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Average Dwell-time Method to Stabilization and L_2-gain Analysis for Uncertain Switched Nonlinear Systems ⋆

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Abstract: This paper is concerned with the problem of stabilization and L_2-gain analysis for a class of switched nonlinear systems with norm-bounded time-varying uncertainties. A system in this class is composed of two parts: a uncertain linear switched part and a nonlinear part, which are also switched systems. When all the subsystem are stabilizable and have a L_2-gain, the switched feedback control law and the switching law are designed respectively using average dwell-time method such that the corresponding closed-loop switched system is exponentially stable and achieves a weighted L_2-gain. We construct the Piecewise Lyapunov functions and design the switching law based on the structural characteristics of the switched system.

1. INTRODUCTION

In recent years, there has been increasing interest in the stability analysis and design methodology of switched systems due to their significance both in theory and applications. Such control systems appear in the modeling of chemical processes, transportation systems, computer controlled systems and power systems, ect. Because of the intricate intersection between continuous and discrete dynamics, switched systems may have very complicated behaviors. This motivated a large and growing body of research work on a diverse array of issues, including the modeling, optimization, stability analysis, and control, among which the stability issues have been a major focus in studying switched systems (Liberzon and Morse (1999), Liberzon et al. (1999), Branicky (1998) and the references therein). Among the stability properties, the uniform asymptotic stability is a desirable property which can be guaranteed by a common Lyapunov function (Zhao and Dimirovski (2004), Cheng (2004), Cheng et al. (2003)). But a common Lyapunov function may not exist or is too difficult to find. In this case, the switched system may be still asymptotically stabilized under some properly chosen switching law. The multiple Lyapunov function method (Branicky (1998), Peleties and Decarlo (1999), Wicks et al (1998)), the single Lyapunov function method (Pettersson and Lennartson (2001), Zhai (2001)), and the average dwell-time method (Hespanha and Morse (1999), Hu et al. (1999), Zhai et al. (2000)) are generally effective tools when studying asymptotic stability problem of switched systems under a certain switching law. All these methods and several other methods such as programming method, convex combination method and so on are summarized in the recent books (Liberzon (2003), Pettersson (1999)). Among these methods, the average dwell-time method is relatively more important, as the switching law designed by using this method only depends on the time.

The L_2-gain property analysis of switched systems is a valuable issue deserving us pay more attention to among the growing body of research works that focus on switched systems. To date, the research works analyzing the L_2-gain property about switched systems are mainly about switched linear systems. Zhai et al. (2001) investigated the disturbance attenuation properties for a class of switched linear systems by using the average dwell-time method incorporated with a piecewise Lyapunov function, and a weighted L_2-gain property is achieved. Sun et al. (2006) analyzed the stability and L_2-gain for switched linear delay systems. Zhao and Hill (accepted) addressed the L_2-gain analysis for switched systems via multiple Lyapunov functions method. In these papers mentioned above, the switched system studied has no uncertainties and no control input. Due to the uncertainties is a common phenomenon in practice, the stabilization and L_2-gain synthesis problem for non-autonomous switched systems with uncertainties is obviously more preferable and challenging.

In this paper, we discuss the stabilization and L_2-gain analysis problem for a class of cascade nonlinear switched systems by using the average dwell-time approach incorporated with a piecewise Lyapunov function. The switched system under consideration is composed of a nonlinear part and a uncertain linear part. Sufficient conditions for globally exponentially stability and $e^{-\lambda t}$-weighted L_2-gain are developed for all admissible uncertainties when all the

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subsystems are stabilizable and have an L_2-gain from the disturbance input to the controlled output. The piecewise Lyapunov function and the switched feedback controller are constructed based on the characteristic of the switched nonlinear cascade system.

The outline of the paper is as follows: Section 2 gives the description of the switched system we study and the preparative knowledge. Section 3 presents the main result. An example is shown in section 4 to illustrate the feasibility of our results. Some conclusions end the paper.

2. PROBLEM STATEMENT AND PRELIMINARIES

2.1 Average Dwell-time Method

Consider linear switched systems described by equations of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t). \tag{1}$$

where $x(t) \in R^n$ is the state, $\sigma(t) : [t_0, \infty] \rightarrow I_N = \{1, \ldots, N\}$ is the switching signal, which is a piecewise constant function of time. $A_i (i \in I_N)$ are constant matrices of appropriate dimension describing the subsystems, and $N > 1$ is the number of subsystems.

Definition 1. The switched system (1) is said to be globally exponentially stable with stability degree $\lambda \geq 0$ if $\|x(t)\| \leq e^{\alpha-\lambda(t-t_0)}$ holds for all $t \geq t_0$ and a constant α.

We now briefly review the average dwell-time method proposed by Hespanha and Morse (1999).

For any switching signal $\sigma(t)$ and any $t > \tau \geq 0$, let $N_\sigma(\tau, t)$ denote the number of switchings of $\sigma(t)$ on the interval (τ, t). For given N_0, $\tau_a > 0$, let $S_a[\tau_a, N_0]$ denote the set of all switching signal satisfying

$$N_\sigma(\tau, t) \leq N_0 + \frac{t-\tau}{\tau_a}. \tag{2}$$

where the constant τ_a is called the "average dwell time" and N_0 the "chatting bounded". The idea is that there may exist consecutive switchings separated by less than τ_a, but the average time interval between consecutive switchings is not less than τ_a. Inequality (2) also indicates that if we ignore the first N_0 switchings, then the average time interval between consecutive switchings is at least τ_a. As commonly used in the literature, we choose $N_0 = 0$. It has been proven in Hespanha and Morse (1999) that when all subsystems matrices A_i in (1) are Hurwitz, and if τ_a is sufficiently large, then the switched system (1) is exponentially stable for any switching signal $\sigma(t) \in S_a[\tau_a, N_0]$.

2.2 System Description

In this paper, we consider the uncertain switched nonlinear system described by

$$\begin{cases} \dot{x}_1(t) = \hat{A}_{1\sigma(t)}x_1(t) + A_{2\sigma(t)}x_2(t) + \hat{B}_{\sigma(t)}u_{\sigma(t)}(t) \\ \qquad\quad + G_{\sigma(t)}w(t), \\ \dot{x}_2(t) = f_{2\sigma(t)}(x_2(t)), \\ \ \ y(t) = C_{\sigma(t)}x_1(t), \end{cases} \tag{3}$$

where $x_1(t) \in R^{n-d}$, $x_2(t) \in R^d$ are the states, $u_{\sigma(t)}(t) \in R^m$ is the control input, $w(t) \in R^q$ is the external disturbance input and $w(t) \in L_2[0, \infty)$, $y(t) \in R^p$ is the controlled output. $\sigma(t) : [0, \infty] \rightarrow I_N = \{1, \ldots, N\}$ is the switching signal, which is a piecewise constant function of time and will be determined later. And $\sigma(t) = i$ means that the ith switched subsystem is activated. $\hat{A}_{1i} = A_{1i} + \triangle A_{1i}$, $\hat{B}_i = B_i + \triangle B_i(t)$, A_{1i}, A_{2i}, B_i, G_i and $C_i (i \in I_N)$ are constant matrices of appropriate dimensions that describe the nominal systems. $f_{2i}(x_2(t))$ are smooth vector fields, and we have $f_{2i}(0) = 0$. $\triangle A_{1i}(t)$ and $\triangle B_i(t)$ are uncertain time-varying matrices denoting the uncertainties in the system matrices and having the following form

$$[\triangle A_{1i}(t), \triangle B_i(t)] = E_i \Gamma(t)[F_{1i}, F_{2i}], \qquad i \in I_N. \tag{4}$$

where $E_i \in R^{(n-d)\times l}$, $F_{1i} \in R^{k\times(n-d)}$, and $F_{2i} \in R^{k\times m}$ are given constant matrices which characterize the structure of uncertainty, and F_{2i} is of full column rank. Γ is the norm-bounded time-varying uncertainty, i.e.

$$\Gamma = \Gamma(t) \in \{\Gamma(t) : \Gamma(t)^T\Gamma(t) = I_{k\times k}, \Gamma(t) \in R^{l\times k}, \\ \text{the elements of } \Gamma(t) \text{ are Lebesgue measurable}\}.$$

There are several reasons for assuming that the system uncertainties have the structures given in (4), which can been found in Khargonekar et al. (2001).

Definition 2. System (3) is said to be globally exponentially stabilizable via switching if there exist a switching signal $\sigma(t)$ and an associate switched state feedback $u_{\sigma(t)}(t) = K_{\sigma(t)}x_1(t)$ such that the corresponding closed-loop system (3) with $w(t) \equiv 0$ is globally exponentially stable for all admissible uncertainties.

Consider the switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t))w(t), \\ y(t) = h_{\sigma(t)}(x(t)), \end{cases} \tag{5}$$

where $x(t) \in R^n$, $w(t)$, $y(t)$, $\sigma(t)$ are the same as stated in (3), $f_i(x(t))$, $g_i(x(t))$, $h_i(x(t))$ $(1 \leq i \leq N)$ are smooth vector fields.

Definition 3. System (5) is said to have a $e^{-\lambda t}$-weighted L_2-gain over $\sigma(t)$, from the disturbance input $w(t)$ to the controlled output $y(t)$, if the following inequality holds for each $\sigma(t)$ and some real-valued function $\beta(t)$ with $\beta(0) = 0$

$$\int_0^\infty e^{-\lambda t}y^T(t)y(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt + \beta(x(0)), \tag{6}$$

along the solutions to (5). Where $x(0) \neq 0$ is the initial state, and $w(t) \in L_2[0, +\infty)$.

The aim of the paper is to find a switched state feedback controller and a class of average dwell-time based switching laws, such that the corresponding closed-loop system (3) is globally exponentially stable with $w(t) = 0$ and has a $e^{-\lambda t}$-weighted L_2-gain under the designed switching law.

The following lemma will be used in the development of the main results.

Lemma 1. (Petersen (1987)) Given any constant $\lambda > 0$ and any matrices M, Γ, N of compatible dimensions, then

$$2x^T M\Gamma N x \leq \frac{1}{\lambda}x^T MM^T x + \lambda x^T N^T N x.$$

for all $x \in R^n$, where Γ is an uncertain matrix satisfying $\Gamma^T\Gamma \leq I$.

3. MAIN RESULTS

This section presents the sufficient condition for the stabilization and $e^{-\lambda t}$-weighted L_2-gain of switched system (3). The switching law satisfying one average dwell time and the switched state feedback controller are also designed.

Theorem 1. Given any constant $\gamma > 0$, suppose that the switched system(3) satisfies the following conditions

(i) if there exist constants $\varepsilon_i > 0$, $\lambda_0 > 0$, $\mu \geq 1$, such that the following inequalities

$$\begin{aligned} A_{1i}^T P_i + P_i A_{1i} + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i G_i G_i^T P_i \\ + \varepsilon_i^2 F_{1i}^T F_{1i} + C_i^T C_i + \lambda_0 P_i + I - (\varepsilon_i^{-1} P_i B_i \\ + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T < 0, \end{aligned} \tag{7}$$

$$P_i \leq \mu P_j, \quad i, j = 1, \ldots, N. \tag{8}$$

have positive definite solutions P_i.

(ii) there exists a proper, positive definite, and radially unbounded function $W(x_2)$ such that

$$\frac{\partial W(x_2)}{\partial x_2} f_{2i}(x_2) \leq -\beta \|x_2\|^2, \tag{9}$$

$$a_1 \|x_2\|^2 \leq \|W(x_2)\| \leq a_2 \|x_2\|^2. \tag{10}$$

for some constants $\beta > 0$, $a_1 > 0$, $a_2 > 0$.

Then, the closed-loop system (3) with $w(t) = 0$ is globally exponentially stable and has a $e^{-\lambda t}$-weighted L_2-gain under arbitrary switching law satisfying the average dwell time

$$\tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda}, \quad \lambda \in [0, \lambda_0). \tag{11}$$

and the corresponding switched state feedback controller is given by

$$u_i = -(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-2} B_i^T P_i + F_{2i}^T F_{1i}) x_1(t). \tag{12}$$

Proof. For switched system (3), define the following piecewise Lyapunov function candidate

$$V(x) = V_{\sigma(t)}(x_1, x_2) = x_1^T P_{\sigma(t)} x_1 + l W(x_2). \tag{13}$$

where P_i are the solutions of (7) and (8), and it switches in accordance with the piecewise constant switching law $\sigma(t)$, constant $l > 0$ will be defined later.

Then, based on Lemma 1, when the ith subsystem is activated, the time derivative of $V(x_1, x_2)$ along the trajectory of the switched system (3) is

$$\begin{aligned} \dot{V} &= x_1^T (A_{1i}^T P_i + A_{1i} P_i) x_1 + 2x_1^T P_i \Delta A_{1i} x_1 + 2x_1^T P_i B_i u_i \\ &\quad + 2x_1^T P_i \Delta B_i u_i + 2x_1^T P_i A_{2i} x_2 + l \frac{\partial W(x_2)}{\partial x_2} f_{2i}(x_2) \\ &\quad + 2x_1^T P_i G_i w \\ &\leq x_1^T (A_{1i}^T P_i + A_{1i} P_i) x_1 + 2x_1^T P_i E_i \Gamma (F_{1i} x_1 + F_{2i} u_i) \\ &\quad + 2x_1^T P_i B_i u_i + 2x_1^T P_i A_{2i} x_2 - l\beta \|x_2\|^2 + 2x_1^T P_i G_i w \\ &\leq x_1^T (A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i) x_1 + \varepsilon_i^2 (F_{1i} x_1 \\ &\quad + F_{2i} u_i)^T (F_{1i} x_1 + F_{2i} u_i) + 2x_1^T P_i B_i u_i - l\beta \|x_2\|^2 \\ &\quad + 2x_1^T P_i A_{2i} x_2 + 2x_1^T P_i G_i w \\ &\leq x_1^T (A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i}) x_1 \\ &\quad + 2\varepsilon_i^2 x_1^T F_{1i}^T F_{2i} u_i + \varepsilon_i^2 u_i^T F_{2i}^T F_{2i} u_i + 2x_1^T P_i B_i u_i \\ &\quad + 2x_1^T P_i A_{2i} x_2 - l\beta \|x_2\|^2 + 2x_1^T P_i G_i w \\ &\leq x_1^T (A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i}) x_1 \\ &\quad + 2x_1^T P_i A_{2i} x_2 - l\beta \|x_2\|^2 + 2x_1^T P_i G_i w + \big[\varepsilon_i u_i \\ &\quad + (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} B_i^T P_i x_1 + \varepsilon_i F_{2i}^T F_{1i} x_1)\big]^T (F_{2i}^T F_{2i}) \\ &\quad \cdot \big[\varepsilon_i u_i + (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} B_i^T P_i x_1 + \varepsilon_i F_{2i}^T F_{1i} x_1)\big] \\ &\quad - x_1^T \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i \\ &\quad + \varepsilon_i F_{1i}^T F_{2i})^T\big] x_1, \end{aligned}$$

From (12), we obtain

$$\begin{aligned} \dot{V} &\leq x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\ &\quad - \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i \\ &\quad + \varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 + 2x_1^T P_i A_{2i} x_2 - l\beta \|x_2\|^2 \\ &\quad + 2x_1^T P_i G_i w. \end{aligned}$$

It is easy to see that there exist constants $l_i > 0$, $m_i > 0$, $i \in I_N$ such that

$$\|A_{2i} x_2\| \leq l_i \|x_2\|, \qquad \|x_1^T P_i\| \leq m_i \|x_1\|.$$

Let $p = \max\{l_i m_i, i \in I_N\}$, we have

$$\begin{aligned} \dot{V} &\leq x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\ &\quad - \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T \\ &\quad \cdot F_{2i})^T\big] \Big\} x_1 + 2p \|x_1\| \|x_2\| - l\beta \|x_2\|^2 + 2x_1^T P_i G_i w, \end{aligned}$$

It is easy to calculate that

$$\begin{aligned} &\dot{V} + y^T y - \gamma^2 w^T w \\ &\leq x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\ &\quad - \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i \\ &\quad + \varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 + 2p \|x_1\| \|x_2\| - l\beta \|x_2\|^2 \\ &\quad + 2x_1^T P_i G_i w + x_1^T C_i^T C_i x_1 - \gamma^2 w^T w \\ &\leq x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\ &\quad + \gamma^{-2} P_i G_i G_i^T P_i + C_i^T C_i - \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) \\ &\quad \cdot (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 + 2p \|x_1\| \|x_2\| \\ &\quad - l\beta \|x_2\|^2 - (\gamma^{-1} G_i^T P_i x_1 - \gamma w)^T (\gamma^{-1} G_i^T P_i x_1 - \gamma w) \\ &\leq x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\ &\quad + \gamma^{-2} P_i G_i G_i^T P_i + C_i^T C_i - \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) \\ &\quad \cdot (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 + 2p \|x_1\| \|x_2\| \\ &\quad - l\beta \|x_2\|^2 \end{aligned}$$

From (7), we know

$$\begin{aligned} &\dot{V} + y^T y - \gamma^2 w^T w \\ &\leq -\lambda_0 x_1^T P_i x_1 - l\lambda_0 W(x_2) + l\lambda_0 W(x_2) - x_1^T x_1 \\ &\quad + 2p \|x_1\| \|x_2\| - l\beta \|x_2\|^2 \\ &\leq -\lambda_0 V + l a_2 \lambda_0 \|x_2\|^2 - x_1^T x_1 + 2p \|x_1\| \|x_2\| - l\beta \|x_2\|^2 \\ &\leq -\lambda_0 V - (\|x_1\| - p\|x_2\|)^2 + p^2 \|x_2\|^2 + l a_2 \lambda_0 \|x_2\|^2 \end{aligned}$$

$$-l\beta\|x_2\|^2$$
$$\leq -\lambda_0 V - (l\beta - la_2\lambda_0 - p^2)\|x_2\|^2.$$

Choose $l \geq \frac{p^2}{\beta - a_2\lambda_0}$, we have

$$\dot{V} + y^T y - \gamma^2 w^T w \leq -\lambda_0 V. \tag{14}$$

When $w(t) = 0$, from the above inequality, we obtain

$$\dot{V} \leq -\lambda_0 V, \tag{15}$$

Moreover, from (8) and (13), it is easy to get

$$V_i(t) \leq \mu V_j(t), \qquad i, j \in I_N. \tag{16}$$

For arbitrary $t > 0$, denote $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \ldots \leq t_{N_\sigma(0,t)}$ as the switching instants of $\sigma(t)$ over the interval $(0, t)$, then

$$\begin{aligned} V(t) &\leq e^{-\lambda_0(t - t_{N_\sigma(0,t)})} V(t_{N_\sigma(0,t)}) \leq \mu e^{-\lambda_0(t - t_{N_\sigma(0,t)})} \\ &\cdot V(t^-_{N_\sigma(0,t)}) \leq \mu e^{-\lambda_0(t - t_{N_\sigma(0,t)-1})} V(t_{N_\sigma(0,t)-1}) \\ &\leq \ldots \leq \mu^{N_\sigma(0,t)} e^{-\lambda_0 t} V(0) = e^{N_\sigma(0,t)\ln\mu - \lambda_0 t} V(0). \end{aligned}$$

If τ_a satisfies (11), i.e. for arbitrary $\tau > 0$

$$N_\sigma(0, \tau) \leq \frac{\tau}{\tau_a^*}, \quad \tau_a^* = \frac{\ln\mu}{\lambda}, \tag{17}$$

then, we have $N_\sigma(0, \tau)\ln\mu \leq \lambda\tau$. Thus

$$V(t) \leq e^{-(\lambda_0 - \lambda)t} V(0). \tag{18}$$

Based on (10) and (13), we know that there exist constants $\lambda_1 > 0$, $\lambda_2 > 0$ such that

$$\lambda_1\|x_1\|^2 + la_1\|x_2\|^2 \leq V(t) \leq \lambda_2\|x_1\|^2 + la_2\|x_2\|^2,$$

where $\lambda_1 = \min\{\lambda_{\min}(P_i) | i \in I_N\}$, $\lambda_2 = \max\{\lambda_{\max}(P_i) | i \in I_N\}$.

Let $b_1 = \min\{\lambda_1, la_1\}$, $b_2 = \max\{\lambda_2, la_2\}$, we have

$$b_1(\|x_1\|^2 + \|x_2\|^2) \leq V(t) \leq b_2(\|x_1\|^2 + \|x_2\|^2). \tag{19}$$

The following inequality follows from (16), (18) and (19),

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{1}{b_1} V(t) \leq \frac{1}{b_1} e^{-(\lambda_0 - \lambda)t} V(0) \\ &\leq \frac{b_2}{b_1} e^{-(\lambda_0 - \lambda)t} \|x(0)\|^2, \end{aligned}$$

Thus,

$$\|x(t)\| \leq \sqrt{\frac{b_2}{b_1}} e^{-\frac{(\lambda_0 - \lambda)}{2}t} \|x(0)\|. \tag{20}$$

Hence, the globally exponential stability of the closed-loop system (3) with $w(t) = 0$ follows.

Integrating both sides of (14), and taking (16) into consideration, we can get

$$\begin{aligned} V(t) &\leq V(t_{N_\sigma(0,t)}) e^{-\lambda_0(t - t_{N_\sigma(0,t)})} - \int_{t_{N_\sigma(0,t)}}^t e^{-\lambda_0(t-\tau)} \\ &\cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\ &\leq \mu V(t^-_{N_\sigma(0,t)}) e^{-\lambda_0(t - t_{N_\sigma(0,t)})} - \int_{t_{N_\sigma(0,t)}}^t e^{-\lambda_0(t-\tau)} \\ &\cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\ &\leq \mu\big[V(t_{N_\sigma(0,t)-1}) e^{-\lambda_0(t_{N_\sigma(0,t)} - t_{N_\sigma(0,t)-1})} - \int_{t_{N_\sigma(0,t)-1}}^{t_{N_\sigma(0,t)}} \\ &\cdot e^{-\lambda_0(t_{N_\sigma(0,t)} - \tau)} [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau\big] \\ &\cdot e^{-\lambda_0(t - t_{N_\sigma(0,t)})} - \int_{t_{N_\sigma(0,t)}}^t e^{-\lambda_0(t-\tau)} [y^T(\tau)y(\tau) \\ &-\gamma^2 w^T(\tau)w(\tau)]d\tau \\ &\vdots \\ &\leq \mu^{N_\sigma(0,t)} e^{-\lambda_0 t} V(0) - \mu^{N_\sigma(0,t)} \int_0^{t_1} e^{-\lambda_0(t-\tau)} \\ &\cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau - \mu^{N_\sigma(0,t)-1} \\ &\cdot \int_{t_1}^{t_2} e^{-\lambda_0(t-\tau)} [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\ &- \ldots - \mu^0 \int_{t_{N_\sigma(0,t)}}^t e^{-\lambda_0(t-\tau)} [y^T(\tau)y(\tau) \\ &-\gamma^2 w^T(\tau)w(\tau)]d\tau \\ &= \mu^{N_\sigma(0,t)} e^{-\lambda_0 t} V(0) - \int_0^t \mu^{N_\sigma(\tau,t)} e^{-\lambda_0(t-\tau)} \\ &\cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\ &= e^{-\lambda_0 t + N_\sigma(0,t)\ln\mu} V(0) - \int_0^t e^{-\lambda_0(t-\tau) + N_\sigma(\tau,t)\ln\mu} \\ &\cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau, \end{aligned}$$

Multiplying both sides of the above inequality by $e^{-N_\sigma(0,t)\ln\mu}$, results in

$$\begin{aligned} e^{-N_\sigma(0,t)\ln\mu} V(t) &\leq e^{-\lambda_0 t} V(0) - \int_0^t e^{-\lambda_0(t-\tau) - N_\sigma(0,\tau)\ln\mu} \\ &\cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau. \end{aligned} \tag{21}$$

Moreover, in view of $V(t) > 0$, the following inequality follows from (17)

$$\begin{aligned} &\int_0^t e^{-\lambda_0(t-\tau) - \lambda\tau} y^T(\tau)y(\tau)d\tau \\ &\leq e^{-\lambda_0 t} V(0) + \gamma^2 \int_0^t e^{-\lambda_0(t-\tau)} w^T(\tau)w(\tau)d\tau. \end{aligned} \tag{22}$$

Integrating both sides of (22) from $t = 0$ to ∞ and rearranging the double-integral area, we obtain

$$\begin{aligned} &\int_0^\infty e^{-\lambda\tau} y^T(\tau)y(\tau) \left(\int_\tau^\infty e^{-\lambda_0(t-\tau)}dt\right) d\tau \leq \int_0^\infty e^{-\lambda_0 t} \\ &\cdot V(0)dt + \gamma^2 \int_0^\infty w^T(\tau)w(\tau) \left(\int_\tau^\infty e^{-\lambda_0(t-\tau)}dt\right) d\tau, \end{aligned}$$

i.e.

$$\int_0^\infty e^{-\lambda\tau} y^T(\tau)y(\tau)d\tau \leq V(0) + \gamma^2 \int_0^\infty w^T(\tau)w(\tau)d\tau. \tag{23}$$

From Definition 3 we know that the closed-loop switched system achieves a $e^{-\lambda t}$-weighted L_2-gain.

Remark 2. Applying Shur complement formula, the first matrix inequality of condition (i) can be easily transformed into the LIMs form. The second inequality of condition (i) is trivial, as long as we let $\mu = \sup_{i,j \in I_N} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}$.

Remark 1. When $\mu \equiv 1$, $\tau_a^* \equiv 0$, and (13) becomes a common Lyapunov function for switched system (3). In this case, the stabilization and L_2-gain analysis problem can be solved under arbitrary switching law.

Remark 3. Condition 2 implies that the second part of the switched system (3) is uniformly exponentially stable. Since the second part of the switched system (3) has a lower dimension, its Lyapunov function is relatively easier to find than that of the whole switched system. A number of methods available for finding the common Lyapunov function for such switched systems are provided by Cheng et al. (2003) and Zhao and Dimirovski (2004).

When the switched system (3) is in the following linear form

$$\begin{cases} \dot{x}_1(t) = \hat{A}_{11\sigma(t)}x_1(t) + A_{12\sigma(t)}x_2(t) + \hat{B}_{\sigma(t)}u_{\sigma(t)}(t) \\ \qquad\quad + G_{\sigma(t)}w(t), \\ \dot{x}_2(t) = A_{22\sigma(t)}x_2(t), \\ \ y(t) = C_{\sigma(t)}x_1(t), \end{cases} \tag{24}$$

where $x_1(t)$, $x_2(t)$, $w(t)$, $y(t)$, $u_{\sigma(t)}(t)$, $\sigma(t)$, $\hat{A}_{11i}$, $\hat{B}_i$, A_{12i}, G_i, C_i ($i \in I_N$) are the same as stated in (3). A_{22i} ($i \in I_N$) are $d \times d$ real matrix. We have the following Corollary.

Corollary 1. Given any constant $\gamma > 0$, suppose that the switched system(24) satisfies the following conditions

(i) if there exist constants $\varepsilon_i > 0$, $\delta > 0$, $\mu \geq 1$, such that the following inequalities

$$\begin{aligned} A_{11i}^T P_i + P_i A_{11i} + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i G_i G_i^T P_i \\ + \varepsilon_i^2 F_{1i}^T F_{1i} + C_i^T C_i + \lambda_0 P_i + I - (\varepsilon_i^{-1} P_i B_i \\ + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T < 0, \end{aligned} \tag{25}$$

$$P_i \leq \mu P_j, \quad i, j = 1, \ldots, N. \tag{26}$$

have positive definite solutions P_i.

(ii) All the matrix A_{22i}, $i \in I_N$ are Hurwitz and share a common Quadratical Lyapunou function

Then, the closed-loop system (24) with $w(t) = 0$ is globally exponentially stable and has a $e^{-\lambda t}$-weighted L_2-gain under arbitrary switching law satisfying the average dwell time

$$\tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda}, \quad \lambda \in [0, \lambda_0). \tag{27}$$

and the corresponding switched state feedback controller is given by

$$u_i = -(F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-2} B_i^T P_i + F_{2i}^T F_{1i})x_1(t). \tag{28}$$

Proof. The proof process is similar to that of Theorem 1.

4. EXAMPLE

Consider the switched system (3) with $I_N = \{1, 2\}$, $n - d = 2$, $d = 1$ and

$$A_{11} = \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix}, A_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, F_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, F_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, G_1 = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -5 & -2 \\ 3 & -4 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

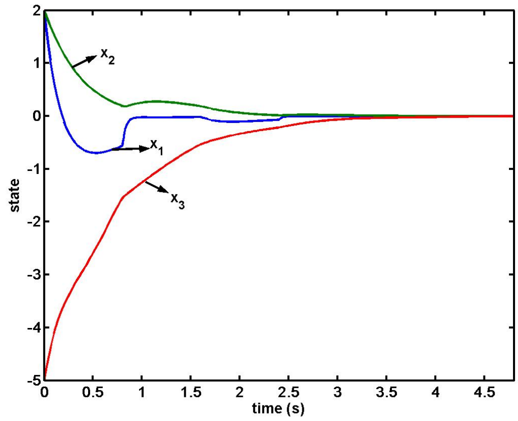

Fig. 1. The state response of the switched system

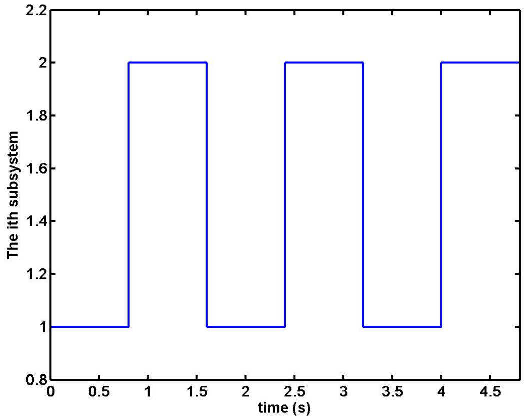

Fig. 2. The switching signal

$$E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, F_{12} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, F_{22} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \Gamma(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \cos t \end{bmatrix},$$

$$f_{22}(x_3) = -x_3 - x_3 \cos^2 x_3, \quad f_{21}(x_3) = -x_3 - x_3 \sin^2 x_3,$$

Choose $W(x_3) = \frac{1}{2}x_3^2$. Let $\gamma = 1$, $\varepsilon_1 = \varepsilon_2 = 1$, $a_1 = 0.3$, a_2=1, $\beta = 1$, $l = 14.9$. It can be verified that all the conditions in Theorem 1 are satisfied. Using Theorem 1, we design the switched state feedback and the average dwell-time based switching law such that the the closed-loop switched system (3) is globally exponentially stable with $w(t) = 0$. Solving (7), we obtain

$$P_1 = \begin{bmatrix} 2.6179 & 0.3342 \\ 0.3342 & 2.3329 \end{bmatrix}, P_2 = \begin{bmatrix} 2.9598 & 0.1577 \\ 0.1577 & 2.0921 \end{bmatrix},$$

Let $\mu = 1.4175$, $\lambda_0 = 0.8$, $\lambda = 0.7$, we can get $\tau_a^* = 0.5$, let $\tau_a^* \leq \tau_a = 0.8$. Design the switching law as

$$\sigma(t) = \begin{cases} 1, & k = 0, 2, 4, \ldots, \\ 2, & k = 1, 3, 5, \ldots, \end{cases} \qquad t_k = 0.8k. \tag{29}$$

and the switched state feedback is given as:

$$u_i = \begin{cases} -1.6432x(1) - 3.5000x(2), & i = 1, \\ -21.8392x(1) - 0.6308x(2), & i = 2. \end{cases} \tag{30}$$

Let the initial state $x(0) = (2, 2, -5)^T$. Figure 1 shows the state response of the switched system, which indicates that with the switched state feedback (30) and under the designed switching law (29) the closed-loop system is globally exponentially stable. Figure 2 gives the switching signal.

The estimate of state decay can be obtained from (20) as:

$$\|x(t)\| \leq 15.4e^{-0.05t}.$$

5. CONCLUSION

In this paper, we have studied the stabilization and L_2-gain analysis problem for a class of uncertain switched nonlinear cascade systems with external disturbances input. The sufficient condition guaranteeing the existence of the switched state feedback controller are presented, the corresponding average-dwell time based switching law has been simultaneously designed. With the switched state feedback controller under the designed switching law the closed-loop switched system is globally exponentially stable and achieves a $e^{-\lambda t}$-weighted L_2 gain. The problem stabilization and L_2-gain analysis for the same class of switched nonlinear cascade systems when both parts are respectively stabilizable under two different average-dwell time deserves further study.

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