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A Discrete-Time Integral Sliding Mode Control Approach for Output Tracking with State Estimation

Xu Jian-Xin * **Khalid Abidi**

* *Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117576 (e-mail: elexujx@nus.edu.sg).*

Abstract: A new integral type sliding surface (ISM) is design for sampled-data systems for output tracking. ISM surface design is based on Output Feedback with a State Observer. Discrete-time control based on ISM achieves good tracking performance while allowing the pole assignment of m poles, which are otherwise zero in a deadbeat design. In particular, the new scheme can avoid overlarge control actions by avoiding the deadbeat response inherent in conventional sliding mode control designed for sampled-data systems. It will be shown in this work that, the discrete-time version of the sliding mode control based on the integral type sliding surface has $O(T^2)$ tracking error for output tracking. An experimental example demonstrates the validity of the proposed scheme.

1. INTRODUCTION

Sliding mode control is a very popular robust control method owing to its ease of design and robustness to 'matched' disturbances. However, full state information is required in the controller design which is a drawback since in most practical applications only the output measurement is available. To solve this problem, focus was placed on output feedback based sliding mode control Żak et al. (1993)-Lai et al. (2004). Two approaches arose: a design based on obervers to construct the missing states, Edwards and Spurgeon (1996), Slotine et al. (1987), the other design focused on using only the output measurement, Żak et al. (1993), El-Khazali and DeCarlo (1995). Both approaches present certain strengths and limitations.

Computer implementation of control algorithms presents a great convenience and has, hence, caused the research in the area of discrete-time control to intensify. This also necessitated a rework in the sliding mode control strategy for sampled-data systems. Most of the discrete-time sliding mode approaches are based on the availability of full state information, Su et al. (2000)-Abidi et al. (2007). A few approaches did focus on the ouput measurement, Lai et al. (2004), Lai et al. (2004). In Lai et al. (2004), the control design was based on the assumption that the state matrix of a discrete-time system is invertible. This is true for sampled-data systems. In this work we will focus on state based approaches as well as expand upon the work of Lai et al. (2004) by focusing on arbitrary reference tracking of a linear time invariant system with matched disturbance.

Delays in the state or disturbance estimation in sampled-data systems is an inevitable phenomenon and must be studied carefully. In Abidi et al. (2007) it was shown that in the case of delayed disturbance estimation a worst case accuracy of $O(T)$ can be guaranteed for deadbeat sliding mode control design and a worst case accuracy of $O(T^2)$ for integral sliding mode control. While deadbeat response is a desired phenomenon, deadbeat control is undesirable in practical implementation due to the overlarge control action required. In Abidi et al. (2007) the integral sliding mode design avoided the deadbeat response by eliminating the poles at zero. In this work, we extend the integral sliding mode design to output tracking problems.

A challenging issue in output tracking control is to perform arbitrary reference tracking when only output measurement is available. To accomplish the task of arbitrary reference tracking a controller based on output feedback with a state observer will be designed. The objective is to drive the output tracking error to a certain neighbourhood of the origin. For this purpose a discrete-time integral sliding surface (ISM) is proposed. The proposed scheme allows full control of the closed-loop error dynamics and the elimination of the reaching phase. The elimination of deadbeat response helps to avoid the generation of overlarge control inputs. It is also worth to highlight that the discrete-time ISM control can achieve the $O(T^2)$ boundary for output tracking error even in the presence of $O(T)$ accuracy in the state estimation.

2. PROBLEM FORMULATION

Consider the following continuous-time system with a nominal linear-time-invariant model and matched disturbance

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B(\mathbf{u}(t) + \mathbf{f}(t)) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned} \tag{1}$$

where the state $\mathbf{x} \in \Re^n$, the output $\mathbf{y} \in \Re^m$, the control $\mathbf{u} \in \Re^m$, and the disturbance $\mathbf{f} \in \Re^m$ is assumed smooth and bounded. The discretized counterpart of (1) can be given by

$$\begin{aligned} \mathbf{x}_{k+1} &= \Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k \\ \mathbf{y}_k &= C\mathbf{x}_k, \quad \mathbf{y}_0 = \mathbf{y}(0) \end{aligned} \tag{2}$$

where

$$\Phi = e^{AT}, \quad \Gamma = \int_0^T e^{A\tau} d\tau B$$

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau)d\tau,$$

and T is the sampling period. Here the disturbance $\mathbf{d}_k$ represents the influence accumulated from kT to $(k+1)T$; in the sequel, it shall directly link to $\mathbf{x}_{k+1} = \mathbf{x}((k+1)T)$. From the definition of Γ it can be shown that

$$\begin{aligned}\Gamma &= BT + \frac{1}{2!}ABT^2 + \frac{1}{3!}A^2BT^3 + \cdots \\ &= BT + MT^2 + O(T^3) \ \Rightarrow\ BT = \Gamma - MT^2 + O(T^3)\end{aligned} \tag{3}$$

where M is a constant matrix. From (4), it can be concluded that the magnitude of Γ is of the order $O(T)$.

Based on the smoothness assumption on the disturbance $\mathbf{f}(t)$, several useful properties were derived in Abidi et al. (2007):

Property 1. The discretized disturbance satisfies:

P 1.

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau)d\tau = \Gamma\mathbf{f}_k + \frac{1}{2}\Gamma\mathbf{v}_k T + O(T^3)$$

, where $\mathbf{v}_k = \mathbf{v}(kT)$, $\mathbf{v}(t) = \frac{d}{dt}\mathbf{f}(t)$. Note that the magnitude of the mismatched part in the disturbance $\mathbf{d}_k$ is of the order $O(T^3)$.

P 2. $\mathbf{d}_k = O(T)$.

P 3. $\mathbf{d}_k - \mathbf{d}_{k-1} = O(T^2)$.

P 4. $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} = O(T^3)$.

Property 2. Assume

$$\mathbf{e}_{k+1} = \Lambda\mathbf{e}_k + \boldsymbol{\delta}_k$$

where matrix Λ is asymptotically stable, the magnitude of $\boldsymbol{\delta}_k$ is of the order $O(T^3)$. Then the magnitude of $\mathbf{e}_k$ is of the order $O(T^2)$.

The primary control objective is to desgign an appropriate controller $\mathbf{u}_k$, such that the output $\mathbf{y}_k$ of (2) can follow an arbitrary trajectory $\mathbf{r}_k$ whose magnitue is of the order $O(1)$.

It is worth to highlight that arbitrary trajectory tracking differs significantly from regulation or set-point control problems. Comparing output tracking for arbitrary trajectory with output regulation or set-point control, the minimum-phase property of the plant (2) is in general a necesary condition for the former but not so for the latter.

Let the control law be $\mathbf{u}_k = -K\mathbf{x}_k + G(q)\mathbf{r}_k$, where $G(q)$ is a design tranfer matrix, q is a forward shifting operator. Substituting the control law into (2) yields

$$\mathbf{y}_k = C\left(qI_m - \Phi + \Gamma K\right)^{-1}\Gamma G(q)\mathbf{r}_k \tag{4}$$

where $I_m \in \Re^m$ is a unity matirx. From (4) we can see that for the precise tracking of an arbitrary reference $\mathbf{r}_k$, $G(q)$ must be the inverse of $C(qI_m-\Phi+\Gamma K)^{-1}\Gamma$. Since K is selected such that $(\Phi-\Gamma K)$ is stable, the only concern is that the inverse $C(qI_m-\Phi+\Gamma K)^{-1}\Gamma$ will contain the zeros of (Φ, Γ, C) and, therefore, will require that the system be minimum-phase.

The control objective is to design a discrete-time integral sliding manifold and a discrete-time SMC law that will stabilize the sampled-data system (2) and achieve as precisely as possible output tracking. Meanwhile the closed-loop dynamics of the sampled-data system has m closed-loop poles assigned to desired locations.

3. OUTPUT TRACKING ISM

In this section we will discuss the state feedback based output tracking controller. The controller will be designed based upon an appropriate integral sliding-surface. Further, the closed-loop system will be analyzed to derive the stability conditions and tracking error-bound.

3.1 Controller Design

Consider the discrete-time integral sliding surface defined below,

$$\begin{aligned}\boldsymbol{\sigma}_k &= \mathbf{e}_k - \mathbf{e}_0 + \boldsymbol{\varepsilon}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\varepsilon}_{k-1} + E\mathbf{e}_{k-1}\end{aligned} \tag{5}$$

where $\mathbf{e}_k = \mathbf{r}_k - \mathbf{y}_k$ is the tracking error, $\boldsymbol{\sigma}_k$, $\boldsymbol{\varepsilon}_k \in \Re^m$ are the sliding function and integral vectors, and $E \in \Re^{m\times m}$ is the integral gain matrix.

By virtue of the concept of equivalent control, a SMC law can be derived by letting $\boldsymbol{\sigma}_{k+1} = 0$. From (5), $-\mathbf{e}_0 + \boldsymbol{\varepsilon}_k = \boldsymbol{\sigma}_k - \mathbf{e}_k$, we have

$$\begin{aligned}\boldsymbol{\sigma}_{k+1} &= \mathbf{e}_{k+1} - \mathbf{e}_0 + \boldsymbol{\varepsilon}_{k+1} = \mathbf{e}_{k+1} - \mathbf{e}_0 + \boldsymbol{\varepsilon}_k + E\mathbf{e}_k \\ &= \mathbf{e}_{k+1} - (I_m - E)\mathbf{e}_k + \boldsymbol{\sigma}_k.\end{aligned} \tag{6}$$

From the system dynamics (2), the output error $\mathbf{e}_{k+1}$ is

$$\mathbf{e}_{k+1} = \mathbf{r}_{k+1} - [C\Phi\mathbf{x}_k + C\Gamma\mathbf{u}_k + C\mathbf{d}_k],$$

and

$$\begin{aligned}\boldsymbol{\sigma}_{k+1} &= \mathbf{r}_{k+1} - [C\Phi\mathbf{x}_k + C\Gamma\mathbf{u}_k + C\mathbf{d}_k] - (I_m - E)\mathbf{e} + \boldsymbol{\sigma}_k. \\ &= \mathbf{a}_k - C\Gamma\mathbf{u}_k - C\mathbf{d}_k\end{aligned} \tag{7}$$

where $\mathbf{a}_k = \mathbf{r}_{k+1} - \Lambda\mathbf{e}_k - C\Phi\mathbf{x}_k + \boldsymbol{\sigma}_k$, and $\Lambda = I_m - E$. Assuming $\boldsymbol{\sigma}_{k+1} = 0$, we can derive the equivalent control

$$\mathbf{u}_k^{eq} = (C\Gamma)^{-1}(\mathbf{a}_k - C\mathbf{d}_k). \tag{8}$$

Note that the control (8) is based on the current value of the disturbance $\mathbf{d}_k$ which is unknown and therefore cannot be implemented in the current form. To overcome this, the disturbance estimate will be used. When the system states are accessible, a delay based disturbance estimate can be easily derived from the plant (2)

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1}. \tag{9}$$

Note that $\mathbf{d}_{k-1}$ is the exogenous disturbance and bounded, therefore $\hat{\mathbf{d}}_k$ is bounded for all k. Using the disturbance estimation (9), the actual ISMC law is given by

$$\mathbf{u}_k = (C\Gamma)^{-1}(\mathbf{a}_k - C\hat{\mathbf{d}}_k). \tag{10}$$

3.2 Stability Analysis

Since the integral switching surface (5) consists of outputs only, it is necessary to examine the closed-loop stability in state space when the ISMC (10) and disturbance esitmate (disturbance-estimation) are used.

Expressing $\mathbf{e}_k = \mathbf{r}_k - C\mathbf{x}_k$, the ISMC law (10) can be rewritten as

$$\begin{aligned}\mathbf{u}_k &= (C\Gamma)^{-1}(\mathbf{r}_{k+1} - \Lambda\mathbf{e}_k - C\Phi\mathbf{x}_k + \boldsymbol{\sigma}_k - C\hat{\mathbf{d}}_k)\\ &= -(C\Gamma)^{-1}(C\Phi - \Lambda C)\mathbf{x}_k - (C\Gamma)^{-1}C\hat{\mathbf{d}}_k\\ &\quad +(C\Gamma)^{-1}(\mathbf{r}_{k+1} - \Lambda\mathbf{r}_k) + (C\Gamma)^{-1}\boldsymbol{\sigma}_k. \qquad (11)\end{aligned}$$

Substituting the above control law (11) into the plant (2) yields the closed-loop state dynamics

$$\begin{aligned}\mathbf{x}_{k+1} = &\;[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]\mathbf{x}_k + \mathbf{d}_k - \Gamma(C\Gamma)^{-1}C\hat{\mathbf{d}}_k\\ &+\Gamma(C\Gamma)^{-1}(\mathbf{r}_{k+1} - \Lambda\mathbf{r}_k) + \Gamma(C\Gamma)^{-1}\boldsymbol{\sigma}_k.\end{aligned} \qquad (12)$$

It can be seen from (12) that the stability of $\mathbf{x}_k$ is determined by the matrix $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ and the boundedness of $\boldsymbol{\sigma}_k$.

According to Xu and Abidi (2008) the eignvalues of $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ are the eigenvalues of Λ and the non-zero eigenvalues of $[\Phi - \Gamma(C\Gamma)^{-1}C\Phi]$, therefore, the matrix $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ has m poles to be placed at desired locations while the remaining $n - m$ poles are the open-loop zeros of the plant (Φ, Γ, C). Since, the plant (2) is assumed to be minimum-phase, the $n - m$ poles are stable. Therefore, stability of the closed-loop state dynamics is guaranteed. Note that if Λ is a zero matrix then m poles are zero and the performance will be the same as the conventional deadbeat sliding-mode controller design.

Since we use disturbance estimate, $\boldsymbol{\sigma}_k \neq 0$. To show the boundedness and facilitate later analysis on the tracking performance, we derive the relationship between the switching surface and the distrubance estimate, as well as the relationship between the output tracking error and the disturbance estimate.

Theorem 1. Assume that the system (2) is minimum-phase and the eigenvalues of the matrix Λ are within the unit circle. Then by the control law (10) we have

$$\boldsymbol{\sigma}_{k+1} = C(\hat{\mathbf{d}}_k - \mathbf{d}_k) \qquad (13)$$

and the error dynamics

$$\mathbf{e}_{k+1} = \Lambda\mathbf{e}_k + \boldsymbol{\delta}_k \qquad (14)$$

where $\boldsymbol{\delta}_k = C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1})$.

Proof:

In order to verify the first part of theorem 1, substitute the control law (10) into (7)

$$\begin{aligned}\boldsymbol{\sigma}_{k+1} &= \mathbf{a}_k - C\Gamma\mathbf{u}_k - C\mathbf{d}_k = \mathbf{a}_k - C\Gamma\mathbf{u}_k^{eq} - C\mathbf{d}_k + C\Gamma(\mathbf{u}_k^{eq} - \mathbf{u}_k)\\ &= C\Gamma(\mathbf{u}_k^{eq} - \mathbf{u}_k),\end{aligned}$$

where we use the property of equivalent control $\boldsymbol{\sigma}_{k+1} = \mathbf{a}_k - C\Gamma\mathbf{u}_k^{eq} - C\mathbf{d}_k = 0$. Comparing two control laws (8) and (10), we obtain

$$\boldsymbol{\sigma}_{k+1} = C(\hat{\mathbf{d}}_k - \mathbf{d}_k).$$

Note that the switching surface $\boldsymbol{\sigma}_{k+1}$ is no longer zero as desired but a function of the difference $\mathbf{d}_k - \hat{\mathbf{d}}_k$. This, however, is acceptable since the difference is $\mathbf{d}_k - \hat{\mathbf{d}}_k = \mathbf{d}_k - \mathbf{d}_{k-1}$ by the delay based disturbance estimation; thus, according to *Property 1* the difference is $O(T^2)$ which is quite small in practical applications.

To derive the second part of theorem 1 regarding the error dynamics, rewritting (7) as

$$\mathbf{e}_{k+1} = \Lambda\mathbf{e}_k + \boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k,$$

and substituting the relationship (13), lead to

$$\begin{aligned}\mathbf{e}_{k+1} &= \Lambda\mathbf{e}_k + C(\hat{\mathbf{d}}_k - \mathbf{d}_k) - C(\hat{\mathbf{d}}_{k-1} - \mathbf{d}_{k-1})\\ &= \Lambda\mathbf{e}_k + C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1}) = \Lambda\mathbf{e}_k + \boldsymbol{\delta}_k.\end{aligned}$$

Since $\hat{\mathbf{d}}_k = \mathbf{d}_{k-1}$, $\boldsymbol{\delta}_k$ is bounded, thus from *Property 2* $\mathbf{e}_k$ is bounded. *Remark:* From (14) we can see that the reference tracking dynamics depends on the choice of Λ which is a design matrix.

3.3 Disturbance Observer Design

Note that according to **Property 1**, the disturbance can be written as

$$\mathbf{d}_k = \Gamma\mathbf{f}_k + \frac{1}{2}\Gamma\mathbf{v}_k T + O(T^3) = \Gamma\boldsymbol{\eta}_k + O(T^3) \qquad (15)$$

where $\boldsymbol{\eta}_k = \mathbf{f}_k + \frac{1}{2}\mathbf{v}_k T$. If $\boldsymbol{\eta}_k$ can be estimated, then the estimation error of $\mathbf{d}_k$ would be $O(T^3)$ which is acceptable in practical applications.

Define the observer

$$\begin{aligned}\mathbf{x}_{d,k} &= \Phi\mathbf{x}_{d,k-1} + \Gamma\mathbf{u}_{k-1} + \Gamma\hat{\boldsymbol{\eta}}_{k-1}\\ \mathbf{y}_{d,k-1} &= C\mathbf{x}_{d,k-1}\end{aligned} \qquad (16)$$

where $\mathbf{x}_{d,k-1} \in \Re^n$ is the observer state vector, $\mathbf{y}_{d,k-1} \in \Re^m$ is the observer output vector, $\hat{\boldsymbol{\eta}}_{k-1} \in \Re^m$ is the disturbance estimate and will act as the 'control input' to the observer, therefore we can write $\hat{\mathbf{d}}_{k-1} = \Gamma\hat{\boldsymbol{\eta}}_{k-1}$. Since the disturbance estimate will be used in the final control signal, it must not be overly large. Therefore, it is wise to avoid a deadbeat design. For this reason we design the disturbance observer based on an integral sliding surface

$$\begin{aligned}\boldsymbol{\sigma}_{d,k} &= \mathbf{e}_{d,k} - \mathbf{e}_{d,0} + \boldsymbol{\varepsilon}_{d,k}\\ \boldsymbol{\varepsilon}_{d,k} &= \boldsymbol{\varepsilon}_{d,k-1} + E_d\mathbf{e}_{d,k-1}\end{aligned} \qquad (17)$$

where $\mathbf{e}_{d,k} = \mathbf{y}_k - \mathbf{y}_{d,k}$ is the output estimation error, $\boldsymbol{\sigma}_{d,k}, \boldsymbol{\varepsilon}_{d,k} \in \Re^m$ are the sliding function and integral vectors, and E_d is an integral gain matrix.

Note that the sliding surface (17) is analogous to (5), that is, the set $(\mathbf{y}_k, \mathbf{x}_{d,k}, \mathbf{u}_k + \hat{\boldsymbol{\eta}}_k, \mathbf{y}_{d,k}, \boldsymbol{\sigma}_{d,k})$ has duality with the set $(\mathbf{r}_k, \mathbf{x}_k, \mathbf{u}_k, \mathbf{y}_k, \boldsymbol{\sigma}_k)$, except for an one-step delay in the observer dynamics (16). Therefore, let $\boldsymbol{\sigma}_{d,k} = 0$ we can derive the virtual equivalent control $\mathbf{u}_{k-1} + \hat{\boldsymbol{\eta}}_{k-1}$, thus

$$\hat{\boldsymbol{\eta}}_{k-1} = (C\Gamma)^{-1}\left[\mathbf{y}_k - \Lambda_d\mathbf{e}_{d,k-1} - C\Phi\mathbf{x}_{d,k-1} + \boldsymbol{\sigma}_{d,k-1}\right] - \mathbf{u}_{k-1} \qquad (18)$$

where $\Lambda_d = I_m - E_d$.

In practice, the quantity $\mathbf{y}_{k+1}$ is not available at the time instance k when computing $\hat{\eta}_k$. Therefore we can only compute $\hat{\eta}_{k-1}$, and in the control law we use the delayed estimate $\hat{\mathbf{d}} = \Gamma\hat{\boldsymbol{\eta}}_{k-1}$.

The stability and convergence properties of the observer (16) and the disturbance estimation (18) are analyzed in the theorem 2.

Theorem 2. The observer outputs $\mathbf{y}_{d,k}$ converge asymptotically to the true outputs $\mathbf{y}_k$, and the disturbance estimate $\hat{\mathbf{d}}_{k-1}$ converges to the actual disturbance $\mathbf{d}_{k-1}$ with the precision order $O(T^2)$.

Proof:

Substituting (18) into (16), and using the relation $\mathbf{e}_{d,k-1} = C(\mathbf{y}_{k-1} - \mathbf{y}_{d,k-1})$, yield

$$\begin{aligned}\mathbf{x}_{d,k} = &\left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]\mathbf{x}_{d,k-1} \\ &+\Gamma(C\Gamma)^{-1}[\mathbf{y}_k - \Lambda_d\mathbf{y}_{k-1}] + \Gamma(C\Gamma)^{-1}\boldsymbol{\sigma}_{d,k-1}.\end{aligned} \tag{19}$$

Since the virtual control $\mathbf{u}_{k-1} + \hat{\boldsymbol{\eta}}_{k-1}$ is chosen such that $\boldsymbol{\sigma}_{d,k} = 0$ for any $k > 0$, (20) renders to

$$\mathbf{x}_{d,k} = \left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]\mathbf{x}_{d,k-1} + \Gamma(C\Gamma)^{-1}[\mathbf{y}_k - \Lambda_d\mathbf{y}_{k-1}]. \tag{20}$$

The second term on the right hand side of (20) can be expressed as

$$\begin{aligned}\Gamma(C\Gamma)^{-1}[\mathbf{y}_k - \Lambda_d\mathbf{y}_{k-1}] = &\Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\mathbf{x}_{k-1} \\ &+\Gamma\mathbf{u}_{k-1} + \Gamma(C\Gamma)^{-1}C\mathbf{d}_{k-1}\end{aligned}$$

by using the relations $\mathbf{y}_k = C\Phi\mathbf{x}_{k-1} + C\Gamma\mathbf{u}_{k-1} + C\mathbf{d}_{k-1}$ and $\mathbf{y}_{k-1} = C\mathbf{x}_{k-1}$. Therefore (20) can be rewritten as

$$\begin{aligned}\mathbf{x}_{d,k} = &\Phi\mathbf{x}_{d,k-1} + \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\Delta\mathbf{x}_{d,k-1} \\ &+\Gamma\mathbf{u}_k + \Gamma(C\Gamma)^{-1}C\mathbf{d}_{k-1}\end{aligned} \tag{21}$$

where $\Delta\mathbf{x}_{d,k-1} = \mathbf{x}_{k-1} - \mathbf{x}_{d,k-1}$.

Further subtracting (21) from the system (2) we obtain

$$\begin{aligned}\Delta\mathbf{x}_{d,k} = &\left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]\Delta\mathbf{x}_{d,k-1} \\ &+[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1}\end{aligned} \tag{22}$$

where $[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1}$ is $O(T^3)$ because
$[I - \Gamma(C\Gamma)^{-1}C][\Gamma\boldsymbol{\eta}_{k-1} + O(T^3)] = [I - \Gamma(C\Gamma)^{-1}C]O(T^3) = O(T^3)$.

Applying the *Property 2*, $\Delta\mathbf{x}_{d,k-1} = O(T^2)$.

From (22) we can see that the stability of the disturbance observer depends only on the system matrix $\left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]$ and is guaranteed by the selection of the matrix Λ_d and the fact that system (Φ, Γ, C) is minimum phase. It should also be noted that the residue term $[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1}$ in the state space is orthogonal to the output space, as $C[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1} = 0$. Therefore premultliplication of (22) with C yields the output tracking error dynamics

$$\mathbf{e}_{d,k} = \Lambda_d\mathbf{e}_{d,k-1} \tag{23}$$

which is asymptotically stable through choosing a stable matrix Λ_d.

Finally we discuss the convergence property of the estimate $\hat{\mathbf{d}}_{k-1}$. Subtracting (16) from (2) with one-step delay, we obtain

$$\Delta\mathbf{x}_{d,k} = \Phi\Delta\mathbf{x}_{d,k-1} + \Gamma(\boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1}) + O(T^3). \tag{24}$$

Premultiplying (24) with C, and substituing (23) that describes $C\Delta\mathbf{x}_{d,k}$, yield

$$\hat{\boldsymbol{\eta}}_{k-1} = \boldsymbol{\eta}_{k-1} + (C\Gamma)^{-1}(C\Phi - \Lambda_d C)\Delta\mathbf{x}_{d,k-1} + (C\Gamma)^{-1}O(T^3). \tag{25}$$

The first term on the right hand side of (25) is $O(T)$ because $\Delta\mathbf{x}_{d,k-1} = O(T^2)$ but $(C\Gamma)^{-1} = O(T^{-1})$. As a result, from (25) we can conclude that $\hat{\boldsymbol{\eta}}_{k-1}$ approaches $\boldsymbol{\eta}_{k-1}$ with the precision $O(T)$. In terms of the relationship

$$\mathbf{d} - \hat{\mathbf{d}} = \Gamma(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}) + O(T^3)$$

and $\Gamma = O(T)$, we conclude $\hat{\mathbf{d}}_{k-1}$ converges to $\mathbf{d}_{k-1}$ with the precision of $O(T^2)$.

3.4 State Observer

State estimation will be accomplished with the following state-observer

$$\hat{\mathbf{x}}_{k+1} = \Phi\hat{\mathbf{x}}_k + \Gamma\mathbf{u}_k + L(\mathbf{y}_k - \hat{\mathbf{y}}_k) + \hat{\mathbf{d}}_k \tag{26}$$

where $\hat{\mathbf{x}}_k$, $\hat{\mathbf{y}}_k$ are the state and output estimates and L is a design matrix. Observer (26) is well-known, however, the term $\hat{\mathbf{d}}_k$ has been added to compensate for the disturbance. It is necessary to investigate the effect of the disturbance estimation on the state estimation. Subtracting (26) from (2) we get

$$\tilde{\mathbf{x}}_{k+1} = [\Phi - LC]\tilde{\mathbf{x}}_k + \mathbf{d}_k - \hat{\mathbf{d}}_k. \tag{27}$$

It can be seen that the state estimation is indepdendent of the control inputs. Under the assumption that (Φ, Γ, C) is controllable and observable, we can choose L such that $\Phi - LC$ is asymptotically stable. From Theorem 4, $\mathbf{d}_k - \hat{\mathbf{d}}_k = O(T^2)$, thus, from *Property 2* the ultimate bound of $\tilde{\mathbf{x}}_k$ is $O(T)$. Later we will show that, for systems of relative degree greater than 1, by virtue of the integral action in the ISM control, the state estimation error will be reduced to $O(T^2)$ in the overall closed-loop system.

3.5 Tracking Error Bound

From the error dynamics of the state estimation (27), the solution is

$$\tilde{\mathbf{x}}_k = [\Phi - LC]^k\tilde{\mathbf{x}}_0 + \sum_{i=0}^{k-1}\left([\Phi - LC]^{k-1-i}(\mathbf{d}_i - \hat{\mathbf{d}}_i)\right). \tag{28}$$

The difference $\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1}$ can be calculated

$$\begin{aligned}\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} = &[(\Phi - LC) - I_n](\Phi - LC)^{k-1}\tilde{\mathbf{x}}_0 \\ &+\sum_{i=0}^{k-1}\left([\Phi - LC]^{k-1-i}(\mathbf{d}_i - \hat{\mathbf{d}}_i)\right) \\ &-\sum_{i=0}^{k-2}\left([\Phi - LC]^{k-1-i}(\mathbf{d}_i - \hat{\mathbf{d}}_i)\right)\end{aligned}$$

which can be simplified to

$\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} = [(\Phi - LC) - I_n](\Phi - LC)^{k-1}\tilde{\mathbf{x}}_0 + (\mathbf{d}_k - \hat{\mathbf{d}}_k)$. Since $(\Phi - LC)$ is asymptotically stable, the ultimate bound is

$$\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} = \mathbf{d}_k - \hat{\mathbf{d}}_k, \tag{29}$$

and $\boldsymbol{\delta}_k$ can be expressed ultimately as

$$\begin{aligned}\boldsymbol{\delta}_k &= C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1}) - C\Phi(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1})\\ &= C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1}) - C\Phi(\mathbf{d}_k - \hat{\mathbf{d}}_k). \end{aligned} \tag{30}$$

From theorem 3, the disturbance estimation error $\mathbf{d}_k - \hat{\mathbf{d}}_k$ is $O(T^2)$. Therefore

$$\boldsymbol{\delta}_k = C \cdot \left(O(T^2) + O(T^2)\right) - C\Phi \cdot O(T^2) = O(T^2),$$

the ultimate bound on $\boldsymbol{\sigma}_k$ is $O(T^2)$, and the ultimate bound on $\|\mathbf{e}_k\|$ is $O(T)$.

Remark: Note that the the guaranteed tracking precision is $O(T)$ because the control problem becomes highly challenging in the presence of state estimation and disturbance estimation errors, and meanwhile aiming at arbitrary reference tracking.

In many motion control tasks the system relative degree is 2 when the torque or force control is designed for position tracking. Now we derive an interesting property by the following corollary.

Corollary: For a continuous system of relative degree greater than 1, the ultimate bound of $\|\mathbf{e}_k\|$ is $O(T^2)$.

Proof:

From theorem 2 (14) and *Property 2*, $\|\mathbf{e}_k\|$ is $O(T^2)$ if $\boldsymbol{\delta}_k = O(T^3)$. When the system relative degree is 2, $CB = 0$, and

$$\begin{aligned}C\Gamma &= C\left(BT + \frac{1}{2!}ABT^2 + \frac{1}{3!}A^2BT^3 + \cdots\right)\\ &= \frac{1}{2!}CABT^2 + \frac{1}{3!}CA^2BT^3 + \cdots = O(T^2).\end{aligned}$$

Similarly

$$\begin{aligned}C\Phi\Gamma &= C(I + AT + \frac{1}{2!}A^2T^2 + \cdots)\Gamma\\ &= C(I + O(T))\Gamma = C\Gamma + O(T^2) = O(T^2).\end{aligned}$$

Now rewrite

$$\begin{aligned}\boldsymbol{\delta}_k &= C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1}) - C\Phi(\mathbf{d}_k - \hat{\mathbf{d}}_k) \\ &= C\Gamma(\hat{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k + \boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1}) - C\Phi\Gamma(\boldsymbol{\eta}_k - \hat{\boldsymbol{\eta}}_k) + O(T^3).\end{aligned} \tag{31}$$

Note that the ultimate bound of $\boldsymbol{\eta}_k - \hat{\boldsymbol{\eta}}_k$, derived in Theorem 3, is $O(T)$. Thus we conclude from (32)

$$\boldsymbol{\delta}_k = O(T^2) \cdot (O(T) + O(T)) - O(T^2) \cdot O(T) + O(T^3) = O(T^3)$$

and consequently $\|\mathbf{e}_k\|$ is ultimately $O(T^2)$.

4. EXPERIMENTAL INVESTIGATION

To verify the effectiveness of the discrete-time integral sliding control design, experiments have been carried out using a linear piezoelectric motor which has many promising applications in industries. Piezoelectric motors are mainly applied to high precision control problems as it can easily reach the precision scale of micro-meters or even nano-meters. This gives rise to extra difficulty in establishing an accurate mathematical model for piezoelectric motors: any tiny factors, nonlinear and unknown, will severely affect their characteristics and control performance.

The configuration of the whole control system is outlined in Fig.1. The driver and the motor can be modeled approximately as a second order system shown in (1) with the system matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{k_{fv}}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{k_f}{M} \end{bmatrix}, \quad C = [1\ 0]$$

where $M = 1kg$, $k_{fv} = 144N$ and $k_f = 6N/Volt$.

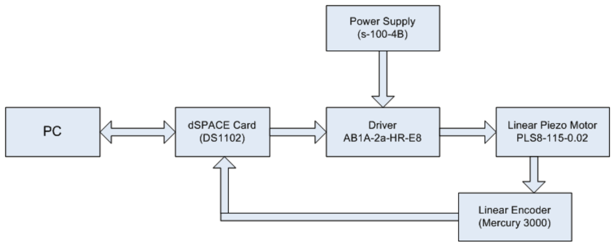


Fig. 1. System Block Diagram

This simple linear model does not contain any nonlinear and uncertain effects such as the frictional force in the mechanical part, high-order electrical dynamics of the driver, loading condition, etc., which are hard to model in practice. In general, producing a high precision model will require more efforts than performing a control task with the same level of precision.

For the state observer approach the system (Φ, Γ, C) is required to be minimum phase. From Fig.2 we see that for a sampling-time between $0.1ms$ and $1s$ the open-loop zero is stable, therefore, the system is minimum phase. From Fig.2 a selection of sampling-time $T = 1ms$ would provide a fast enough convergence while having a good enough tracking error. Upon sampling at $T = 1ms$ the resulting

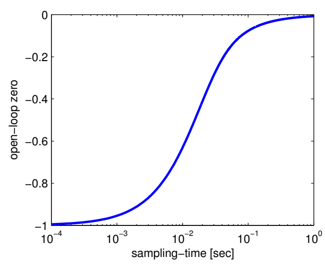


Fig. 2. Open-loop zero of (Φ, Γ, C) w.r.t sampling-time

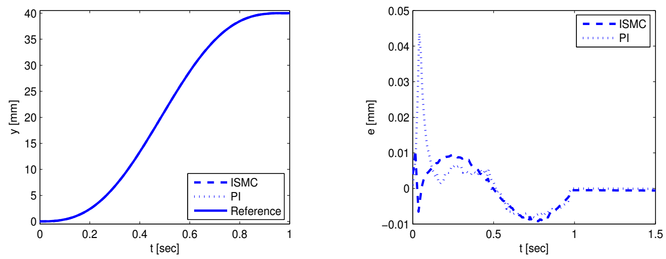
sampled-data system state and gain matrices are

$$\Phi = \begin{bmatrix} 1.0000 & 0.0009 \\ 0 & 0.8659 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2.861 \times 10^{-6} \\ 5.6 \times 10^{-3} \end{bmatrix}$$

The open-loop zero for this sampling-time are -0.954. To proceed with the implementation three sets of parameters need to be designed: the state observer gain matrix L, the disturbance observer integrator gain matrix E_d, and the controller integrator gain E. The state observer gain is selected such that the observer poles are $(0.4, 0.4)$. This selection is arbitrary, but, the poles are selected to ensure quick convergence. Next, the matrix E_d is designed. Note that for this second order system E_d is a scalar. To ensure the quick convergence of the disturbance observer, E_d is selected such that the observer pole is $\lambda_d = 0.9$ which corresponds to $s = -105.4$ in the continuous-time. Since the remaining pole of the observer is the non-zero open-loop zero $z = -0.954$ corresponding to a pole with real part of $s = -47.1$ in the continuous-time, it is the dominant pole. Finally, the controller pole is selected as $\lambda = 0.958$ which is found to be the best possible after some trials. Thus, the design parameters are as follows

$$L = [\,1.06\ 231.05\,],\ E_d = 1 - \lambda_d = 0.1,\ E = 1 - \lambda = 0.042.$$

The reference trajectory r_k used is a sigmoid curve as shown in Fig.3a. The ISM results are compared to that of a PI controller as seen in Fig.3a and Fig.3b. From the results we see that the ISM controller has a better tracking performance compared to a PI controller. The results in Fig.4 are for the control inputs for the PI and ISM. Finally, an extra load of $2.5kg$ is added without modifying the controller parameters. We see from the results that the change of load barely effects the ISM controller performance as seen in Fig.5a, as the error magnitude is only varied slightly around 0.01 mm. The control for this case is seen in Fig.5b. We can observe from the figures that the control input needed to overcome the deadzone is increased from around $1.25V$ to around $1.5V$ when the load is added.



(a) (b)

Fig. 3. Position and tracking error of ISMC and PI controllers

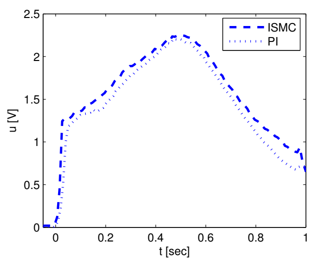
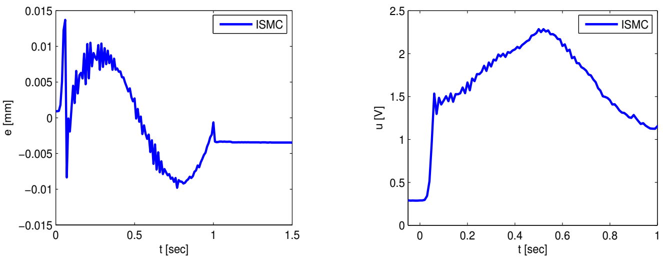


Fig. 4. Comparison of the control inputs of ISMC and PI controllers

5. CONCLUSION

This work presents a form of the discrete-time integral sliding control design for sampled-data systems with output tracking. Proper disturbance and state observers were



(a) (b)

Fig. 5. Tracking error and control input of ISMC loaded with $2.5kg$

presented. The closed-loop stability of the system was not dependent on either observer and is designed seperately. It was shown that the maximum bound on the tracking error is $O(T^2)$ at steady state. It was also shown that even though the state observer produced $O(T)$ estimation error, the ISM state observer approach could still produce $O(T^2)$ tracking error. Experimental comparison with a PI controller proves the effectiveness of the proposed methods.

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