_i) = \underbrace{\frac{k_i x_i}{\epsilon_i + k_i |x_i|}}_{\eta_i(x_i)} (|x_i| + \bar{z}_i) \tag{30}$$

where $k_i > 0$ and $\epsilon_i > 0$. A graph of the composition $\mathcal{Q}_{i,frac} \circ \mathrm{Dz}_i(.)$ with $k_i = 3, \epsilon_i = 0.1$ is shown in Figure 4. Note that as expected $\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i) = 0$ when $z_i \in \mathcal{Z} = [-\bar{z}_i, \bar{z}_i]$, and then the graph approaches the unity gradient line (the boundary of the sector) asymptotically. The value of k_i/ϵ_i dictates the rate at which the graph converges to unity.

Another possible function of a somewhat simpler, although non-smooth, form is

$$\mathcal{Q}_{i,sat}(x_i) = \mathrm{sign}(x_i) \min\{1, k_i|x_i|\} (|x_i| + \bar{z}_i) \tag{31}$$

where again $k_i > 0$. A graph of the composition $\mathcal{Q}_{i,sat} \circ \mathrm{Dz}_i(.)$ with $k_i = 3$ is shown in Figure 4. Note that as expected $\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i) = 0$ when $z_i \in \mathcal{Z} = [-\bar{z}_i, \bar{z}_i]$ but as soon as $|z_i| = \bar{z}_i/k_i$, the composition $\mathcal{Q}_i \circ \mathrm{Dz}_i(z_i) = z_i$ and remains there thereafter. For large value of k_i, the gradient of the line connecting the zero portion of the function to the unity gradient line becomes steeper, and begins to approximate a discontinuous function. However, provided k_i is finite, $\mathcal{Q}_{i,sat}(.)$ remains Lipschitz.

4. EXAMPLE

There are two control strategies to which these results are logically applied: anti-windup control and override control. It has been shown in, for example Zaccarian and Teel (2004) and associated papers, that the anti-windup stabilisation problem can be written as a cascade of the form described in $\bar{\Sigma}$. In that paper, a nonlinear anti-windup compensator which switched anti-windup gains on the basis of positively invariant sets and, as the signal z converged to zero, successively more aggressive compensator gains were used to obtain improved performance. In that paper, promising results were demonstrated but the nonlinear compensator's construction was rather complex as its construction relied on the solution of a set of LMI's, the number of which increased as the number of gains was increased. An alternative strategy, which we explore here, is to design only one compensator but introduce nonlinear activation characteristics, through $\mathcal{Q}(.)$, which allow it to have larger "gain" for smaller signals.

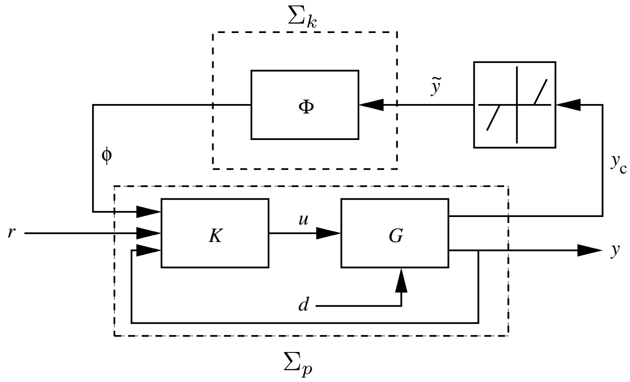

Fig. 5. Override control scheme

Another control strategy which can take advantage of these results is override control, which is rather like anti-windup except, one considers constraints on outputs rather than inputs. A reasonably generic override control scheme is shown in Figure 5 where the measured outputs used for standard linear behaviour are denoted y and the constrained outputs are denoted y_c - these are required to remain below certain limits, $\bar{y}_c$. It can easily be seen that its structure is exactly the same as that of $\bar{\Sigma}$ with the interconnection of the linear plant $G(s)$, and the controller $K(s)$ playing the role of Σ_p and the override controller, $\Phi(s)$ playing the role of Σ_k. The signal y_c is equivalent to z and the signal $\tilde{y}$ is equivalent to v. The aim with override control is to ensure that the constrained output y_c does not exceed imposed limits, $\bar{y}_c$ by designing a suitable override controller $\Phi(s)$, which is activated when the limits are transiently exceeded. The aim here is to improve the linear override controller by using a nonlinear design $\tilde{\Phi} = \Phi \circ \mathcal{Q}$ for suitable $\mathcal{Q}$ which satisfy Lemma 1.

4.1 Exponentially unstable example

We consider the same exponentially unstable example as in Turner and Postlethwaite (2002c) in which the linear plant $G(s)$ has the following state-space realisation

$$G(s) \sim \begin{cases} \dot{x}_1 = x_1 - x_2 + u \\ \dot{x}_2 = x_1 \\ \;\; y = x_1 + 10x_2 \\ y_c = u \end{cases} . \tag{32}$$

A 4th order $\mathcal{H}_\infty$ loop-shaping controller using shaping functions $\tilde{W}_1 = \frac{s+10}{s}, \tilde{W}_2 = 1$ was designed for this plant using the `ncfsyn` command in the Matlab μ-analysis and synthesis toolbox. This yielded a good nominal linear response without constraints. The controller's state-space realisation is given by

$$K(s) \sim \left[\begin{array}{c|cc} A_c & B_c & B_{cr} \\ \hline C_c & D_c & D_{cr} \end{array}\right]$$
$$= \left[\begin{array}{cccc|cc} -28.2357 & -260.6437 & 1.1419 & 0 & 25.0283 & 0 \\ -1.8851 & -28.8506 & 0 & 0 & 2.8851 & 0 \\ -35.9512 & -256.0618 & -6.3891 & 0 & 22.6459 & 0 \\ -13.3052 & -29.6024 & -6.3891 & 0 & 0 & 0.8391 \\ \hline -4.2075 & -9.3611 & -2.0204 & 3.1623 & 0 & 0.2654 \end{array}\right].$$

As $G(s)$ is exponentially unstable, it is well-known that this system cannot be globally asymptotically stabilised with bounded feedback and the positive eigenvalues may cause problems with some LMI based anti-windup design procedures. Thus, as advocated in Turner and Postlethwaite (2002b) and Glattfelder and Schaufelberger (2007) (see also Glattfelder and Schaufelberger (2004)) we use an override controller to ensure, indirectly, that the control limits are respected as much as possible. In particular we assume the control limits are ± 0.25 and that the constrained "output" is taken as the control input i.e $y_c = u$. Limits of $\bar{y}_c = \pm 0.24$ were then applied to y_c to ensure that the override controller would become active just before the control limits were violated. Using the results of Turner and Postlethwaite (2002a), when override control is to be used, the linear controller is augmented to have the form (see Figure 5)

$$K(s) \sim \begin{cases} \dot{x}_c = A_c x_c + B_c y + B_{cr} r + [I \quad 0]\phi \\ u = C_c x_c + D_c y + D_{cr} r + [0 \quad I]\phi \end{cases} \tag{33}$$

In this realisation, the additional signal contributed by the override controller is $\phi \in \mathbb{R}^{n_c+m}$, where n_c and m are the controller state and control input dimensions respectively. When override control is inactive, $\phi \equiv 0$ and the nominal linear controller given above is recovered. The override control signal is computed as $\phi = \Phi(s)\tilde{y}$ and in this case a linear override controller Φ was obtained according to the techniques described in Turner and Postlethwaite (2002a). In particular Φ was obtained as a static gain which ensured the system was stable for all nonlinearities in $\text{Sector}[0, I]$. Thus Φ was computed as

$$\Phi = [\, -3.4844 \;\; -0.1935 \;\; -14.1398 \;\; -11.8456 \;\; 0.1515 \,]'$$

In order to improve the performance of the linear override controller we choose $Q(.)$ in the forms indicated in Section 3. Specifically, $\mathcal{Q}$ was chosen as

$$\mathcal{Q}_{sat}(\tilde{y}) = \text{sign}(\tilde{y}) \min\{1, 20|\tilde{y}|\} (|\tilde{y}| + \bar{y}_c) \tag{34}$$

$$\mathcal{Q}_{frac}(\tilde{y}) = \frac{10\tilde{y}}{0.1 + 10|\tilde{y}|} (|\tilde{y}| + \bar{y}_c) \tag{35}$$

Thus three different override controllers were obtained: Φ, $\Phi_{sat} := \Phi \mathcal{Q}_{sat}(.)$ and $\Phi_{frac} := \Phi \mathcal{Q}_{frac}(.)$

Figure 6 shows the response of the system to a step demand of one unit for various situations. The thin solid line shows the nominal linear response without constraints; observe the fast, smooth response with little overshoot. The dotted line shows the response of the system with constraints but no override control; this response is significantly more oscillatory than the linear response. The linear override (i.e. when Φ alone was used) response is shown by the dashed line; in this case the response is noticeably less oscillatory than without override

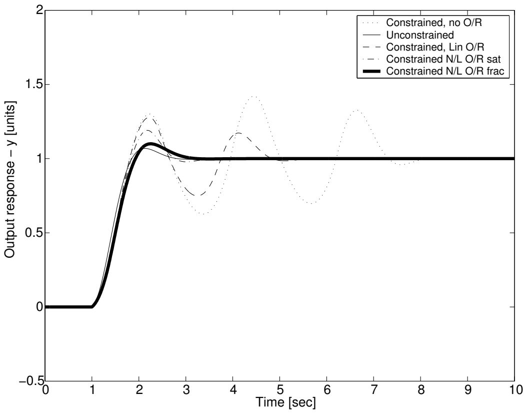

Fig. 6. Step response: $y(t)$

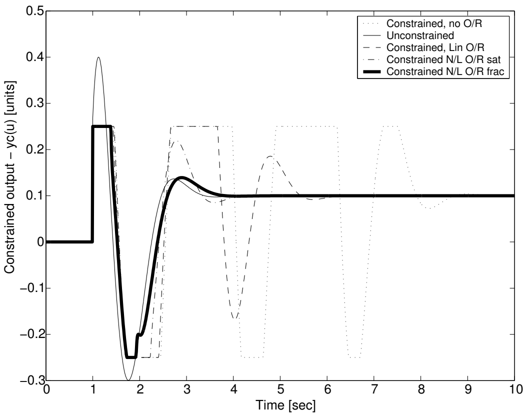

Fig. 7. Step response: $y_c(t)$ $(= u(t))$

but there is significantly more overshoot than the linear case, and the settling time is much longer. The dash-dotted response shows the nonlinear override response when $\Phi \equiv \Phi_{sat}(.)$ is used and the thick solid line shows the response when $\Phi \equiv \Phi_{frac}(.)$ is used. Both these responses are significantly better than the linear override response and actually remarkably close to the linear response, indicating the advantage of using nonlinear override control. The corresponding control responses are shown in Figure 7. Notice that the nonlinear override controllers prevent extended periods of control saturation which can lead to the oscillatory behaviour. It is also interesting to note that when the step demand is increased to 1.2 units, both the constrained system without override and the constrained system with *linear override* become unstable, whereas the *nonlinear override* controllers are able to maintain the stability of the system.

5. CONCLUSION

This paper has proposed a simple modification to sector based stability results for systems containing deadzones. The results show how a simple modification of the "controller" can be made to improve performance without any corresponding increase in computational burden; essentially they allow a linear controller to be transformed into an nonlinear controller without forfeiting stability guarantees. The results have obvious application in constrained control, anti-windup compensation and override control and it appears that they can be particularly useful in the area of override control.

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