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Robust Control of Uncertain Switched Delay Systems: a Sliding Mode Control Design★

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Abstract: This paper investigates the robust sliding mode control problem for a class of uncertain switched delay systems. A single sliding surface is constructed such that the reduced-order equivalent sliding motion restricted to the sliding surface is completely invariant to all admissible uncertainties. For the cases of known delay and unknown delay, the existence conditions of the sliding surface are proposed, respectively. The corresponding hysteresis switching laws are designed to asymptotically stabilize the sliding motion. Furthermore, variable structure controllers are developed to drive the state of the switched system to reach the single sliding surface in finite time and remain on it thereafter. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

1. INTRODUCTION

The sliding mode control (SMC) has various attractive features such as fast response, good transient response and order-reduction (Roh & Oh, 1999; Choi, 2007). It is also insensitive to variations in system parameters and external disturbances. Generally the SMC is to employ a discontinuous control to drive the state from an arbitrarily initial state to along a desired prespecified trajectory. In recent years more and more research in this area has been done (Utkin, 1977; Choi, 2003; Xia & Jia, 2003; Kim, Park & Oh, 2000; Gouaisbaut, Dambrine & Richard, 2002).

For switched systems, only a few research results in which the SMC technique is employed exist due to the complexity of control systems and the excess burden of the control synthesis and switching law design. Akar & Ozguner (1998) proposed a SMC method to make nominal switched systems exponentially stable. In this paper the existence conditions of sliding modes were given and a state feedback controller was designed such that sliding modes occur. Variable structure control with sliding mode sector was presented for a hybrid system in Pan, Suzuki & Furuta (2005). The sliding mode sector was defined as subspace inside which some norm of state decrease for each subsystem of the hybrid system, and a variable structure control law was designed to switch the hybrid system among subsystem to ensure its quadratic stability.

On the other hand, time-delay is often encountered in various industrial systems. Switched systems with time-delay are one of the most useful models and have strong engineering background such as power systems (Meyer, Schroder & Doncker, 2004) and networked control systems (Kim, Prak & Ko, 2004). However, due to the complicated behaviour of switched delay systems, very few results on such systems have appeared. Sufficient conditions of asymptotical stability were established for switched linear delay systems under arbitrary and constructed switching signals respectively in Xie & Wang (2004). Sun, Wang & Xie (2006) investigated the problem of delay-dependent common Lyapunov functions for switched linear delay systems, which established the relationship between delay-dependent common Lyapunov functions and the common Lyapunov functions for corresponding switched systems without delays. The stabilization problem of arbitrary switched linear systems with unknown time varying delays was considered in Hetel, Daafouz & Iung (2006). For uncertain linear discrete-time switched systems with state delays, sufficient conditions of robust stability and stabilizability in terms of matrix inequalities and Riccati-like inequalities were given in Phat (2005). Stability of a class of switched delay systems was shown in Kim Campbell & Liu (2006) by using a common Lyapunov functional method. However, to the best of the authors' knowledge, there are no results for the SMC of switched delay systems in the current literature, which is indeed our motivation.

This paper considers the robust SMC problem for a class of uncertain switched delay systems. A single sliding surface is constructed such that the reduced-order equivalent sliding

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motion restricted to the sliding surface is completely invariant to all admissible uncertainties. For the delay-known case, a sufficient condition of the existence of the sliding surface is given in terms of linear matrix inequalities (LMIs), and by using the information of current state and delay-state, a hysteresis switching law is designed to guarantee the stability of the sliding motion. For the delay-unknown case, a sufficient condition of the existence of the sliding surface is given by solving Riccati inequality, and the corresponding hysteresis switching law that only depend on the current state is designed. Variable structure controllers are developed respectively for two cases such that the state of the switched system reach the single sliding surface in finite time and remain on it thereafter.

Throughout this paper, $\|\bullet\|$ denotes the Euclidean norm for a vector or the matrix induced norm for a matrix.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the uncertain switched delay system of the form

$$\begin{aligned}\dot{x}(t) &= (A_\sigma + \Delta A_\sigma)x(t) + (A_{d\sigma} + \Delta A_{d\sigma})x(t-\tau) \\ &\quad + B[u_\sigma(t) + Z_\sigma(t)u_\sigma(t) + f_\sigma(x,t)], \qquad (1) \\ x(t) &= \varphi(t),\ t\in[-\tau,0],\end{aligned}$$

where $x(t)\in R^n$ is the system state, $\sigma:[0,\infty)\to\Xi=\{1,2,\ldots,l\}$ is the piecewise constant switching signal that might depend on time t or state x, $u_i\in R^m$ is the control input of the $i-th$ subsystem, A_i, A_{di}, B are constant matrices with appropriate dimensions, $\varphi(t)$ is a differentiable vector-valued initial function on $[-\tau,0]$, ΔA_i and ΔA_{di} represent system parameter uncertainties, $Z_i(t)$ and $f_i(x,t)$ represent the input matrix uncertainty and nonlinearity of the system, respectively. The following standard assumptions are introduced.

Assumption 1. The parameter uncertainties can be represented and emulated as

$$[\Delta A_i \quad \Delta A_{di}] = \left[D_{1i}\Sigma_{1i}(t) \quad D_{2i}\Sigma_{2i}(t)\right]E\ ,\ i\in\Xi\ ,$$

where D_{1i}, D_{2i} and E are constant matrices with appropriate dimensions and the matrix E is right invertible. $\Sigma_{1i}(t)$ and $\Sigma_{2i}(t)$ are unknown matrices with Lebesgue measurable elements and satisfy ${\Sigma_{1i}}^{\mathrm{T}}\Sigma_{1i}\le I$, ${\Sigma_{2i}}^{\mathrm{T}}\Sigma_{2i}\le I$.

Assumption 2. The input matrix B has full rank m and $m<n$.

Assumption 3. There exist known nonnegative scalar-valued functions $\phi_i(x,t)$, $i\in\Xi$ such that $\|f_i(x,t)\|\le\phi_i(x,t)$ for all t.

Assumption 4. There exist known nonnegative constants ρ_i, $i\in\Xi$ such that $\|Z_i(t)\|\le\rho_i<1$ for all t.

Remark 1. Assumptions 1~4 are standard assumptions in the study of variable structure control.

Let Γ be an $n\times n$ symmetric matrix satisfying

$$\Gamma = I - E^g E, \qquad (2)$$

where E^g is the Moore-Penrose inverse of E.

Remark 2. If the matrix E is not right invertible, we can make a decomposition of E, that is, to express E as the product of a left invertible matrix and a right invertible matrix. Let (E_1,E_2) is any full-rank factor, i.e., $E=E_1E_2$, where E_1 is a left invertible matrix and E_2 is a right invertible matrix, then we can easily obtain the Moore-Penrose inverse of E as $E^g = E_2^{\mathrm{T}}(E_2E_2^{\mathrm{T}})^{-1}(E_1^{\mathrm{T}}E_1)^{-1}E_1^{\mathrm{T}}$.

We design the single sliding surface for the switched system (1) as

$$\zeta(t) = Sx(t) = B^{\mathrm{T}}(\Gamma X\Gamma + BYB^{\mathrm{T}})^{-1}x(t) = 0\,, \qquad (3)$$

where X and Y are symmetric matrices which will be determined latter.

Remark 3. The single sliding surface $\zeta(t)=Sx(t)=0$ is designed such that the switched delay system (1) is asymptotically stable based on the single Lyapunov function approach in the sliding surface. The purpose of designing the single sliding surface for the switched delay system is to reduce the reaching phase in which systems are sensitive to uncertainties and perturbations, and improve the transient performance and robustness.

Lemma 1. For the system (1) and the sliding surface (3), the sliding motion dynamics restricted to the sliding surface is

$$\begin{aligned}\dot{\xi}_1(t) &= \widetilde{B}^{\mathrm{T}}A_\sigma P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t) \\ &\quad + \widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t-\tau).\end{aligned} \qquad (4)$$

Proof. To get a regular form of the system (1), we define a nonsingular matrix G and an associated vector ξ as follows

$$G = \begin{bmatrix}\widetilde{B}^{\mathrm{T}} \\ S\end{bmatrix} = \begin{bmatrix}\widetilde{B}^{\mathrm{T}} \\ B^{\mathrm{T}}P^{-1}\end{bmatrix}, \qquad (5)$$

where $\widetilde{B}$ is an orthogonal complement of the matrix B, $P=\Gamma X\Gamma + BYB^{\mathrm{T}}$ and

$$\xi(t) = \begin{bmatrix}\xi_1(t) \\ \xi_2(t)\end{bmatrix} = Gx(t) = \begin{bmatrix}\widetilde{B}^{\mathrm{T}} \\ B^{\mathrm{T}}P^{-1}\end{bmatrix}x(t) \qquad (6)$$

with $\xi_1\in R^{n-m}$, $\xi_2=\zeta\in R^m$. Note that the matrix G is invertible. Indeed, it can be checked that

$$G^{-1} = \left[P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1} \quad B(SB)^{-1}\right]. \qquad (7)$$

By the state transformation (6), the system (1) is represented by the following regular form

$$\begin{aligned}\begin{bmatrix}\dot{\xi}_1(t) \\ \dot{\zeta}(t)\end{bmatrix} &= \begin{bmatrix}\overline{A}_{\sigma11} & \overline{A}_{\sigma12} \\ \overline{A}_{\sigma21} & \overline{A}_{\sigma22}\end{bmatrix}\begin{bmatrix}\xi_1(t) \\ \zeta(t)\end{bmatrix} + \begin{bmatrix}\overline{A}_{d\sigma11} & \overline{A}_{d\sigma12} \\ \overline{A}_{d\sigma21} & \overline{A}_{d\sigma22}\end{bmatrix}\begin{bmatrix}\xi_1(t-\tau) \\ \zeta(t-\tau)\end{bmatrix} \\ &\quad + \begin{bmatrix}0 \\ SB\end{bmatrix}(u_\sigma(t) + Z_\sigma u_\sigma(t) + f_\sigma(x,t)), \qquad (8) \\ \xi_1(t) &= \overline{\varphi}_1(t), t\in[-\tau,0], \\ \zeta(t) &= \overline{\varphi}_2(t), t\in[-\tau,0],\end{aligned}$$

where

$$\overline{A}_{\sigma11} = \widetilde{B}^{\mathrm{T}}[A_\sigma + D_{1\sigma}\Sigma_{1\sigma}(t)E]P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\,,$$

$$\overline{A}_{\sigma12} = \widetilde{B}^{\mathrm{T}}[A_\sigma + D_{1\sigma}\Sigma_{1\sigma}(t)E]B(SB)^{-1}\,,$$

$$\overline{A}_{d\sigma11} = \widetilde{B}^{\mathrm{T}}[A_{d\sigma} + D_{2\sigma}\Sigma_{2\sigma}(t)E]P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\,,$$

$$\overline{A}_{d\sigma 12} = \widetilde{B}^{\mathrm{T}}[A_{d\sigma} + D_{2\sigma}\Sigma_{2\sigma}(t)E]B(SB)^{-1},$$
$$\overline{A}_{\sigma 21} = S[A_{\sigma} + D_{1\sigma}\Sigma_{1\sigma}(t)E]P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1},$$
$$\overline{A}_{\sigma 22} = S[A_{\sigma} + D_{1\sigma}\Sigma_{1\sigma}(t)E]B(SB)^{-1},$$
$$\overline{A}_{d\sigma 21} = S[A_{d\sigma} + D_{2\sigma}\Sigma_{2\sigma}(t)E]P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1},$$
$$\overline{A}_{d\sigma 22} = S[A_{d\sigma} + D_{2\sigma}\Sigma_{2\sigma}(t)E]B(SB)^{-1},$$
$$\overline{\varphi}_1(t) = \widetilde{B}^{\mathrm{T}}\varphi(t);\ \overline{\varphi}_2(t) = S\varphi(t).$$

Then the sliding motion dynamics in the sliding surface ($\zeta(t) = \dot{\zeta}(t) = 0$) can be described by following $(n-m)$ dimensional switched system

$$\begin{aligned}\dot{\xi}_1(t) &= \widetilde{B}^{\mathrm{T}}A_{\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t) + \widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1} \\ &\times \xi_1(t-\tau) + \widetilde{B}^{\mathrm{T}}D_{1\sigma}\Sigma_{1\sigma}(t)EP\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t) \\ &+ \widetilde{B}^{\mathrm{T}}D_{2\sigma}\Sigma_{2\sigma}(t)EP\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t-\tau).\end{aligned} \tag{9}$$

By (2), we can easily obtain

$$EP\widetilde{B} = E[(1-E^{g}E)X(1-E^{g}E) + BYB^{\mathrm{T}}]\widetilde{B} = 0.$$

Then the sliding motion (9) can be represented by (4).

Remark 4. We can see that by using the SMC method, the uncertainties ΔA_i ΔA_{di} and the nonlinearities $f_i(x,t)$ disappear in the sliding motion (4) and the order of the considered system is reduced. Therefore we only need to study stabilization of the $(n-m)$ dimensional linear switched delay system (4) without uncertainties.

Definition 1. The sliding motion (4) is said to be asymptotically stable if there exist a Lyapunov function $V(x)$ and a switching law σ such that the derivative V along the trajectory of the system (4) satisfies

$$L(t) = \dot{V}(t) < 0$$

for all $t \in R^{+}$.

The objective in this paper is how to determine the sliding matrix S, design the switching law $\sigma(t)$ and variable structure controllers u_i, $i \in \Xi$ such that

1). the $(n-m)$ dimensional sliding motion (4) restricted to the sliding surface (3) is robustly asymptotically stable under the switching law $\sigma(t)$;

2). the state of the system (1) is driven towards the sliding surface (3) and stays there for all the future time.

The design of the switched delay system (1) is split into the known and unknown time-delay cases. The corresponding results will be described in the next sections.

3. MAIN RESULTS

In this section, we give the design method. In general, variable structure control design methodology comprises two steps. First, the sliding surface is designed, so that the controlled system will yield the desired dynamic performance in the sliding surface. The second phase is to design the variable structure controllers such that the trajectory of the system arrive the sliding surface and remain on the sliding surface for all subsequent time.

3.1 τ is a Known Constant

In this subsection, the system (1) with known time-delay τ is considered.

The following theorem shows that the system (1) in the sliding surface (3) is robustly asymptotically stabilizable under the switching law σ.

Theorem 1. The sliding motion (4) based on the sliding surface (3) is asymptotically stabilizable via switching. If there exist symmetric matrices X, Y matrix $Q_0 > 0$ and scalars $\alpha_i \geq 0, i \in \Xi$, $\sum_{i=1}^{l}\alpha_i = 1$ satisfying the following LMIs

$$\begin{gathered}\Gamma X\Gamma + BYB^{\mathrm{T}} > 0, \\ \begin{bmatrix}\widetilde{B}^{\mathrm{T}}(\overline{A}\Gamma X\Gamma + \Gamma X\Gamma\overline{A}^{\mathrm{T}})\widetilde{B} + Q_0 & \widetilde{B}^{\mathrm{T}}\overline{A}_d\Gamma X\Gamma\widetilde{B} \\ \widetilde{B}^{\mathrm{T}}\Gamma X\Gamma\overline{A}_d^{\mathrm{T}}\widetilde{B} & -Q_0\end{bmatrix} < 0,\end{gathered} \tag{10}$$

where $\overline{A} = \sum_{i=1}^{l}\alpha_i A_i$, $\overline{A}_d = \sum_{i=1}^{l}\alpha_i A_{di}$.

Proof. We define regions

$$\Omega_i = \left\{\begin{bmatrix}\xi_1(t) \\ \xi_1(t-\tau)\end{bmatrix}\begin{bmatrix}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t) \\ (\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t-\tau)\end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix}\widetilde{B}^{\mathrm{T}}(A_i\Gamma X\Gamma + \Gamma X\Gamma A_i^{\mathrm{T}})\widetilde{B} + Q_0 & \widetilde{B}^{\mathrm{T}}A_{di}\Gamma X\Gamma\widetilde{B} \\ \widetilde{B}^{\mathrm{T}}\Gamma X\Gamma A_{di}^{\mathrm{T}}\widetilde{B} & -Q_0\end{bmatrix} \times \begin{bmatrix}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t) \\ (\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t-\tau)\end{bmatrix} < 0, i \in \Xi\right\}.$$

Obviously, $\bigcup_{i\in\Xi}\Omega_i = R^{2(n-m)} \setminus \{0\}$.

Then, we design the following hysteresis switching law

$$\sigma(0) = \min\arg\{\Omega_i \mid \hat{\xi}_1(0) \in \Omega_i\},$$

for $t > 0$,

$$\sigma(t) = \begin{cases} i, & \textit{if } \hat{\xi}_1(t) \in \Omega_i \textit{ and } \sigma(t^{-}) = i, \\ \min\arg\{\Omega_k \mid \hat{\xi}_1(t) \in \Omega_k\}, & \textit{if } \hat{\xi}_1(t) \notin \Omega_i \textit{ and } \sigma(t^{-}) = i,\end{cases} \tag{11}$$

where $\hat{\xi}_1(t) = [\xi_1^{\mathrm{T}}(t), \xi_1^{\mathrm{T}}(t-\tau)]^{\mathrm{T}}$.

Take symmetric positive-define matrices P_1, Q and choose a Lyapunov functional candidate

$$V = \xi_1^{\mathrm{T}}(t)P_1\xi_1(t) + \int_{t-\tau}^{t}\xi_1^{\mathrm{T}}(\theta)Q\xi_1(\theta)d\theta. \tag{12}$$

Then the derivative of the Lyapunov functional (12) along the trajectory of the system (4) is

$$\begin{aligned}\dot{V} &= \begin{bmatrix}\xi_1(t) \\ \xi_1(t-\tau)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\widetilde{A}_{\sigma 11}^{\mathrm{T}}P_1 + P_1\widetilde{A}_{\sigma 11} + Q & P_1\widetilde{A}_{d\sigma 11} \\ \widetilde{A}_{d\sigma 11}^{\mathrm{T}}P_1 & -Q\end{bmatrix}\begin{bmatrix}\xi_1(t) \\ \xi_1(t-\tau)\end{bmatrix} \\ &= \begin{bmatrix}P_1\xi_1(t) \\ P_1\xi_1(t-\tau)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}P_1^{-1}\widetilde{A}_{\sigma 11}^{\mathrm{T}} + \widetilde{A}_{\sigma 11}P_1^{-1} + P_1^{-1}QP_1^{-1} & \widetilde{A}_{d\sigma 11}P_1^{-1} \\ P_1^{-1}\widetilde{A}_{d\sigma 11}^{\mathrm{T}} & -P_1^{-1}QP_1^{-1}\end{bmatrix} \\ &\times \begin{bmatrix}P_1\xi_1(t) \\ P_1\xi_1(t-\tau)\end{bmatrix},\end{aligned}$$

with $\widetilde{A}_{\sigma 11} = \widetilde{B}^{\mathrm{T}}A_{\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}$, $\widetilde{A}_{d\sigma 11} = \widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}$.

Take the matrices $P_1 = (\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}$, $Q = (\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}Q_0(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}$, then we have $\dot{V} < 0$. By the single Lyapunov function

method, the sliding motion (4) based on the sliding surface (3) is asymptotically stable under the switching law (11). This completes the proof.
Next, the result of controller design of reaching motion is given.

Theorem 2. Suppose LMIs (10) are feasible and the single sliding surface is given by (3). Then the state of the system (1) can enter the sliding surface in finite time, and subsequently remain on it by employing the following variable structure controllers

$$\begin{aligned} u_i = &-(SB)^{-1}(SA_i x(t) + SA_{di}x(t-\tau)) - \frac{(SB)^{-1}}{1-\rho_i}\{\rho_i\|SA_i x(t)\| \\ &+ \rho_i\|SA_{di}x(t-\tau)\| + \|SD_{1i}\|\|Ex(t)\| + \|SD_{2i}\|\|Ex(t-\tau)\| \\ &+ \|SB\|\phi_i(x,t) + \mu_1\}sign(\zeta), i \in \Xi, \end{aligned} \quad (13)$$

where μ_1 is a positive scalar.

Proof. Consider the following Lyapunov function candidate

$$V(\zeta) = \frac{1}{2}\zeta^{\mathrm{T}}\zeta . \quad (14)$$

Its derivative along the trajectory of the system (1) is given

$$\begin{aligned} \dot{V} &= \zeta^{\mathrm{T}}(S(A_i + \Delta A_i)x(t) + S(A_{di} + \Delta A_{di})x(t-\tau) \\ &\quad + SB[u_i + Z_i(t)u_i + f_i(x,t)]) \\ &\le \zeta^{\mathrm{T}}\{SA_i x(t) + SA_{di}x(t-\tau) + SB[u_i + Z_i(t)u_i]\} \quad (15) \\ &\quad + \|SD_{1i}\|\|Ex(t)\|\|\zeta\| + \|SD_{2i}\|\|Ex(t-\tau)\|\|\zeta\| \\ &\quad + \|SB\|\|\zeta\|\phi_i(x,t). \end{aligned}$$

Applying the variable structure controllers (13) to the inequality (15) results in $\zeta^{\mathrm{T}}\dot{\zeta} \le -\mu_1\|\zeta\|$. Hence the state of the system (1) will reach the single sliding surface (3) in finite time and subsequently remain on it. This completes the proof.

3.2 τ is an Unknown Constant

When time-delay τ is an unknown constant, the switching law (11) and the controllers (13) are not applicable. We assume the time-delay is an unknown, but it is bounded by the known constant $\bar{\tau}$.

The following theorem shows that the system (1) with unknown time-delay τ in the sliding surface (3) is robust asymptotically stabilizable under the switching law σ.

Theorem 3. The sliding motion (4) based on the sliding surface (3) is asymptotically stablilizable via switching. If there exists a positive number ε, matrix $Q_1 > 0$ symmetric matrices X, Y and scalars $\beta_i \ge 0, i \in \Xi$, $\sum_{i=1}^{l}\beta_i = 1$ satisfying the following inequalities

$$\begin{aligned} &\Gamma X\Gamma + BYB^{\mathrm{T}} > 0, \\ &\widetilde{B}^{\mathrm{T}}\hat{A}\Gamma X\Gamma\widetilde{B} + \widetilde{B}^{\mathrm{T}}\Gamma X\Gamma\hat{A}^{\mathrm{T}}\widetilde{B} + \varepsilon Q_1 \quad (16) \\ &\quad + (\widetilde{B}^{\mathrm{T}}\hat{A}_d\Gamma X\Gamma\widetilde{B})\varepsilon^{-1}R^{-1}(\widetilde{B}^{\mathrm{T}}\hat{A}_d\Gamma X\Gamma\widetilde{B})^{\mathrm{T}} < 0, \end{aligned}$$

where $\hat{A}_d = \left[\sqrt{\beta_1}\widetilde{B}^{\mathrm{T}}A_{d1}\Gamma X\Gamma\widetilde{B}, \ \ldots, \ \sqrt{\beta_l}\widetilde{B}^{\mathrm{T}}A_{dl}\Gamma X\Gamma\widetilde{B}\right]$, $\hat{A} = \sum_{i=1}^{l}\beta_i A_i$, $R = diag\{Q_1,\ldots,Q_1\}$.

Proof. We define regions

$$\begin{aligned} \Phi_i = \{&\xi_1(t) | \xi_1^{\mathrm{T}}(t)\{\widetilde{B}^{\mathrm{T}}A_i\Gamma X\Gamma\widetilde{B} + \widetilde{B}^{\mathrm{T}}\Gamma X\Gamma A_i^{\mathrm{T}}\widetilde{B} \\ &+ \varepsilon Q_1 + (\widetilde{B}^{\mathrm{T}}A_{di}\Gamma X\Gamma\widetilde{B})\varepsilon^{-1}Q_1^{-1}\widetilde{B}^{\mathrm{T}}A_{di}\Gamma X\Gamma\widetilde{B})^{\mathrm{T}} \\ &\times \xi_1(t) < 0, i \in \Xi\}. \end{aligned}$$

Obviously, $\bigcup_{i\in\Xi}\Phi_i = R^{(n-m)} \setminus \{0\}$.

Then, we design the following hysteresis switching law

$$\sigma(0) = \min\arg\{\Phi_i | \xi_1(0) \in \Phi_i\},$$

for $t > 0$,

$$\sigma(t) = \begin{cases} i, & if\ \xi_1(t) \in \Phi_i\ and\ \sigma(t^-) = i, \\ \min\arg\{\Phi_k | \xi_1(t) \in \Phi_k\}, & if\ \xi_1(t) \notin \Phi_i\ and\ \sigma(t^-) = i. \end{cases} \quad (17)$$

Take symmetric positive-define matrix P_2 and define the following Lyapunov-Krasovskii functional

$$V_1(t) = \xi_1^{\mathrm{T}}P_2\xi_1 + \int_{t-\tau}^{t}\xi_1^{\mathrm{T}}(\theta)P_2\varepsilon Q_1P_2\xi_1(\theta)d\theta . \quad (18)$$

where Q_1 satisfy (16).

Then the derivative of (18) along the trajectory of the system (4) is

$$\begin{aligned} \dot{V}(t) = &\xi_1^{\mathrm{T}}(t)\{(\widetilde{B}^{\mathrm{T}}A_\sigma P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1})^{\mathrm{T}}P_2 \\ &+ P_2(\widetilde{B}^{\mathrm{T}}A_\sigma P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}) + \varepsilon P_2Q_1P_2\}\xi_1(t) \\ &+ 2\xi_1^{\mathrm{T}}(t)P_2\widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t-\tau) \\ &- \xi_1^{\mathrm{T}}(t-\tau)\varepsilon P_2Q_1P_2\xi_1(t-\tau). \end{aligned} \quad (19)$$

Applying the standard bounding relation

$$a^{\mathrm{T}}b \le a^{\mathrm{T}}\mathrm{Z}a + b^{\mathrm{T}}\mathrm{Z}^{-1}b, \ \forall a,b \in R^n, \forall \mathrm{Z} \in R^{n\times n}, \mathrm{Z} > 0,$$

gives

$$\begin{aligned} &2\xi_1^{\mathrm{T}}(t)P_2\widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\xi_1(t-\tau) \\ &\le \xi_1^{\mathrm{T}}(t)P_2\widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}P_2^{-1}\varepsilon^{-1}Q_1^{-1}P_2^{-1}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1} \quad (20) \\ &\times \widetilde{B}^{\mathrm{T}}PA_{d\sigma}^{\mathrm{T}}\widetilde{B}P_2\xi_1(t) + \xi_1^{\mathrm{T}}(t-\tau)\varepsilon P_2Q_1P_2\xi_1(t-\tau). \end{aligned}$$

Substituting the right side of the inequality (20) into (19), we have

$$\begin{aligned} \dot{V}(t) = &\xi_1^{\mathrm{T}}(t)P_2\{P_2^{-1}(A_\sigma P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1})^{\mathrm{T}} + (A_\sigma P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}) \\ &\times P_2^{-1} + \varepsilon Q_1 + \widetilde{B}^{\mathrm{T}}A_{d\sigma}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}P_2^{-1}\varepsilon^{-1}Q_1^{-1}P_2^{-1} \\ &\times(\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}\widetilde{B}^{\mathrm{T}}PA_{d\sigma}^{\mathrm{T}}\widetilde{B}\}P_2\xi_1(t). \end{aligned} \quad (21)$$

Choosing $P_2 = (\widetilde{B}^{\mathrm{T}}P\widetilde{B})^{-1}$, we get $\dot{V}(t) < 0$. By the single Lyapunov function method, the sliding motion (4) based on the sliding surface (3) is asymptotically stable under the switching law (17). This completes the proof.
Next, the result of controller design of reaching motion is given.

Theorem 4. Suppose inequalities (16) are feasible and the single sliding surface is given by (3). Then the state of the system (1) can enter the sliding surface in finite time, and subsequently remain on it by employing the following variable structure controllers

$$\begin{aligned} u_i = &-(SB)^{-1}SA_i x(t) - \frac{(SB)^{-1}}{1-\rho_i}\{\rho_i\|SA_i x(t)\| + \lambda\|SA_{di}\|\|x(t)\| \\ &+ \|SD_{1i}\|\|Ex(t)\| + \lambda\|SD_{2i}\|\|E\|\|x(t)\| + \|SB\|\phi_i(x,t) \\ &+ \mu_2\}sign(\zeta), i \in \Xi, \end{aligned} \quad (22)$$

where μ_2 is a positive scalar.

Proof. It follows from the Razumikin theorem (Hale & Lunel, 1993) that for any solution $x(t+\theta)$ of (1), there exist a constant $\lambda > 1$ such that

$$\|x(t+\theta)\| \le \lambda \|x(t)\|, -\bar{\tau} \le \theta \le 0 . \tag{23}$$

Consider the Lyapunov function candidate

$$V(\zeta) = \frac{1}{2}\zeta^{\mathrm{T}}\zeta . \tag{24}$$

Its derivative along the trajectory of the system (1) is

$$\begin{aligned}\dot{V} &= \zeta^{\mathrm{T}}(S(A_i+\Delta A_i)x(t) + S(A_{di}+\Delta A_{di})x(t-\tau) \\ &\quad + SB[u_i + Z_i(t)u_i + f_i(x,t)]) \\ &\le \zeta^{\mathrm{T}}\{SA_i x(t) + SB[u_i + Z_i(t)u_i]\} \\ &\quad + \|SD_{1i}\|\|Ex(t)\|\|\zeta\| + \|SA_{di}\|\|x(t-\tau)\|\|\zeta\| \\ &\quad + \|SD_{2i}\|\|E\|\|x(t-\tau)\|\|\zeta\| + \|SB\|\|\zeta\|\phi_i(x,t).\end{aligned} \tag{25}$$

Applying the variable structure controllers (22) to the inequality (25) results in $\zeta^{\mathrm{T}}\dot{\zeta} \le -\mu_2\|\zeta\|$. Hence the state of the system (1) will reach the single sliding surface (3) in finite time and subsequently remain on it. This completes the proof.

4. EXAMPLES

In this section, we present a numerical example to demonstrate the effectiveness of the proposed design method. Consider the following uncertain switched delay system

$$\begin{aligned}\dot{x}(t) &= (A_\sigma + \Delta A_\sigma)x(t) + (A_{d\sigma} + \Delta A_{d\sigma})x(t-\tau) \\ &\quad + B[u_\sigma + Z_\sigma(t)u_\sigma + f_\sigma(x,t)], \\ x(t) &= \varphi(t),\ t \in [-\tau, 0],\end{aligned} \tag{26}$$

where $\sigma(t) \in \Xi = \{1,2\}$, $\tau \le 0.5$,

$$A_1 = \begin{bmatrix} 0.2 & 1 & -0.5 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & -0.5 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.1 & 1 & 0.2 \\ 1 & 1 & -1 \\ 0.5 & 1 & 0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ -0.5 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 1 & 0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ 1 & 1 & -0.5 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 \end{bmatrix},\ Z_1 =$$

$Z_2 = 0$, $f_1 = f_2 = 0$, the parameter uncertainties $\Delta A_i = D_{1i} \times \Sigma_{1i}E$, $\Delta A_{di} = D_{2i}\Sigma_{2i}E$, where $D_{11} = D_{12} = [0\ \ 1\ \ 0]^{\mathrm{T}}$, $D_{21} = D_{22} = [1\ \ 0\ \ 0]^{\mathrm{T}}$, $E = [1\ \ 1\ \ 0]$, $\Sigma_{1i} = \nu_{1i} \in [-1,1]$, $\Sigma_{2i} = \nu_{2i} \in [-1,1]$.

We select $\beta_1 = 0.4$, $\beta_2 = 0.6$, $\varepsilon = 0.1$, $\mu_2 = 5$.

By solving inequality (16), we can obtain the following solutions

$$X = \begin{bmatrix} 1015.2 & -33.4 & -375.5 \\ -33.4 & -1073.1 & -380.6 \\ -375.5 & -380.6 & 6.6 \end{bmatrix},\ Q_1 = \begin{bmatrix} 8.4973 & -6.423 \\ -6.423 & 20.1922 \end{bmatrix},$$

$$Y = 4.9448 .$$

By (3) the single sliding function is

$$\zeta(t) = [0.2022\quad 0.2022\quad 0]x(t) . \tag{27}$$

From Theorem 4, the reaching control laws are taken as follows

$$\begin{aligned}u_1 &= -0.7x_1(t) - 2x_2(t) - (7.2426\|x(t)\| \\ &\quad + \|x_1(t) + x_2(t)\| + 5)sign(\zeta(t)), \\ u_2 &= -0.9x_1(t) - 2x_2(t) + 0.8x_3(t) - (7.2426\|x(t)\| \\ &\quad + \|x_1(t) + x_2(t)\| + 5)sign(\zeta(t)).\end{aligned}$$

It is easy to verify that the conditions of Theorem 3 and 4 are satisfied. Following the proposed design method use Theorem 3 and 4.

The hysteresis switching law is

$$\sigma(t) = \begin{cases} 1, if(x(0) \in \Phi_1) or(x(t) \in \Phi_1 and \sigma(t^-) = 1) \\ \quad or(x(t) \notin \Phi_2 and \sigma(t^-) = 2), \\ 2, if(x(0) \notin \Phi_1) or(x(t) \in \Phi_2 and \sigma(t^-) = 2) \\ \quad or(x(t) \notin \Phi_1 and \sigma(t^-) = 1), \end{cases} \tag{28}$$

where

$$\Phi_1 = \{x(t) | x^{\mathrm{T}}(t)\begin{bmatrix} -1.0992 & -0.862 & -1.724 \\ -0.862 & 0.0063 & 0.0126 \\ -1.724 & 0.0126 & 0.0252 \end{bmatrix}x(t) < 0\},$$

$$\Phi_2 = \{x(t) | x^{\mathrm{T}}(t)\begin{bmatrix} 0.2443 & 0.3891 & 0.7782 \\ 0.3891 & -0.0811 & -0.1621 \\ 0.7782 & -0.1621 & -0.3242 \end{bmatrix}x(t) < 0\}.$$

The simulation results for the switched system (26) are depicted in Fig. 1-Fig. 4.

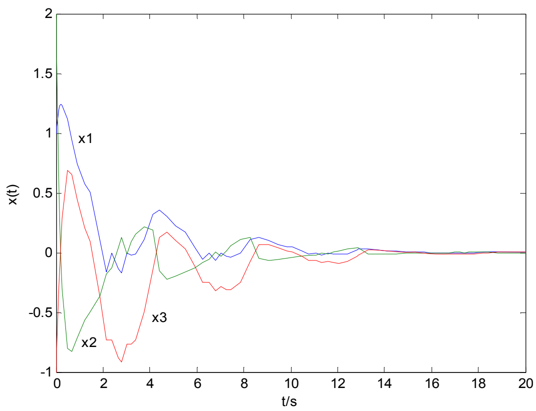

Fig. 1. The state responses of the switched system (26)

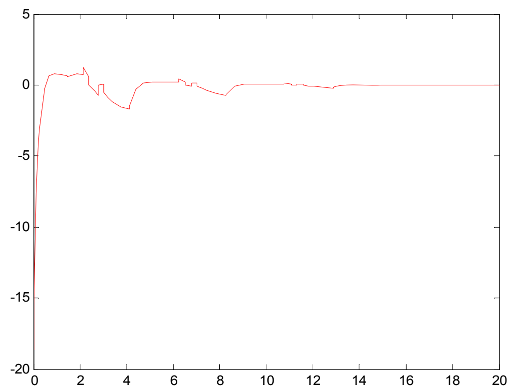

Fig. 2. The input signal of the switched system (26)

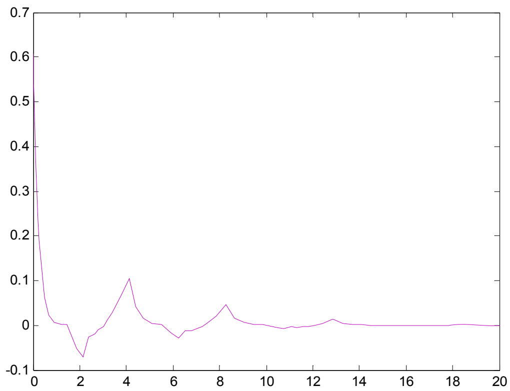

Fig. 3. The trajectory of the sliding function (27)

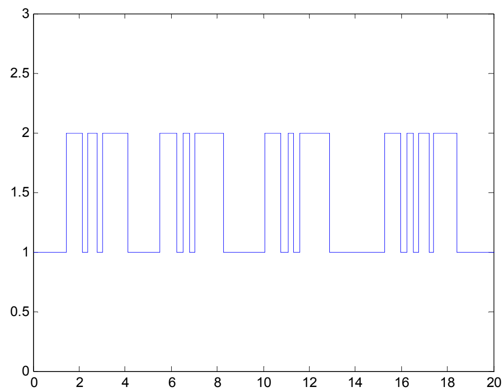

Fig. 4. The switching signal (28)

The state of the system (26) in the closed-loop and with the same initial state vector $x_0 = [1, 2, -1]^T$ is shown in Fig. 1. It is clearly seen that the closed-loop system of the switched system (26) with the designed controllers and the switching law (28) is asymptotically stable.

5. CONCLUSION

In this paper, the problem of robust sliding mode variable structure control has been studied for a class of uncertain switched delay systems. The single sliding surface has been constructed. The existence conditions of the sliding surface have been proposed for delay-known and delay-unknown cases, respectively. The corresponding hysteresis switching laws and variable structure controllers have been developed such that the resulting closed-loop system is robust stable and completely invariant to all admissible uncertainties in the sliding surface.

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