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Delay-Distribution-Dependent Stability and Stabilization for Wireless Networked Control System with Data Quantization ⋆

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Abstract: In this paper, the problem of stability and stabilization is studied for wireless networked control system (WiNCS) with data quantization. Considering the mobile characteristics of wireless network environment, it is assumed in this paper that the network-induced delay is random and its probability distribution is known in advance. In terms of the probability distribution of the delay, a new type of system model with stochastic parameter matrices is proposed. The purpose of this paper is to design the state feedback controller which can guarantee the system is exponentially mean-square stable when considering the effect of both network conditions and data quantizations. By using linear matrix inequality (LMI) technology, sufficient conditions for the solvability of the addressed problem are obtained. It should be noted that, different from the traditional methods in the existing references, the solvability of the derived conditions depends on not only the size of the delay, but also the probability of the delay taking values in some intervals. Finally, a numerical example is provided to demonstrate the effectiveness and applicability of the proposed approach.

1. INTRODUCTION

System communicating through wireless links in the forward and backward loops is called wireless networked control system (WiNCS) Akyildiz et al. [2002], Drew et al. [2005]. In recent years, the investigation of WiNCS has received considerable attention since the employment of wireless network in the control systems has many advantages, such as fully mobile operation, flexible installation, rapid deployment and low maintenance costs, see Akyildiz et al. [2002], Colandairaj et al. [2007], Drew et al. [2005], Kawka and Alleyne [June, 2005] and the references therein. However, since the usage of wireless network in the system, new problems shown below will be inevitably faced in a practical WiNCS.

(1) Random delays and high rate packet losses. Because of the mobility of a wireless network, a high rate of packet losses is often encountered in WiNCS. Moreover, the transmission delay is often of a random characteristic Drew et al. [2005], Liu et al. [2005]. As pointed out in Wang et al. [2004a], for a given wireless network, it can be measured that there exists a small number ε such that $Prob\{\tau(k) > d\} < \varepsilon$, where $\tau(k)$ denotes the transmission delay and d is a constant. For this case, what we need to investigate is, for a given ε, to find the upper bound for d, or, for a given d, to find the upper bound for ε. Obviously, the stochastic characteristic of the time delay will affect the size of the allowable variation range of the delay.
(2) Data quantization. In the WiNCS, data quantization is required from the following aspects. Firstly, since the controller in WiNCS is implemented digitally, the signals that take values in a continuous set need to be represented with finite precision. Secondly, the use of A/D and D/A converters also determines that the data transmissions can not be performed with infinite precision. Thirdly, the wireless communication bandwidth between the plant and controller is limited due to the capacity or security constraints. In order to save the limited bandwidth, data quantization is necessary before it is sent through the wireless network to the next node. From the existing references on the study of WiNCS Colandairaj et al. [2007], Wang et al. [2004a], Ploplys et al. [2004], Subramanian and Sayed [2005], it can be seen that little attention has been paid to the study of WiNCS considering the effect of the randomness of the transmission delay and data quantization.

In this paper we consider the design problem of state feedback controller for the WiNCS with data quantization. The network-induced delay considered in this paper is assumed to vary randomly in an interval with its probability distribution known in advance. In terms of the probability distribution of the delay, a new type of system model with stochastic parameter matrices is proposed. In addition, the effect of the data quantization is also considered in this paper, where a time-varying quantizer is designed to quantize the sensor data. Considering the effect of both

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wireless network conditions and data quantization, delay-distribution-dependent stability and stabilization conditions are derived for the WiNCS. It should be noted that the solvability of the obtained criteria depends on not only the size of the delay, but also its probability distribution. At last, a numerical example is proposed to illustrate the applicability of the method.

2. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control input respectively, A and B are system matrices with compatible dimensions and $(\ A\ B\ )$ is a controllable pair. Therefore, there exists a constant matrix K such that $A + BK$ is a stable matrix. In this paper, the designed controller is of the following linear form

$$u(t) = Kx(t) \tag{2}$$

Assumption 1. In this paper, it is assumed that the system (1) and the controller (2) are connected over a wireless network and the data is quantized before it is transmitted through the network.

Firstly, we consider the modelling problem when only the effect of the network-induced delay and packet losses on the system is concerned. In this case, the practical control signal reaching the system can be represented as

$$u(t) = Kx(i_k h), t \in [\, i_k h + \tau_{i_k},\ i_{k+1}h + \tau_{i_{k+1}}\,) \tag{3}$$

where K is the feedback gain to be determined later, h is the sampling period of the sensor, $\tau_{i_k}(k = 1, 2, \cdots)$ is the transmission delay from sensor to controller. Substituting the controller (3) into (1), we obtain the closed-loop system

$$\dot{x}(t) = Ax(t) + BKx(i_k h), t \in [\, i_k h + \tau_{i_k},\ i_{k+1}h + \tau_{i_{k+1}}\,) \tag{4}$$

Remark 1. $\{i_1, i_2, i_3, \cdots\}$ is subset of $\{1, 2, 3, \cdots\}$, which contains the information of packet losses and wrong packet sequence. If $\{i_1, i_2, i_3, \cdots\} = \{1, 2, 3, \cdots\}$, $i_{k+1} = i_k + 1$, which means no packet losses and wrong packet sequence happen. If $i_{k+1} - i_k = n(\geq 2)$, it means $n - 1$ continuous packets are lost. It can be easily seen that $\cup_{k=1}^{\infty} [\, i_k h + \tau_{i_k},\ i_{k+1}h + \tau_{i_k}\,) = [\,0,\ \infty\,)$.

Remark 2. As pointed out in Remark 3 in Yue et al. [2004], when $i_{k+1} < i_k$, that is, the new packet reaches the destination before the old one, discarding the old packet may result in a less conservative result. Therefore we assume $i_{k+1} > i_k$ in the following discussion.

Since the sensor data will be quantized before being sent to the controller, except for the impact of the network conditions, we should also consider the impact of data quantization on the performance of the system. Defining a function $q(z) : \mathbb{R}^l \to \mathbb{D}$ as a piecewise constant function, where z is the variable to be quantized and $\mathbb{D}$ is a finite subset of $\mathbb{R}^l$. This leads to a partition of $\mathbb{R}^l$ into a finite number of quantization regions of the form $\left\{z \in \mathbb{R}^l : q(z) = i, i \in \mathbb{D}\right\}$. It is assumed that $q(z)$ satisfies the following conditions Liberzon and Nešić [2007]:

<a> if $\|z\| \leq \mathscr{M}$, then $\|q(z) - z\| \leq \Delta$,
<b> if $\|z\| \geq \mathscr{M}$, then $\|q(z)\| \geq \mathscr{M} - \Delta$,

where $\mathscr{M}$ and Δ are the quantization range and quantization error of $q(\cdot)$.

In this paper, we will use quantized measurements of the form

$$q(z) = \mu q(\frac{z}{\mu}), \mu > 0 \tag{5}$$

Then the conditions (a) and (b) can be rewritten as

$$\text{if } \|z\| \leq \mu\mathscr{M}, \text{then } \left\|\mu q(\frac{z}{\mu}) - z\right\| \leq \mu\Delta \tag{6}$$

$$\text{if } \|z\| \geq \mu\mathscr{M}, \text{then } \left\|\mu q(\frac{z}{\mu})\right\| \geq \mu\,(\mathscr{M} - \Delta) \tag{7}$$

where μ can be viewed as a zoom variable: for the same z and $q(z)$, large μ will lead to large quantization range and quantization error, little μ will lead to little quantization range and quantization error.

If we also consider the effect of quantization of the sensor data $x(i_k h)$, the practical signal receiving at the controller is $\mu_k q\left(\frac{x(i_k h)}{\mu_k}\right)$. Therefore, the controller signal can be written as

$$u(t) = K\mu_k q\left(\frac{x(i_k h)}{\mu_k}\right), t \in [\, i_k h + \tau_{i_k},\ i_{k+1}h + \tau_{i_{k+1}}\,) \tag{8}$$

The closed-loop system of (1) with controller (8) is

$$\dot{x}(t) = Ax(t) + BK\mu_k q\left(\frac{x(i_k h)}{\mu_k}\right), \tag{9}$$
$$t \in [\, i_k h + \tau_{i_k},\ i_{k+1}h + \tau_{i_{k+1}}\,)$$
$$x(t) = \psi(t), t \in [\,-\eta_2,\ 0\,] \tag{10}$$

Define $\eta(t) = t - i_k h$ and $\delta(t) = \mu_k q\left(\frac{x(t-\eta(t))}{\mu_k}\right) - x\left(t - \eta(t)\right)$, (9) can be written as

$$\dot{x}(t) = Ax(t) + BKx\left(t - \eta(t)\right) + B\delta(t) \tag{11}$$

From the definition of $\eta(t)$, $\eta(t)$ changes from τ_{i_k} to $\left(i_{k+1} - i_k\right)h + \tau_{i_{k+1}}$ for $t \in [\, i_k h + \tau_{i_k},\ i_{k+1}h + \tau_{i_{k+1}}\,)$. Since the delay $\tau_{i_k}(k = 1, 2, 3, \cdots)$ is random, for any $t \in \mathcal{R}^+$, the value of $\eta(t)$ is also random. In this paper, we assume $\eta(t)$ satisfies the following conditions.

Assumption 2. There exist three constants η_1, η_2 and $\beta_0 \in [\,0,\ 1\,]$, where $\eta_1 < \eta_2$, such that

i> $0 \leq \eta(t) \leq \eta_2$,
ii> the probability of $\eta(t)$ taking values in $[\,0,\ \eta_1\,)$ is β_0 and in $[\,\eta_1,\ \eta_2\,]$ is $1 - \beta_0$.

Define two sets

$$\Omega_1 = \{t : \eta(t) \in [\,0,\ \eta_1\,)\}, \Omega_2 = \{t : \eta(t) \in [\,\eta_1,\ \eta_2\,]\} \tag{12}$$

Obviously, $\Omega_1 \cup \Omega_2 = \mathcal{R}^+$ and $\Omega_1 \cap \Omega_2 = \Phi$(empty set). From the definitions of Ω_1 and Ω_2, it can be seen that $t \in \Omega_1$ means the event $\eta(t) \in [\,0,\ \eta_1\,)$ occurs and $t \in \Omega_2$ means the event $\eta(t) \in [\,\eta_1,\ \eta_2\,]$ occurs. Based on the above two sets, the following two functions are defined

$$\eta_1(t) = \begin{cases} \eta(t), \text{for } t \in \Omega_1 \\ 0, \text{for } t \in \Omega_2 \end{cases} \tag{13}$$

$$\eta_2(t) = \begin{cases} \eta(t), \text{for } t \in \Omega_2 \\ \eta_1, \text{for } t \in \Omega_1 \end{cases} \tag{14}$$

Furthermore, we can define a stochastic variable $\beta(t)$ as

$$\beta(t) = \begin{cases} 1, \ t \in \Omega_1 \\ 0, \ t \in \Omega_2 \end{cases} \tag{15}$$

Assumption 3. $\beta(t)$ is a Bernoulli distributed sequence with

$$\text{Prob}\{\beta(t) = 1\} = \mathcal{E}\{\beta(t)\} = \beta_0, \tag{16}$$

$$\text{Prob}\{\beta(t) = 0\} = 1 - \mathcal{E}\{\beta(t)\} = 1 - \beta_0, \tag{17}$$

Remark 3. The introduction of $\beta(t)$ is motivated by Wang et al. [2004b, 2006], where the Bernoulli distributed sequence $\beta(t)$ is used to model the missing message of the system. Different from Wang et al. [2004b, 2006], $\beta(t)$ is used in this paper to describe the probability of the random delays appearing in different intervals.

By using the new functions $\eta_i(t)$ $(i = 1, 2)$ and $\beta(t)$, the system (11) can be rewritten as

$$\begin{aligned} \dot{x}(t) = & Ax(t) + \beta(t)BKx(t - \eta_1(t)) \\ & +(1 - \beta(t))BKx(t - \eta_2(t)) + B\delta(t) \end{aligned}$$

which can further be expressed as

$$\mathcal{A}\zeta(t) + B\delta(t) + (\beta(t) - \beta_0)\mathcal{B}\zeta(t) = 0 \tag{18}$$

where

$$\begin{aligned} \mathcal{A} &= [\, A \ \ \beta_0 BK \ \ 0 \ \ (1-\beta_0)BK \ \ 0 \ \ -I \,] \\ \mathcal{B} &= [\, 0 \ \ BK \ \ 0 \ \ -BK \ \ 0 \ \ 0 \,] \\ \zeta^T(t) &= [\, x^T(t) \ \ x^T(t - \eta_1(t)) \ \ x^T(t - \eta_1) \\ & \quad x^T(t - \eta_2(t)) \ \ x^T(t - \eta_2) \ \ \dot{x}^T(t) \,] \end{aligned}$$

Since the stochastic variable $\beta(t)$ is introduced, the transformed system (18) is a stochastic differential equation. To analyze the dynamic property of the system (18), we need the following definitions.

Definition 1. For a given function $V : C^b_{F_0}([-\eta_2, 0], \mathbb{R}^n) \times \mathbb{S}$, its infinitesimal operator $\mathcal{L}$Mao [2002] is defined as

$$\mathcal{L}V(x_t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}\left[\mathcal{E}(V(x_{t+\Delta}|x_t)) - V(x_t)\right] \tag{19}$$

Definition 2. System (18) is said to be exponentially stable in the mean square sense(ESMSS) if there exist constants $\alpha > 0$ and $\theta > 0$ such that for $t \geq 0$

$$\mathcal{E}\left\{\|x(t)\|^2\right\} \leq \alpha e^{-\theta t} \sup_{-2\eta_2 \leq s \leq 0} \mathcal{E}\left\{\|\psi(s)\|^2\right\}$$

The following two lemmas are necessary in the proof of the main results.

Lemma 1. Ω_1, Ω_2 and Ω are constant matrices of appropriate dimensions, $\tau(t)$ is function of t and satisfies $0 \leq \tau(t) \leq \tau_M$, then

$$\tau(t)\Omega_1 + (\tau_M - \tau(t))\Omega_2 + \Omega < 0 \tag{20}$$

if and only if

$$\tau_M\Omega_1 + \Omega < 0 \tag{21}$$

$$\tau_M\Omega_2 + \Omega < 0 \tag{22}$$

The proof of Lemma 1 can be seen in Appendix A.

Lemma 2. Ω_{1i}, Ω_{2i} and Ω are constant matrices of appropriate dimensions, $\tau_i(t)$ are functions of t and satisfy $0 \leq \tau_1(t) \leq \tau_0 \leq \tau_2(t) \leq \tau_M, i = 1, 2$, then

$$\begin{aligned} &[\tau_1(t)\Omega_{11} + (\tau_0 - \tau_1(t))\Omega_{21}] \\ &+ [(\tau_2(t) - \tau_0)\Omega_{12} + (\tau_M - \tau_2(t))\Omega_{22}] + \Omega < 0 \end{aligned} \tag{23}$$

if and only if the following 4 inequalities hold

$$\tau_0\Omega_{11} + (\tau_M - \tau_0)\Omega_{12} + \Omega < 0 \tag{24}$$

$$\tau_0\Omega_{11} + (\tau_M - \tau_0)\Omega_{22} + \Omega < 0 \tag{25}$$

$$\tau_0\Omega_{21} + (\tau_M - \tau_0)\Omega_{12} + \Omega < 0 \tag{26}$$

$$\tau_0\Omega_{21} + (\tau_M - \tau_0)\Omega_{22} + \Omega < 0 \tag{27}$$

The proof of Lemma 2 can be seen in Appendix B.

3. MAIN RESULTS

Theorem 1. For given scalars η_1, η_2 and matrices K, W, the system (18) is ESMSS if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, N, M, V, T and $S_i(i = 1, 2)$ of appropriate dimensions such that the following LMIs hold

$$\Xi(l) = \begin{bmatrix} \Xi_{11} + W & \Xi_{21}^{(l)} & \Xi_{31} \\ * & \Xi_{22} & 0 \\ * & * & \Xi_{33} \end{bmatrix} < 0 \tag{28}$$

$$l = 1, 2, 3, 4$$

where

$$\begin{aligned} \Xi_{11} &= \Gamma + \Gamma^T + diag(\, Q_1 + Q_2 \ \ 0 \ \ -Q_1 \ \ 0 \ \ -Q_2 \ \ \Sigma_1 \,) \\ \Gamma &= [\, \Sigma_2 \ \ -N + M \ \ -M + V \ \ -V + T \ \ -T \ \ \mathcal{A}^T S_2^T \,] \\ \Sigma_1 &= \eta_1 R_1 + (\eta_2 - \eta_1) R_2, \Sigma_2 = N + \mathcal{A}^T S_1^T + I_6 P \\ I_6^T &= [\, 0 \ 0 \ 0 \ 0 \ 0 \ I \,], \Xi_{21}^{(1)} = [\, \sqrt{\eta_1}N \ \ \sqrt{\eta_2 - \eta_1}V \,], \\ \Xi_{21}^{(2)} &= [\, \sqrt{\eta_1}N \ \ \sqrt{\eta_2 - \eta_1}T \,], \Xi_{21}^{(3)} = [\, \sqrt{\eta_1}M \ \ \sqrt{\eta_2 - \eta_1}V \,], \\ \Xi_{21}^{(4)} &= [\, \sqrt{\eta_1}M \ \ \sqrt{\eta_2 - \eta_1}T \,], \Xi_{22} = diag(\, -R_1 \ \ -R_2 \,), \\ \Xi_{31} &= [\, S_{12} \ \ \sqrt{\beta_0(1 - \beta_0)}R_3\mathcal{B}^T \,], \\ \Xi_{33} &= diag(\, -R_3 \ \ -R_3 \,), S_{12}^T = [\, S_1^T \ \ 0 \ 0 \ 0 \ 0 \ \ S_2^T \,] \end{aligned}$$

where the wireless network conditions and the corresponding quantizer parameters satisfy

$$(i_{k+1} - i_k)h + \tau_{i_{k+1}} \leq \eta_2 \tag{29}$$

$$\frac{4}{\lambda_1}\|S_{12}B\|\Delta < \mathscr{M} \tag{30}$$

Proof. Choose the Lyapunov-Krasovskii functional candidate as

$$V(x_t) = x^T(t)Px(t) + \int_{t-\eta_1}^{t} x^T(s)Q_1x(s)ds$$

$$+\int_{t-\eta_2}^{t-\eta_1} x^T(s)Q_2x(s)ds + \int_{t-\eta_1}^{t}\int_{s}^{t} \dot{x}^T(v)R_1\dot{x}(v)dvds$$
$$+\int_{t-\eta_2}^{t-\eta_1}\int_{s}^{t} \dot{x}^T(v)R_2\dot{x}(v)dvds \quad (31)$$

Using the infinitesimal operator (19) and employing the free matrix method He et al. [2004], we obtain

$$\begin{aligned}\mathcal{L}V(x_t) &= 2\dot{x}^T(t)Px(t) + x^T(t)\left(Q_1+Q_2\right)x(t)\\ &-x^T(t-\eta_1)Q_1x(t-\eta_1) - x^T(t-\eta_2)Q_2x(t-\eta_2)\\ &+\dot{x}^T(t)\left(\eta_1R_1+(\eta_2-\eta_1)R_2\right)\dot{x}(t)\\ &-\int_{t-\eta_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds - \int_{t-\eta_2}^{t-\eta_1}\dot{x}^T(s)R_2\dot{x}(s)ds\\ &+2\zeta^T(t)N\left[x(t)-x(t-\eta_1(t)) - \int_{t-\eta_1(t)}^{t}\dot{x}(s)ds\right]\\ &+2\zeta^T(t)M\left[x(t-\eta_1(t))-x(t-\eta_1) - \int_{t-\eta_1}^{t-\eta_1(t)}\dot{x}(s)ds\right]\\ &+2\zeta^T(t)V\left[x(t-\eta_1)-x(t-\eta_2(t)) - \int_{t-\eta_2(t)}^{t-\eta_1}\dot{x}(s)ds\right]\\ &+2\zeta^T(t)T\left[x(t-\eta_2(t))-x(t-\eta_2) - \int_{t-\eta_2}^{t-\eta_2(t)}\dot{x}(s)ds\right]\\ &+2\zeta^T(t)S_{12}\left[\mathcal{A}\zeta(t)+B\delta(t)+(\beta(t)-\beta_0)\mathcal{B}\zeta(t)\right] \quad (32)\end{aligned}$$

It can be shown from (32) that there exist scalars $R_i > 0 (i = 1, 2, 3)$ such that

$$-2\zeta^T(t)N\int_{t-\eta_1(t)}^{t}\dot{x}(s)ds \le \Pi_1 + \int_{t-\eta_1(t)}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \quad (33)$$
$$-2\zeta^T(t)M\int_{t-\eta_1}^{t-\eta_1(t)}\dot{x}(s)ds \le \Pi_2 + \int_{t-\eta_1}^{t-\eta_1(t)}\dot{x}^T(s)R_1\dot{x}(s)ds \quad (34)$$
$$-2\zeta^T(t)V\int_{t-\eta_2(t)}^{t-\eta_1}\dot{x}(s)ds \le \Pi_3 + \int_{t-\eta_2(t)}^{t-\eta_1}\dot{x}^T(s)R_2\dot{x}(s)ds \quad (35)$$
$$-2\zeta^T(t)T\int_{t-\eta_2}^{t-\eta_2(t)}\dot{x}(s)ds \le \Pi_4 + \int_{t-\eta_2}^{t-\eta_2(t)}\dot{x}^T(s)R_2\dot{x}(s)ds \quad (36)$$

$$\begin{aligned}&2\zeta^T(t)S_12\left(\beta(t)-\beta_0\right)\mathcal{B}\zeta(t)\\ &\le \zeta^T(t)\left(S_{12}R_3^{-1}S_{12}^T + \left(\beta(t)-\beta_0\right)^2\mathcal{B}^TR_3\mathcal{B}\right)\zeta(t) \quad (37)\end{aligned}$$

where $\Pi_1 = \eta_1(t)\zeta^T(t)NR_1^{-1}N^T\zeta(t)$, $\Pi_2 = (\eta_1 - \eta_1(t))\cdot \zeta^T(t)MR_1^{-1}M^T\zeta(t)$, $\Pi_3 = (\eta_2(t) - \eta_1)\zeta^T(t)VR_2^{-1}V^T\zeta(t)$, $\Pi_4 = (\eta_2 - \eta_2(t))\zeta^T(t)TR_2^{-1}T^T\zeta(t)$. Substituting (33)-(37) into (32) and taking expectation on it, we have

$$\begin{aligned}\mathcal{E}\left\{\mathcal{L}V(x_t)\right\} \le \mathcal{E}\Big\{\zeta^T(t)\Big[\Xi_{11} + S_{12}R_3^{-1}S_{12}^T\\ +\left(\beta(t)-\beta_0\right)^2\mathcal{B}^TR_3\mathcal{B}\Big]\zeta(t) + \Pi_1\\ +\Pi_2+\Pi_3+\Pi_4+2\zeta^T(t)S_{12}B\delta(t)\Big\} \quad (38)\end{aligned}$$

Using Lemma 2 and Schur complements, we can conclude from (28) that

$$\begin{aligned}&\Xi_{11} + S_{12}R_3^{-1}S_{12}^T + \beta_0\left(1-\beta_0\right)\mathcal{B}^TR_3\mathcal{B} + \eta_1(t)NR_1^{-1}N^T\\ &+\left(\eta_1-\eta_1(t)\right)MR_1^{-1}M^T + \left(\eta_2(t)-\eta_1\right)VR_2^{-1}V^T\\ &+\left(\eta_2-\eta_2(t)\right)TR_3^{-1}T^T < -W \quad (39)\end{aligned}$$

Combining (38) and (39), we have

$$\begin{aligned}\mathcal{E}\left\{\mathcal{L}V(x_t)\right\} \le -\lambda_1\mathcal{E}\left\{\|\zeta(t)\|^2 + 2\zeta^T(t)S_{12}B\delta(t)\right\}\\ -\lambda_2\mathcal{E}\left\{\|\zeta(t)\|^2\right\} \quad (40)\end{aligned}$$

where $\lambda_1 = \lambda_{\min}[W]$, $\lambda_2 = \lambda_{\min}[\Xi_{11}]$. Recalling (30), for arbitrary $x(t-\eta_1(t))$ and $x(t-\eta_2(t))$, we can always find positive μ_k such that

$$\frac{4\mu_k\Delta}{\lambda_1}\|S_{12}B\| \le \|x(t-\eta_1(t))\| + \|x(t-\eta_2(t))\| \le \mu_k\mathscr{M} \quad (41)$$

Furthermore, from the relationship of $x(t-\eta(t))$ with $x(t-\eta_1(t))$ and $x(t-\eta_2(t))$, we have

$$\begin{aligned}&\|x(t-\eta_1(t))\| + \|x(t-\eta_2(t))\|\\ &\ge \|\beta(t)x(t-\eta_1(t))\| + \|(1-\beta(t))x(t-\eta_2(t))\|\\ &\ge \|\beta(t)x(t-\eta_1(t)) + (1-\beta(t))x(t-\eta_2(t))\|\\ &= \|x(t-\eta(t))\| \quad (42)\end{aligned}$$

Combining (41) with (42) and recalling (6), we have

$$\|\delta(t)\| \le \mu_k\Delta \quad (43)$$

From (40) and (43) we can show that

$$\begin{aligned}&\mathcal{E}\left\{\mathcal{L}V(x_t)\right\} + \lambda_2\mathcal{E}\left\{\|\zeta(t)\|^2\right\}\\ &\le -\lambda_1\|\zeta(t)\|\mathcal{E}\left\{\|\zeta(t)\| - \frac{2\mu_k}{\lambda_1}\|S_{12}B\|\Delta\right\}\\ &\le -\lambda_1\|\zeta(t)\|\mathcal{E}\Big\{\frac{1}{2}\|x(t-\eta_1(t))\| + \frac{1}{2}\|x(t-\eta_2(t))\|\\ &-\frac{2\mu_k}{\lambda_1}\|S_{12}B\|\Delta\Big\} \le 0\end{aligned}$$

Define a new function as

$$W(x_t) = e^{\varepsilon t}V(x_t) \quad (44)$$

Its infinitesimal operator $\mathcal{L}$ is given by

$$\mathcal{L}W(x_t) = \varepsilon e^{\varepsilon t}V(x_t) + e^{\varepsilon t}\mathcal{L}V(x_t) \quad (45)$$

By the generalized *Itô* formula Mao [2002], we can obtain from (45) that

$$\begin{aligned}&\mathcal{E}\left\{W(x_t)\right\} - \mathcal{E}\left\{W(x_0)\right\}\\ &= \int_0^t \varepsilon e^{\varepsilon s}\mathcal{E}\left\{V(x_s)\right\}ds + \int_0^t e^{\varepsilon s}\mathcal{E}\left\{\mathcal{L}V(x_s)\right\}ds \quad (46)\end{aligned}$$

Then, using the similar method of Yue and Han [2005], we can see that there exists a positive number α such that for $t \geq 0$

$$\mathcal{E}\left\{V(x_t)\right\} \leq \alpha \sup_{-2\eta_2 \leq s \leq 0} \mathcal{E}\left\{\|\psi(s)\|^2\right\} e^{-\varepsilon t} \tag{47}$$

Since $V(x_t) \geq \{\lambda_{\min}(P)\} x^T(t)x(t)$, it can be shown from (47) that for $t \geq 0$

$$\mathcal{E}\left\{x^T(t)x(t)\right\} \leq \bar{\alpha} e^{-\varepsilon t} \sup_{-2\eta_2 \leq s \leq 0} \mathcal{E}\left\{\|\psi(s)\|^2\right\} \tag{48}$$

where $\bar{\alpha} = \frac{\alpha}{\lambda_{\min}(P)}$. This completes the proof.

Remark 4. Removing W from (28) and the condition (30) from Theorem 1, Theorem 1 reduces to the stability conditions for the wireless networked control system (18) without considering the effect of data quantization.

The design criterion for the feedback control gain K is derived based on Theorem 1.

Theorem 2. For given scalars η_1, η_2, ρ and matrix W, the system (18) is ESMSS if there exist matrices $\hat{P} > 0$, $\hat{Q}_1 > 0$, $\hat{Q}_2 > 0$, $\hat{R}_1 > 0$, $\hat{R}_2 > 0$, $\hat{R}_3 > 0$, $\hat{N}$, $\hat{M}$, $\hat{V}$, $\hat{T}$, X and Y of appropriate dimensions such that the following LMIs hold

$$\hat{\Xi}(l) = \begin{bmatrix} \hat{\Xi}_{11} & \hat{\Xi}_{21}^{(l)} & \hat{\Xi}_{31} & \mathcal{X} \\ * & \hat{\Xi}_{22} & 0 & 0 \\ * & * & \hat{\Xi}_{33} & 0 \\ * & * & * & -W^{-1} \end{bmatrix} < 0, l = 1,2,3,4 \tag{49}$$

where

$$\begin{aligned}
\hat{\Xi}_{11} &= \hat{\Gamma} + \hat{\Gamma}^T + diag\left(\hat{Q}_1 + \hat{Q}_2 \;\, 0 \;\, -\hat{Q}_1 \;\, 0 \;\, -\hat{Q}_2 \;\, \hat{\Sigma}_1\right),\\
\hat{\Gamma} &= \left[\hat{\Sigma}_2 \;\; -\hat{N} + \hat{M} \;\; -\hat{M} + \hat{V} \;\; -\hat{V} + \hat{T} \;\; -\hat{T} \;\; \rho\mathbb{A}^T\right],\\
\hat{\Sigma}_1 &= \eta_1 \hat{R}_1 + (\eta_2 - \eta_1)\hat{R}_2, \hat{\Sigma}_2 = \hat{N} + \mathbb{A}^T + I_6\hat{P}\\
\hat{\Xi}_{21}^{(1)} &= [\sqrt{\eta_1}\hat{N} \;\; \sqrt{\eta_2 - \eta_1}\hat{V}], \hat{\Xi}_{21}^{(2)} = [\sqrt{\eta_1}\hat{N} \;\; \sqrt{\eta_2 - \eta_1}\hat{T}],\\
\hat{\Xi}_{21}^{(3)} &= [\sqrt{\eta_1}\hat{M} \;\; \sqrt{\eta_2 - \eta_1}\hat{V}], \hat{\Xi}_{21}^{(4)} = [\sqrt{\eta_1}\hat{M} \;\; \sqrt{\eta_2 - \eta_1}\hat{T}],\\
\hat{\Xi}_{22} &= diag(-\hat{R}_1 \;\; -\hat{R}_2), \hat{\Xi}_{31} = [F \;\; \sqrt{\beta_0(1-\beta_0)}\mathbb{B}^T],\\
\hat{\Xi}_{33} &= diag(-\hat{R}_3 \;\; -\hat{R}_3), F^T = [\hat{R}_3 \;\, 0 \;\, 0 \;\, 0 \;\, 0 \;\, \rho\hat{R}_3],\\
\mathbb{A} &= [AX^T \;\; \beta_0 BY \;\; 0 \;\; (1-\beta_0)BY \;\; 0 \;\; -X^T],\\
\mathbb{B} &= [0 \;\; BY \;\; 0 \;\; -BY \;\; 0 \;\; 0], \mathcal{X} = diag(X, X, X, X, X, X)\\
\hat{S}_{12}^T &= \left[X^{-T} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \rho X^{-T}\right]
\end{aligned}$$

where the wireless network conditions and the corresponding quantizer parameters satisfy

$$(i_{k+1} - i_k)h + \tau_{i_{k+1}} \leq \eta_2 \tag{50}$$

$$\frac{4}{\lambda_1}\left\|\hat{S}_{12}B\right\|\Delta < \mathscr{M} \tag{51}$$

Proof. Defining $S_1 = X^{-1}$ and $S_2 = \rho X^{-1}$, where $\rho \neq 0$ and X is a nonsingular matrix, pre and post-multiplying (28) with $diag\left(X \; X \; X \; X \; X \; X \; X \; X \; R_3^{-1} \; I\right)$ and its transpose, and defining $\hat{N} = XNX^T, \hat{M} = XMX^T, \hat{V} = XVX^T, \hat{T} = XTX^T, \hat{P} = XPX^T, \hat{Q}_1 = XQ_1X^T, \hat{Q}_2 = XQ_2X^T, \hat{R}_1 = XR_1X^T, \hat{R}_2 = XR_2X^T, \hat{R}_3 = R_3^{-1}$ and $Y = KX^T$, (49) can be obtained from (28) by using Schur complements.

4. NUMERICAL EXAMPLES

Example 1. Consider the system (1) with parameter matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \tag{52}$$

Under the assumption that the effect of the data quantization can be neglected, the stability of system (52) with a networked controller $u(t) = [-3.75 \;\; -11.5]\,x(t)$ was investigated in Kim et al. [2003], Yue et al. [2004, 2005], Zhang et al. [2001]. The maximum allowable η_2 obtained in the existing references that guarantees the stability of system (52) is given in Table 1.

Table 1. Maximum allowable η_2 obtained in the existing references

	η_2
Zhang et al. [2001]	4.5×10^{-4}
Kim et al. [2003]	0.7805
Yue et al. [2004]	0.8695
Yue et al. [2005]	0.8871

When the transmission delay is random and its probability distribution is known a prior, using Theorem 1 and Remark 4, we can obtain Table 2.

Table 2. Maximum allowable η_2 for different $\eta_1$and β_0

η_1	0.1	0.3	0.5	0.7
$\beta_0 = 0.7$	1.40	1.30	1.17	1.02
$\beta_0 = 0.9$	3.60	2.95	2.24	1.46

From Table 1 and Table 2, it can be found that when the information of probability distribution of the delay is known, the maximum allowable value of η_2 and the corresponding feedback gain as well as the quantization parameters can be greatly improved.

Given values $W = I$ and $\rho = 0.8$ as well as a probability distribution $\beta_0 = 0.9$, the maximum allowable η_2 can be solved by using Theorem 2 when considering the effect of both wireless network conditions and data quantization, which are shown in Table 3.

The maximum allowable continuous packet losses can also be computed when the maximum η_2 is obtained. For example, for $\eta_1 = 15$, we derive that $\eta_2 = 89$, that is $(i_{k+1} - i_k)h + \tau_{i_{k+1}} \leq 89$, if we set $h = 1$ and $\{\tau_{i_k} \leq 10 \,|k = 1, 2, \cdots\}$, we have $i_{k+1} - i_k \leq 79$, that means system (52) still remains stable when 78 continuous packets are lost.

5. CONCLUSION

This paper has investigated the problem of state feedback controller design for wireless networked control system (WiNCS) with data quantization. Considering the effect of both wireless network conditions and data quantization, a new type of system model with stochastic parameter

Table 3. Maximum allowable η_2 for $\beta_0 = 0.9$

η_1	0.1	0.5	1.0	15	50
Maximum allowable η_2	124	116	105	89	64
Controller gain K	$\begin{bmatrix} -0.0008 \\ -0.0089 \end{bmatrix}^T$	$\begin{bmatrix} -0.0009 \\ -0.0105 \end{bmatrix}^T$	$\begin{bmatrix} -0.0011 \\ -0.0125 \end{bmatrix}^T$	$\begin{bmatrix} -0.0012 \\ -0.0129 \end{bmatrix}^T$	$\begin{bmatrix} -0.0012 \\ -0.0143 \end{bmatrix}^T$
Quantizationparameters	$M \geq 11.2048\Delta$	$M \geq 8.9293\Delta$	$M \geq 8.1262\Delta$	$M \geq 6.1178\Delta$	$M \geq 5.8363\Delta$

matrices has been proposed. In terms of linear matrix inequality technique, delay-distributed conditions have been obtained to guarantee the exponentially mean-square stable of the WiNCS. A numerical example has been given to show the applicability of the proposed method.

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Appendix A. PROOF OF LEMMA 1

Proof. 1. *Sufficiency*

For $\eta(t) = 0$ and η_M in (20), we can obtain (21) and (22).

2. *Necessity*

Define a function as

$$f(\eta(t)) = \eta(t)\Xi_1 + (\eta_M - \eta(t))\,\Xi_2 + \Omega \qquad \text{(A.1)}$$

which can be further rewritten as

$$f\,(\eta(t)) = \frac{\eta(t)}{\eta_M}\,(\eta_M\Xi_1 + \Omega) + \frac{\eta_M - \eta(t)}{\eta_M}\,(\eta_M\Xi_2 + \Omega) \text{(A.2)}$$

From (21) and (22) we can conclude that $f\,(\eta(t)) < 0$ for all $\eta(t) \in [\,0,\ \eta_M\,]$, that is

$$\eta(t)\Xi_1 + (\eta_M - \eta(t))\,\Xi_2 + \Omega < 0$$

This completes the proof.

Appendix B. PROOF OF LEMMA 2

Proof. Set

$$f\,(\eta_2(t)) = [(\eta_2(t) - \eta_0)\,\Xi_{12} + (\eta_M - \eta_2(t))\,\Xi_{22}] + \Omega \text{ (B.1)}$$

then (23) can be rewritten as

$$[\eta_1(t)\Xi_{11} + (\eta_0 - \eta_1(t))\,\Xi_{21}] + f\,(\eta_2(t)) < 0 \qquad \text{(B.2)}$$

Using Lemma 1, (B.2) is equivalent to

$$\eta_0\Xi_{11} + f\,(\eta_2(t)) < 0 \qquad \text{(B.3)}$$

$$\eta_0\Xi_{21} + f\,(\eta_2(t)) < 0 \qquad \text{(B.4)}$$

Then (B.3) can be rewritten as

$$[(\eta_2(t) - \eta_0)\,\Xi_{12} + (\eta_M - \eta_2(t))\,\Xi_{22}] + \eta_0\Xi_{11} + \Omega < 0 \text{(B.5)}$$

Denote $\hat{\Omega} = \eta_0\Xi_{11} + \Omega$ and use lemma 1 again, we can show that (23) holds if and only if (24) and (25) hold. Similarly, (B.4) holds if and only if (26) and (27) hold. This completes the proof.