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Optimal Precision of Quantum System Parameter Estimation Subject to Instrumentation Constraints

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Abstract:
Maximizing the precision in estimating parameters in a quantum system subject to instrumentation constraints is cast as a convex optimization problem.

1. INTRODUCTION

Making measurements based on quantum mechanics is generally referred to as *quantum metrology*. Some of the current and envisioned applications include: measuring physical properties, (*e.g.*, electrical, thermal, photonic), atomic clocks, imaging, communications, quantum computation, biometrics, magentometry, and gravimetry. Quite often the information of interest is contained in a *single* parameter which cannot be measured directly, *e.g.*, the phase difference between the arms of an optical interferometer, or the transition frequency of an atomic clock. For single parameter estimation the limit of theoretical accuracy has been examined in depth, *e.g.*, Holevo (1982), Braunstein and Caves (1994), Sarovar and Milburn (2006), Giovannetti et al. (2006), Boixo et al. (2007). These studies reveal that special preparation of the instrumentation – the probe – can achieve an asymptotic variance smaller than the Cramér-Rao lower bound, the so-called Quantum Cramér-Rao bound, or the Quantum Fisher Information. In addition, the unique quantum property of *entanglement* can increase the parameter estimation convergence rate for N repeated experiments from the shot-noise limit of $1/\sqrt{N}$ to the Heisenberg limit $1/N$, which arises from the uncertainty principle (see, *e.g.*, Brif and Mann (1996)).

It is reasonable to expect that the theoretical Quantum Fisher Information bound will not be obtained with imperfect and limited instrumentation resources, *i.e.*, not all states can be prepared and not all measurement schemes are possible. Under these conditions what exactly is the best that can be done?

In this paper we present an approach which maximizes the parameter estimation accuracy in the presence of limits on instrumentation, specifically, input states and measurement schemes. The method is based on the convex optimization approach to optimal experiment design as developed in Boyd and Vandenberghe (2004) and as applied to quantum system identification in Kosut et al. (2004). We develop a worst-case and average case objective for optimizing the precision. Focusing on the single parameter case, we show that the optimization problems are *linear programs*. For the average case the solution to the linear program can be expressed analytically and involves a simple search, *i.e.*, finding the largest element in a list. This means that an enormous number of combinations of state and sensor configurations can be examined. In principal, with enough computer resources, the multi-parameter case can also be addressed. We leave that for a future effort.

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Limited space does not permit any in-depth discussion of the origin of the quantum mechanical expressions. As this work is closely related to quantum information processes we recommend M.A. Nielsen and I.L. Chuang (2000) for further exposition.

The paper is organized as follows: Section §2 describes the underlying method and introduces the worst-case and average-case criteria. Section §3 applies the method to the single parameter estimation problem which is further specialized in Section §4 to quantum parameter estimation. The quantum Fisher bound is described in Section §5 and an example is presented in Section §6. Some concluding remarks are in Section §8 with a few suggestions for further studies.

2. OPTIMAL EXPERIMENT DESIGN

What is presented in this section is completely classical; no quantum mechanics is required. Following this we will show how the ideas apply to quantum metrology.

Consider a system dependent on an *unknown* $n_\theta \times 1$ vector of real parameters θ which is known *a priori* to be in a set Θ. The parameter θ is to be estimated from a set of measured data from repeated *independent, identical* experiments. In each experiment the system can be put in any one of $N_{\rm config}$ *configurations* x selected from the set of configurations,

$$X = \{\, x_k \mid k = 1, \ldots, N_{\rm config} \,\} \tag{1}$$

The outcome from each experiment in configuration x_k is one of $N_{\rm out}$ possibilities with outcome probability denoted by,

$$p_i(x_k, \theta),\ i = 1, \ldots, N_{\rm out} \tag{2}$$

Let $n_i(x_k, \theta)$ denote the number of times outcome i is obtained from $N(x_k)$ identical experiments in configuration x_k. Thus,

$$\mathbf{E}\, n_i(x_k, \theta) = N(x_k) p_i(x_k, \theta),\ \sum_{i=1}^{N_{\rm out}} n_i(x_k, \theta) = N(x_k) \tag{3}$$

where $\mathbf{E}$ is the expected value operator with respect to the probability distribution (2). Let N denote the total number of experiments and $\lambda(x_k)$ the *distribution of experiments* in configuration x_k. Thus,

$$\lambda(x_k) = N(x_k)/N \ \Rightarrow\ \sum_{k=1}^{N_{\rm config}} \lambda(x_k) = 1 \tag{4}$$

The problem is to select the distribution of experiments per configuration, $\lambda(x_k), k = 1, \ldots, N_{\rm config}$, or equivalently the

number of experiments per configuration, $N(x_k)$, so as to obtain an estimate of $\theta \in \Theta$ with the best accuracy from N experiments. The "best" attainable estimation accuracy is defined here as the smallest possible Cramér-Rao bound on the estimation variance Cramér (1946).

Specifically, if $\widehat{\theta}_N$ is an unbiased estimate of θ, then the estimation error variance satisfies,

$$\mathbf{E}\,\|\widehat{\theta}_N - \theta\|^2 \geq \frac{1}{N}\mathrm{Tr}\; F(\lambda,\theta)^{-1}$$

$$F(\lambda,\theta) = \sum_{k=1}^{N_{\mathrm{config}}} \lambda(x_k) G(x_k,\theta) \qquad (5)$$

$$G(x,\theta) = \sum_{i=1}^{N_{\mathrm{out}}} \frac{\left(\nabla_\theta\, p_i(x,\theta)\right)\left(\nabla_\theta\, p_i(x,\theta)\right)^T}{p_i(x,\theta)}$$

$F(\lambda,\theta)$ is the To achieve the best accuracy we would like to make the Cramér-Rao lower bound as small as possible, or more generally, maximize a measure of the size of the *Fisher information matrix*, $F(\lambda,\theta)$, for all possible $\theta \in \Theta$. One way to obtain such an estimate is to solve the *worst-case experiment design problem*:

$$\begin{array}{ll}\text{minimize} & V(\lambda) = \max\limits_{\theta\in\Theta}\ \mathrm{Tr}\; F(\lambda,\theta)^{-1} \\ \text{subject to} & \sum_{k=1}^{N_{\mathrm{config}}} \lambda(x_k) = 1 \\ & \lambda(x_k) \text{ is an integrer multiple of } 1/N\end{array} \qquad (6)$$

Although $F(\lambda,\theta)$ is convex in λ it is not, in general, convex in θ. In addition, the integer constraint on λ is not a convex set. Utilizing the optimal experiment design method presented in (Boyd and Vandenberghe, 2004§7.5), we can *relax* the integer constraints to the convex constraints $\lambda(x_k) \geq 0$. In addition, suppose we take a finite number of samples from the set Θ, say,

$$\{\, \theta_r \mid r = 1,\ldots,N_\theta \,\} \qquad (7)$$

Then the non-convex integer optimization (6) is approximated by,

$$\begin{array}{ll}\text{minimize} & V(\lambda) = \max\limits_{r=1,\ldots,N_\theta}\ \mathrm{Tr}\; F(\lambda,\theta_r)^{-1} \\ \text{subject to} & \sum_{k=1}^{N_{\mathrm{config}}} \lambda(x_k) = 1,\ \lambda(x_k) \geq 0\end{array} \qquad (8)$$

This is a convex optimization problem in λ. The solution to this (relaxed and approximated) problem provides upper and lower bounds to the unknown solution of (8) with the integer constraint on the $\lambda(x_k)$ from (6). Specifically, let λ^{opt} denote a solution to (8) with the integer constraint, a solution which is not known. Let λ^{rlx} be a solution to the convex optimization (8). From the latter we can determine a nearby solution which satisfies the integer constraint, *e.g.*,

$$\widehat{\lambda}(x_k) = \mathbf{round}\left(N\lambda^{\mathrm{rlx}}(x_k)\right)/N \qquad (9)$$

Then,

$$V(\lambda^{\mathrm{rlx}}) \leq V(\lambda^{\mathrm{opt}}) \leq V(\widehat{\lambda}) \qquad (10)$$

Thus $N(x_k) = N\widehat{\lambda}(x_k)$ is the number of experiments to repeat in configuration x_k. This method was applied to the experiment design for quantum system identification in Kosut et al. (2004).

A variation on this problem is the *average-case experiment design problem*:

$$\begin{array}{ll}\text{minimize} & V(\lambda) = \mathrm{Tr}\; F_{\mathrm{avg}}(\lambda)^{-1} \\ \text{subject to} & F_{\mathrm{avg}}(\lambda) = \sum_{r=1}^{N_\theta} p(\theta_r) F(\lambda,\theta_r) \\ & \sum_{k=1}^{N_{\mathrm{config}}} \lambda(x_k) = 1,\ \lambda(x_k) \geq 0\end{array} \qquad (11)$$

where $p(\theta_r)$ is the probability that θ is θ_r. For example, if all that is known is that $\theta \in \Theta$, then it suffices to assume that θ is uniformly distributed in Θ, and hence, $p(\theta_r) = 1/N_\theta$.

3. SINGLE PARAMETER ESTIMATION

If θ is a scalar, then problems (8) and (11) become, respectively,

$$\begin{array}{ll}\text{minimize} & V(\lambda) = \max\limits_{r=1,\ldots,N_\theta} 1/F(\lambda,\theta_r) \\ \text{subject to} & F(\lambda,\theta_r) = \lambda^T g(\theta_r),\ r = 1,\ldots,N_\theta \\ & \lambda \geq 0,\ \mathbf{1}^T\lambda = 1\end{array} \qquad (12)$$

and

$$\begin{array}{ll}\text{minimize} & V(\lambda) = 1/F_{\mathrm{avg}}(\lambda) \\ \text{subject to} & F_{\mathrm{avg}}(\lambda) = \lambda^T g_{\mathrm{avg}} \\ & \lambda \geq 0,\ \mathbf{1}^T\lambda = 1\end{array} \qquad (13)$$

with the k-th element of the $N_{\mathrm{config}} \times 1$ vectors $g(\theta)$ and g_{avg} are given by,

$$\begin{array}{rcl}(g(\theta))_k &=& g(x_k,\theta) \;=\; \sum_{i=1}^{N_{\mathrm{out}}} \left(\partial p_i(x_k,\theta)/\partial\theta\right)^2 / p_i(x_k,\theta) \\ (g_{\mathrm{avg}})_k &=& g_{\mathrm{avg}}(x_k) \;=\; \sum_{r=1}^{N_\theta} p(\theta_r) g(x_k,\theta_r)\end{array}$$

Problem (12) can be expressed more compactly as,

$$\begin{array}{ll}\text{maximize} & \min\limits_{r=1,\ldots,N_\theta} \lambda^T g(\theta_r) \\ \text{subject to} & \lambda \geq 0,\ \mathbf{1}^T\lambda = 1\end{array} \qquad (14)$$

Similarly, problem (13) is equivalent to,

$$\begin{array}{ll}\text{maximize} & \lambda^T g_{\mathrm{avg}} \\ \text{subject to} & \lambda \geq 0,\ \mathbf{1}^T\lambda = 1\end{array} \qquad (15)$$

These latter formulations reveal that these optimization problems (12)-(15) are *linear programs* (LP) in λ which can be solved efficiently for a very large number of configurations N_{config}. In particular, it is easy to see that a solution to (13), or equivalently (15), is given explicitly by,

$$\begin{array}{rcl}\widehat{\lambda}_k &=& \begin{cases} 1 & k = \arg\max\limits_{k'=1,\ldots,N_{\mathrm{config}}} g_{\mathrm{avg}}(x_{k'}) \\ 0 & \text{otherwise}\end{cases} \\ & \Downarrow & \\ \widehat{x} &=& \arg\max\limits_{x\in X} g_{\mathrm{avg}}(x) \\ V(\widehat{\lambda}) &=& 1/g_{\mathrm{avg}}(\widehat{x})\end{array} \qquad (16)$$

It is possible that there is more than one optimal distribution because $\max_k g_{\mathrm{avg}}(x_k)$ may not be unique. It is more likely that there are other choices which give similar results. Nonetheless, an advantage of the average-case solution over the worst-case solution is that only a *single* configuration is required. As we will see in the example to follow, the two solutions can be quite different.

4. QUANTUM SYSTEM PARAMETER ESTIMATION

Consider the quantum system in Figure 1.

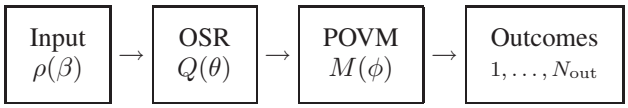

Fig. 1. Quantum system for estimating parameter θ.

- **Input** The $n \times n$ state $\rho(\beta)$ is dependent on the input configuration parameter $\beta \in \mathbf{R}^{n_\beta}$, for which the available values are,
$$\{\, \beta_\ell \mid \ell = 1, \ldots, N_{\rm input} \,\} \tag{17}$$
- **OSR** The quantum system dynamics depends on the unknown parameter $\theta \in \mathbf{R}$ and is described in terms of what is known as the Kraus Operator Sum Representation (OSR) (M.A. Nielsen and I.L. Chuang, 2000§8.2).
$$Q(\theta)\{\rho(\beta)\} = \sum_k Q_k(\theta)\rho(\beta)Q_k(\theta)^\dagger \tag{18}$$
The $n \times n$ matrices Q_k, called operation elements, satisfy $\sum_k Q_k^\dagger Q_k = I_n$. A special case is when there is one OSR element, *i.e.*, the unitary channel, $U(\theta)$, where $Q(\theta)\{\rho(\beta)\} = U(\theta)\rho(\beta)U(\theta)^\dagger$ and $U(\theta)^\dagger U(\theta) = I_n$. A further special case is when,
$$U(\theta) = \exp(-i\theta H_0) \tag{19}$$
where H_0 is a known $n \times n$ Hamiltonian.
- **POVM** (Positive Operator Value Measure) provides for a generalization of the quantum measurement postulate (M.A. Nielsen and I.L. Chuang, 2000§2.2). Any measurement scheme can be represented by a POVM with elements satisfying the *completeness relation*,
$$M_i(\phi) \geq 0, \ \sum_{i=1}^{N_{\rm out}} M_i(\phi) = I_n \tag{20}$$
Here the elements, $M_i\phi)$ are $n \times n$ matrices which depend on the POVM configuration parameter $\phi \in \mathbf{R}^{n_\phi}$ for which the available values are,
$$\{\, \phi_k \mid k = 1, \ldots, N_{\rm povm} \,\} \tag{21}$$
- **Outcomes** The probability of recording the i-th outcome with $i \in \{1, \ldots, N_{\rm out}\}$ is given by,
$$\begin{aligned} p_i(\phi, \beta, \theta) &= \mathrm{Tr}\, M_i(\phi)\sigma(\theta, \beta) \\ \sigma(\theta, \beta) &= \textstyle\sum_k Q_k(\theta)\rho(\beta)Q_k(\theta)^\dagger \end{aligned} \tag{22}$$
The state $\sigma(\theta, \beta)$ is the output of the quantum channel $Q(\theta)$ and the input to the POVM.

We ought to mention that the form of the system of Figure 1 is not the most general. For example, it may be that the "OSR" block depends on ϕ as well as θ, and that the POVM is not at all adjustable. The method, however, remains the same. Hence, under the stated conditions, the worst-case experiment design problem (14) becomes,
$$\begin{aligned} &\text{maximize} \ \min_{r=1,\ldots,N_\theta} \textstyle\sum_{k=1}^{N_{\rm povm}} \sum_{\ell=1}^{N_{\rm input}} \lambda_{k\ell}\, g(\phi_k, \beta_\ell, \theta_r) \\ &\text{subject to} \ \lambda_{k\ell} \geq 0, \ \textstyle\sum_{k=1}^{N_{\rm povm}} \sum_{\ell=1}^{N_{\rm input}} \lambda_{k\ell} = 1 \end{aligned} \tag{23}$$
with
$$g(\phi, \beta, \theta) = \textstyle\sum_{i=1}^{N_{\rm out}} \left(\partial p_i(\phi, \beta, \theta)/\partial\theta\right)^2 / p_i(\phi, \beta, \theta) \tag{24}$$
Similarly, the average-case experiment design problem (15) becomes,
$$\begin{aligned} &\text{maximize} \ \textstyle\sum_{k=1}^{N_{\rm povm}} \sum_{\ell=1}^{N_{\rm input}} \lambda_{k\ell}\, g_{\rm avg}(\phi_k, \beta_\ell) \\ &\text{subject to} \ \lambda_{k\ell} \geq 0, \ \textstyle\sum_{k=1}^{N_{\rm povm}} \sum_{\ell=1}^{N_{\rm input}} \lambda_{k\ell} = 1 \end{aligned} \tag{25}$$
with
$$g_{\rm avg}(\phi_k, \beta_\ell) = \sum_{r=1}^{N_\theta} p(\theta_r) g(\phi_k, \beta_\ell, \theta_r) \tag{26}$$
Following (16), the solution to (25) is,
$$\begin{aligned} \widehat{\lambda}_{k\ell} &= \begin{cases} 1 \ k, \ell = \arg\max\limits_{k', \ell'} g_{\rm avg}(\phi_{k'}, \beta_{\ell'}) \\ 0 \ \text{otherwise} \end{cases} \\ V(\widehat{\lambda}) &= 1/\max_{k,\ell} g_{\rm avg}(\phi_k, \beta_\ell) \end{aligned} \tag{27}$$
Solutions to (23) and (27) give, respectively, the worst-case and average-case levels of Fisher information as a function of the uncertain parameter θ:
$$F_{\rm wc}(\theta) = \textstyle\sum_{k=1}^{N_{\rm povm}} \sum_{\ell=1}^{N_{\rm input}} \ \lambda_{k\ell}^{\rm wc}\, g(\phi_k, \beta_\ell, \theta) \tag{28}$$
$$F_{\rm ac}(\theta) = \lambda_{k\ell}^{\rm ac}\, g(\phi_k, \beta_\ell, \theta) \tag{29}$$
In addition, as a benchmark we can also compute the maximum possible subject to the constraints on the input and measurement scheme,
$$F_{\max}(\theta) = \max_{\phi_k, \beta_\ell} g(\phi_k, \beta_\ell, \theta) \tag{30}$$

5. QUANTUM FISHER UPPER BOUND

The *quantum Fisher bound* represents the maximum possible with no measurement constraints, *i.e.*, the POVMs do not depend upon a configuration parameter as in (20). For the system of Figure 1 with a single parameter to be estimated, this maximum achievable Fisher information is given by solving,
$$\begin{aligned} &\text{maximize} \ F(\theta, \beta) = \textstyle\sum_{i=1}^{N_{\rm out}} \left(\partial p_i/\partial\theta\right)^2 / p_i \\ &\text{subject to} \ p_i = \mathrm{Tr} M_i \sigma(\theta, \beta) \\ &\qquad M_i \geq 0, \ \textstyle\sum_{i=1}^{N_{\rm out}} M_i = I_n \end{aligned} \tag{31}$$
with $\sigma(\theta, \beta)$ from (22). The optimization variables are the POVM matrices M_i. For the single parameter case, a solution method has been developed, see, *e.g.*, Holevo (1982), Braunstein and Caves (1994). Specifically,
$$\begin{aligned} &F_{\rm Qmax}(\theta, \beta) = \mathrm{Tr}\ S(\theta, \beta)^2 \sigma(\theta, \beta) \\ &S(\theta, \beta)\sigma(\theta, \beta) + \sigma(\theta, \beta)S(\theta, \beta) = 2\frac{\partial\sigma(\theta, \beta)}{\partial\theta} \end{aligned} \tag{32}$$
with $S(\theta, \beta)$ the solution to the above Lyapunov equation. The quantum Fisher bound, $F_{\rm Qmax}(\theta, \beta)$ generally depends on the unknown parameter value θ, and in this case also on the input configuration parameter β. As developed in the previous references, for arbitrary pure state inputs $\rho = |\psi\rangle\langle\psi|$, and with a unitary channel given by (19),
$$F_{\rm Qmax}(\theta) = \left(\lambda_{\max}(H_0) - \lambda_{\min}(H_0)\right)^2 \tag{33}$$
with $\lambda_{\max}, \ \lambda_{\min}$ here denoting the maximum and minimum eigenvalues of the Hamiltonian H_0. Although the Fisher bound does not depend on θ, the optimizing pure state input does.

6. EXAMPLE: PERTURBED UNITARY CHANNEL

To illustrate the optimization methods we assume the quantum channel in Figure 1 is a unitary channel whose output is corrupted by *amplitude damping*, an effect attributed to energy

dissipation, *e.g.*, spontaneous photon emission. The unitary is given by (19), specifically,

$$U(\theta) = \exp(-i\theta H_0),\ H_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{34}$$

with the unknown parameter θ in the range,

$$0.2 \leq \theta/(\pi/2) \leq 0.8 \tag{35}$$

The amplitude damping channel can be described by an OSR with two elements (see, *e.g.*, M.A. Nielsen and I.L. Chuang (2000)),

$$A_1(\gamma) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix},\ A_2(\gamma) = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix},$$

with γ the probability of dissipation, *e.g.*, photon loss. It follows that the OSR of $Q(\theta)$ in Figure 1 has two elements, $Q_k(\theta) = A_k(\gamma)U(\theta),\ k = 1, 2$.

The available input for the experiment is the 2×1 pure state $|\psi(\beta)\rangle$ which can be adjusted via an angle β as follows:

$$|\psi(\beta)\rangle = \cos\beta|0\rangle + \sin\beta|1\rangle,\ 0 \leq \beta \leq \pi \tag{36}$$

with the standard basis,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The POVMs can be adjusted via an angle ϕ,

$$\left.\begin{array}{rcl} M_1(\phi) & = & |z(\phi)\rangle\langle z(\phi)| \\ M_2(\phi) & = & I_2 - M_1(\phi) \\ |z(\phi)\rangle & = & \cos\phi|0\rangle + \sin\phi|1\rangle \end{array}\right\}\ 0 \leq \phi \leq \pi \tag{37}$$

We determine the Fisher information for three amplitude damping probabilities: $\gamma = 0,\ 0.05,\ 0.25$ with $N_\theta = 100$ uniformly spaced samples of θ from (35). The POVM and input configuration angles β, ϕ are selected from the ranges in (36) and (37) with $N_{\text{input}} = 10$ and $N_{\text{povm}} = 10$ uniformly spaced samples for each of the following three configuration constraints:

(1) POVM configured ($0 \leq \phi \leq \pi$), input fixed ($\beta = 0$)
(2) POVM fixed ($\phi = 0$), input configured ($0 \leq \beta \leq \pi$)
(3) POVM & input configured ($0 \leq \phi, \beta \leq \pi$)

Figures 2-4 show the Fisher information as a function of the parameter θ for the three values of amplitude damping and the three configuration constraints. In each figure the solid line is the maximum achievable for each value of θ that maximizes the Fisher information under the configuration constraints from (30). The dashed-line is what is achieved by using the worst-case distribution of experiments from (28), and the dotted line is the average-case distribution of experiments from (29). The dot-dash line is the quantum Fisher upper bound for each θ from (32).

In all cases, the Fisher information with the average-case distribution, $F_{\text{ac}}(\theta)$ from (29) perfectly matches the maximum possible, $F_{\max}(\theta)$ from (30), over a portion of the θ range. The constrained information is of course always lower than the quantum Fisher bound. When both POVM and input are jointly configured the constrained information begins to approach the ideal upper bound. The curves for the case where only the POVM is configured are generally below those where only the input is configured. This clearly shows that the constraints can have a significant impact on accuracy.

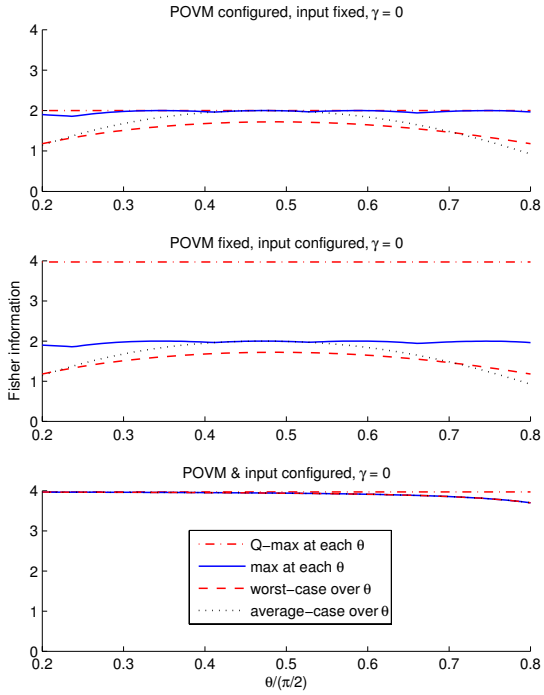

Fig. 2. Comparison of configuration constraints with $\gamma = 0$

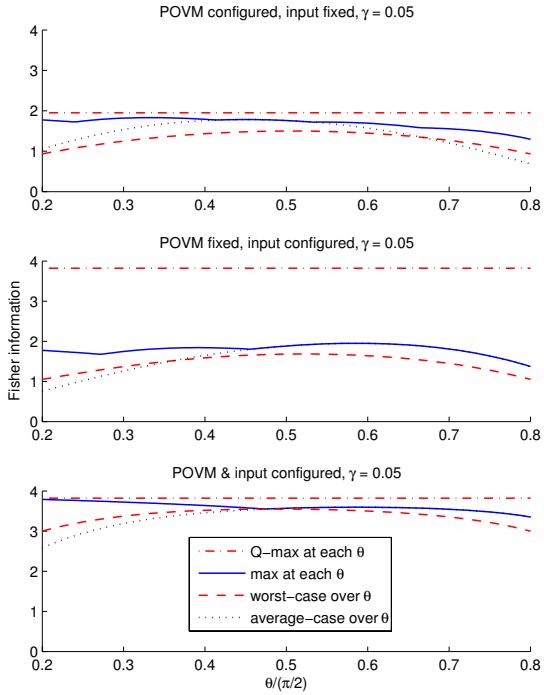

Fig. 3. Comparison of configuration constraints with $\gamma = 0.05$

Input configuration also an effect on the quantum Fisher bound. The solid lines in Figure 5 show the bound for each value of γ vs. the input configuration angle β for a large number of samples in the range. It turns out that the bound is independent of θ – the solid lines reflect that value. (Why it is constant in this example, or if this is a more general feature, is not understood at this time.) The triangles show the $N_{\text{input}} = 10$ available values. The solid lines indicate that multiple inputs can achieve the bound whereas the restricted set forces a unique maximizer

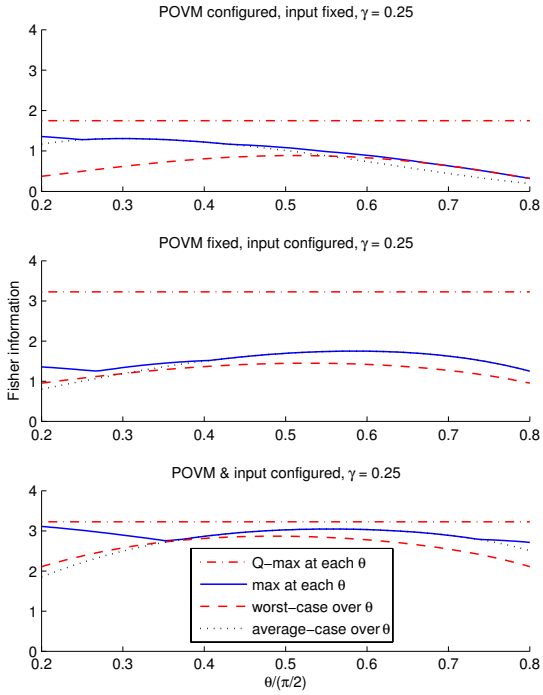

Fig. 4. Comparison of configuration constraints with $\gamma = 0.25$

which does not necessarily occur at the true maximum. For example, as seen in the top plot for $\gamma = 0$, the constrained maximum is near the global value, but not quite. The global value for $\gamma = 0$, computed from (33) with H_0 from (34) is $F_{\mathrm{Qmax}}(\theta) = 4$. This value is achieved only in the case with $\gamma = 0$ and clearly over bounds all the other plots where $\gamma > 0$. As might be expected, a perturbation of the unitary channel, in this case via amplitude damping, makes it harder to attain the quantum Fisher bound. Observe also that if the inputs were further constrained, say $\beta/\pi \in \{0, 0.2, 0.5, 0.8\}$, then the achieved quantum Fisher bound would not be nearly as close to the maximum possible. Figure 5 provides the designer with information about the limit of performance and if the potential performance increase is significant, then new instrumentation might be considered.

Another way to see the results in Figures 2-4 is presented in Table 1 which gives the minimum number of experiments to achieve an estimation accuracy of 0.01 as predicted by the Cramér-Rao bound (5). For the three test values of amplitude damping and the three configurations, the table shows, $N_{\min} = 1/\left((0.01^2 \min_\theta F(\theta)\right)$ for $F(\theta)$ from the quantum Fisher bound (32), the maximum subject to the constraints (30), the worst-case (28), and the average-case (29). As the Cramér-Rao bound is asymptotic in the number of experiments, there is no guaranty that these are the number of experiments *actually* required to saturate the bound. The table thus only gives a representation of the effort required in each case.

The numerically non-zero elements of the worst-case and average-case optimal distributions for all the cases are shown in Table 2. By construction, only one input configuration is required for the average-case distribution (27). The worst-case distribution requires up to 3 configurations when $\gamma > 0$. In this example the configuration angles remain relatively unchanged exhibiting some robustness to the amplitude damping probability γ. The worst-case distributions change more significantly.

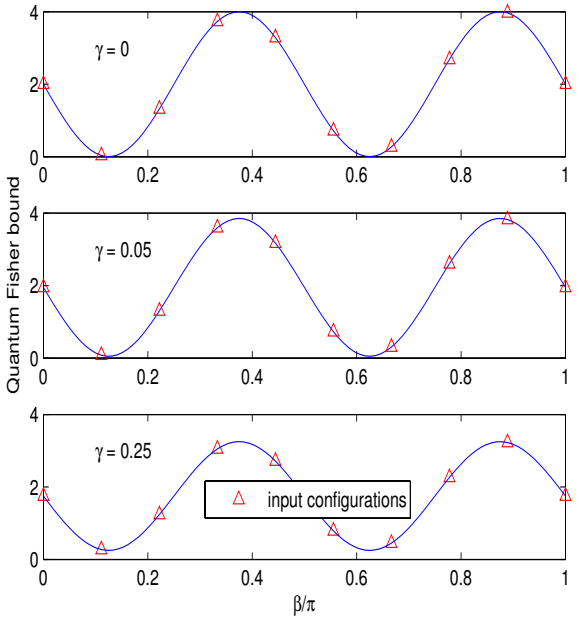

Fig. 5. Quantum Fisher bound vs. all possible input configurations β for each γ. △ are those available.

γ	Configured	Qmax	Max	Worst-Case	Average-Case
0	POVM	5000	5380	8480	10,822
	input	2519	5380	8485	10,824
	both	2519	2703	2703	2703
0.05	POVM	5128	7719	10,725	14,569
	input	2617	7294	9506	13,225
	both	2617	2983	3335	3873
0.25	POVM	5714	31,097	31,097	51,625
	input	3099	7991	10,508	12,486
	both	3099	3687	4743	5401

Table 1. Minimum number of experiments to achieve 0.01 deviation in estimation accuracy

γ	Configured	Average-Case			Worst-Case		
		$\frac{\phi}{\pi/2}$	$\frac{\beta}{\pi/2}$	λ_{ac}	$\frac{\phi}{\pi/2}$	$\frac{\beta}{\pi/2}$	λ_{wc}
0	POVM	.89	0	1	.44	0	.57
	input	0	.89	1	0	.44	.57
					0	.78	.43
	both	.89	.89	1	.89	.89	.89
0.05	POVM	.89	0	1	.44	0	.43
					.78	0	.57
	input	0	.33	1	0	0	.15
					0	.33	.70
					0	1	.15
	both	.89	.33	1	.89	.33	.65
					.89	.89	.35
0.25	POVM	.44	0	1	.78	0	1
	input	0	.33	1	0	0	.14
					0	.33	.72
					0	1	.14
	both	.89	.33	1	.89	.33	.80
					.89	.89	.20

Table 2. Optimal distributions

7. COMPUTING THE ESTIMATE

Figures 2-4 and/or Tables 1-2 are clearly useful in assessing the effort required to obtain an estimate of the parameter. In

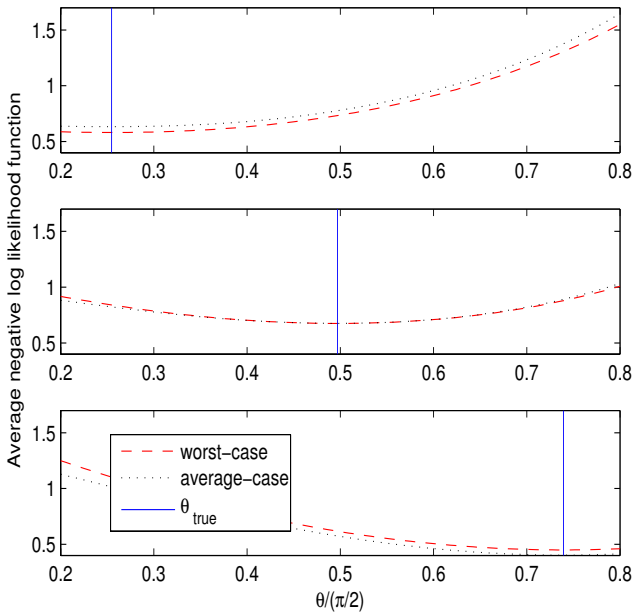

Fig. 6. Average negative log-likelihood function for $\gamma = 0.25$

addition, it is also necessary to know if the search space is easy or hard, *e.g.*, convex or not. Using the notation from Section §2, consider the Maximum Likelihood (ML) estimate. That is, solve for $\theta \in \Theta$ which minimizes the negative log-likelihood function,

$$L(\theta) = -\sum_{i=1}^{N_{\text{out}}} \sum_{k=1}^{N_{\text{config}}} n_i(x_k) \log p_i(x_k, \theta) \qquad (38)$$

For analysis purposes consider the average log-likelihood function, $L_{\text{avg}}(\theta) = (1/N) < L(\theta) >$, which when evaluated for the previous example gives,

$$\begin{aligned} L_{\text{avg}}(\theta) &= \sum_{k=1}^{N_{\text{povm}}} \sum_{\ell=1}^{N_{\text{input}}} \lambda_{k\ell} L_{k\ell}(\theta) \\ L_{k\ell}(\theta) &= -\sum_{i=1}^{N_{\text{out}}} p_i(\phi_k, \beta_\ell, \theta_0) \log\ p_i(\phi_k.\beta_\ell, \theta) \end{aligned} \qquad (39)$$

where θ_0 is the true parameter and θ is the optimization variable. Figure 6 shows this function for $\gamma = 0.25$ and for three values of θ_0 (vertical lines), with both the POVM and input jointly configured, and with $\lambda_{k\ell}$ optimized for θ in the range (35). The search space is clearly convex, which it also is for all the other cases in this example. *In general the search space is not guaranteed to be convex.*

To increase the precision would require a more refined knowledge of the parameter range. A standard approach to circumvent not knowing the true value, or a more refined range, is to proceed adaptively, by "bootstrapping." Use the current estimate to solve for the corresponding optimal configuration, and then repeat the estimation procedure. This clearly introduces additional computational effort, not only increasing the overall rate of convergence, but also potentially decreasing the precision since the configuration is tuned to the current estimate. A potential advantage to the approach presented here is the capability of accounting for a parameter range which could also be estimated adaptively. In addition, the set of samples of θ (35) used for calculating the experiment design does not likely contain either the true parameter or the estimate obtained from maximum likelihood. The latter, however, for the next iteration, can be added to the sample set as well as refining the range, perhaps by also estimating the variance. Clearly issues of convergence – both rate and precision – need to be investigated for this approach.

8. CONCLUSION

We have shown that maximizing the precision in estimating a single parameter in a quantum system subject to input and POVM constraints reduces to a linear program for both what is defined here as a worst-case and average-case objective. For the average-case, the solution to the linear program can be expressed analytically and involves a simple search, *i.e.*, find the largest element of an easily computed vector. Both solutions provide different levels of Fisher information over the range of anticipated parameter variation. Comparing these constrained solutions to the best possible under the constraints as well as to the quantum Fisher bound gives an indication of the performance limitations imposed by the constraints.

There are several directions worth pursuing. Perhaps most importantly is the overall rate of parameter estimation convergence. This should include not just the asymptotic rate from the classical or quantum Cramér-Rao inequalities, but also the effort required to compute the estimate, *e.g.*, with bootstrapping or other adaptive approaches. It will also be interesting to see how constraints effect the rate of convergence, with or without bootstrapping, when entanglement is introduced, *e.g.*, see Giovannetti et al. (2006) and the references therein. A further refinement of what has been presented here is to solve a robust optimization problem where both the parameters of interest and the parameters associated with loss (amplitude damping probability in the example) are taken into account.

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