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Estimation of the Uncertainty in a Helicopter Dynamic Model Identified by the Subspace-based Method Using Bootstrap Techniques ⋆

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Abstract: Knowledge of the uncertainty in an identified model is of great importance, in particular for robust controller design. This paper presents a method for estimating the uncertainty in a state-space helicopter dynamical model identified from the subspace-based method using bootstrap techniques. Computer simulations are carried out to illustrate the operation and performance of the method using concatenated data sets generated from an unmanned rotorcraft model. The results obtained are in good agreement with those from conventional Monte Carlo simulations demonstrating the effectiveness of the method.

Keywords: State-space model; Subspace methods; Helicopter dynamic modeling; Bootstrap method; Uncertainty estimation

1. INTRODUCTION

The problem of helicopter dynamic modelling via system identification has been studied for many years and a large number of papers have been published over the last two decades (see e.g. Tischler and Remple (2006), Li et al (2007a) and references therein). Of great importance, in particular for robust controller design, is knowledge of the uncertainty in the identified model.

Traditionally, the parameter estimation-based "grey-box" modelling approach is adopted for building helicopter dynamic models, where the model structure and parametrization are assumed to be known and the problem of dynamic modelling is transformed into a problem of parameter estimation. With such an approach, the amount of uncertainty in the identified model is measured by the variance or standard deviation of the parameter estimates which can be estimated from measurement data (see e.g. Chapter 9 of Ljung (1999)) on the basis of asymptotic results for these parameter estimates.

There are a number of difficulties (see Li et al (2007b)) associated with the application of the above parameter estimation-based method in the identification of a fully coupled MIMO state-space helicopter dynamic model, such as the identifiability of the model parameters for the given measurements, sensitivity of the parameter estimates to the initial values *etc.* To overcome these difficulties, the subspace-based "black-box" modelling approach has recently been applied to helicopter dynamic modelling (see e.g. Li et al (2007b)). However, the problem of evaluating the uncertainty in such an identified model is not discussed. This is mainly because of the complexity of the statistical theory underlying the subspace identification method. Therefore, this paper will focus on the development of practical methods (rather than the derivation of closed-form asymptotic expressions) for estimating uncertainty in a helicopter dynamic model identified using the subspace-based method. A "bootstrap-based" simulation method will be presented. The rest of the paper is organized as follows. Section 2 briefly describes the PO-MOESP subspace identification method for state-space helicopter dynamic modelling, followed in Section 3 by an introduction to the bootstrap method for determining uncertainty of a statistic. A bootstrap-based method for estimating the model uncertainty in PO-MOESP framework is derived in Section 4. The results from a simulation study are presented in Section 5 with concluding remarks in Section 6.

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2. SUBSPACE SYSTEM IDENTIFICATION FOR HELICOPTER DYNAMIC MODELLING

2.1 Preliminaries

The linearized helicopter dynamic model about a trim condition is of the following state-space form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v} \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are known as the stability and control matrices that are derived from the partial derivatives of the nonlinear model function (see Padfield (1996)); $\mathbf{x}$ and $\mathbf{u}$ are respectively the perturbed state and input about a known trim point. The term $\mathbf{v}$ represents measurement noise. For most practical applications, the measurements are usually sampled-data (i.e. discrete), and so the system identification algorithm will then be implemented in the

discrete-time domain. Suppose that the input is constant over the sampling interval T, the sampling/discrete version of the model (1)∼(2) is then given by (see e.g. Ljung (1999)):

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k + \mathbf{w}_k \tag{3}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{v}_k \tag{4}$$

where $\mathbf{x}_k \in \Re^n$, $\mathbf{u}_k \in \Re^m$, $\mathbf{y}_k \in \Re^l$, $\mathbf{\Phi} = e^{\mathbf{A}T}$ and $\mathbf{G} = \int_0^T e^{\mathbf{A}\tau}\mathbf{B}d\tau$; $\mathbf{w}_k$ and $\mathbf{v}_k$ are zero-mean white Gaussian sequences of appropriate strength and are independent of the input $\mathbf{u}_k$. The additional term $\mathbf{w}_k$ is added in the state equation (3) as process noise to represent possible atmospheric and other disturbances or the modelling errors due to approximations. Helicopter dynamic model identification then amounts to the determination of matrices $\mathbf{\Phi}$, $\mathbf{G}$ (thus $\mathbf{A}$, $\mathbf{B}$), $\mathbf{C}$ and $\mathbf{D}$ using the measurement sequences $\mathcal{D}_N = \{\mathbf{u}_k, \mathbf{y}_k\}_{k=1}^N$.

2.2 PO-MOESP type subspace-based state-space model identification

The starting point for subspace system identification is the following structured input-output equation that is derived by repeated substitution of (3) and (4):

$$\mathbf{Y}_{k,s,j} = \mathbf{\Gamma}_s\mathbf{X}_{k,j} + \mathbf{H}_s\mathbf{U}_{k,s,j} + \mathbf{E}_s\mathbf{W}_{k,s,j} + \mathbf{V}_{k,s,j} \tag{5}$$

where

$$\mathbf{Y}_{k,s,j} = \begin{bmatrix} \mathbf{y}_k & \mathbf{y}_{k+1} & \cdots & \mathbf{y}_{k+j-1} \\ \mathbf{y}_{k+1} & \mathbf{y}_{k+2} & \cdots & \mathbf{y}_{k+j} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{k+s-1} & \mathbf{y}_{k+s} & \cdots & \mathbf{y}_{k+s+j-2} \end{bmatrix}_{sl\times j} \tag{6}$$

$$\mathbf{H}_s = \begin{bmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} & \cdots \mathbf{0} \\ \mathbf{C}\mathbf{G} & \mathbf{D} & \mathbf{0} & \cdots \mathbf{0} \\ \mathbf{C}\mathbf{\Phi}\mathbf{G} & \mathbf{C}\mathbf{G} & \mathbf{D} & \cdots \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots \vdots \\ \mathbf{C}\mathbf{\Phi}^{s-2}\mathbf{G} & \mathbf{C}\mathbf{\Phi}^{s-3}\mathbf{G} & \mathbf{C}\mathbf{\Phi}^{s-4}\mathbf{G} & \cdots \mathbf{D} \end{bmatrix}_{sl\times sm} \tag{7}$$

$\mathbf{U}_{k,s,j}$, $\mathbf{W}_{k,s,j}$ and $\mathbf{V}_{k,s,j}$ are constructed in a manner similar to $\mathbf{Y}_{k,s,j}$ and $\mathbf{E}_s$ similar to $\mathbf{H}_s$ (see e.g. Verhaegen (1994) for details). $\mathbf{\Gamma}_s$ is the extended observability matrix for the system to be identified and $\mathbf{X}_{k,j}$ is formed by consecutive state vectors:

$$\mathbf{\Gamma}_s = \begin{bmatrix} \mathbf{C}^T & (\mathbf{C}\mathbf{\Phi})^T & \cdots & (\mathbf{C}\mathbf{\Phi}^{s-1})^T \end{bmatrix}^T \tag{8}$$

$$\mathbf{X}_{k,j} = \begin{bmatrix} \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_{k+j-1} \end{bmatrix}_{n\times j} \tag{9}$$

The indices (k, s, j) of the data Hankel matrices $\mathbf{Y}_{k,s,j}$ and $\mathbf{U}_{k,s,j}$ determine their size and what part of the I/O sequences is stored in them.

The PO-MOESP identification scheme described here was originally proposed in Verhaegen (1994) and addresses the problem of identification of the deterministic part of a MIMO state-space model given by (3) and (4). With this identification scheme, the I/O equation (5) is split into two parts, a "past" one denoted by subscript p and a "future" one denoted by subscript f:

$$\mathbf{Y}_p = \mathbf{\Gamma}_s\mathbf{X}_p + \mathbf{H}_s\mathbf{U}_p + \mathbf{E}_s\mathbf{W}_p + \mathbf{V}_p \tag{10}$$

$$\mathbf{Y}_f = \mathbf{\Gamma}_s\mathbf{X}_f + \mathbf{H}_s\mathbf{U}_f + \mathbf{E}_s\mathbf{W}_f + \mathbf{V}_f \tag{11}$$

where the "past" data Hankel matrices are $\mathbf{Y}_p = \mathbf{Y}_{1,s,j}$ and $\mathbf{U}_p = \mathbf{U}_{1,s,j}$ as defined in (6) with $k = 1$; the "future" data Hankel matrices are $\mathbf{Y}_f = \mathbf{Y}_{s+1,s,j}$ and $\mathbf{U}_f = \mathbf{U}_{s+1,s,j}$ as defined in (6) with $k = s + 1$; $\mathbf{W}_p$, $\mathbf{W}_f$ and $\mathbf{V}_p$, $\mathbf{V}_f$ are noise Hankel matrices formed from the noise sequences $\mathbf{w}_k$ and $\mathbf{v}_k$ in a manner similar to $\mathbf{Y}_p$, $\mathbf{Y}_f$, and $\mathbf{X}_p$, $\mathbf{X}_f$ are defined as in (9) with $k = 1$ and $k = s + 1$ respectively. The identification scheme is based on the use of both the past input and past output as instrumental variables to remove the effect of noise. The overall algorithm contains two steps: (1) identification of the extended observability matrix $\mathbf{\Gamma}_s$, and (2) calculation of state-space model matrices $\mathbf{\Phi}, \mathbf{G}, \mathbf{C}$ and $\mathbf{D}$ (see Verhaegen (1994) for details). As a first step in the algorithm, the following RQ decomposition is performed:

$$\begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \\ \mathbf{Y}_p \\ \mathbf{Y}_f \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{21} & \mathbf{R}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} & \mathbf{0} \\ \mathbf{R}_{41} & \mathbf{R}_{42} & \mathbf{R}_{43} & \mathbf{R}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^{(ms,j)} \\ \mathbf{Q}_2^{(ms,j)} \\ \mathbf{Q}_3^{(ls,j)} \\ \mathbf{Q}_4^{(ls,j)} \end{bmatrix} \tag{12}$$

A consistent estimate of the column space of $\mathbf{\Gamma}_s$ can be retrieved from the following SVD:

$$\begin{matrix} & & n & ls\text{–}n \end{matrix}$$
$$[\mathbf{R}_{42}\ \mathbf{R}_{43}] = \begin{matrix} ls \end{matrix} \begin{pmatrix} \mathbf{U}_n & \mathbf{U}_n^{\perp} \end{pmatrix} \underbrace{\begin{matrix} \begin{matrix} n & ls\text{–}n & ms \end{matrix} \\ \begin{pmatrix} \mathbf{S}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 & \mathbf{0} \end{pmatrix} \end{matrix}}_{\mathbf{S}} \mathbf{V}^T \tag{13}$$

where n is the actual order of the model to be identified. When the input is sufficiently persistently exciting, the column space of the matrix $\mathbf{U}_n$ is a consistent estimate of that of $\mathbf{\Gamma}_s$ and the estimates of the matrices $\mathbf{\Phi}, \mathbf{G}, \mathbf{C}$ and $\mathbf{D}$ can subsequently be calculated (see e.g. Verhaegen (1994), Li et al (2007b)). The above identification scheme can be straightforwardly adapted to processing concatenated data sets from multiple tests and this is very helpful for helicopter dynamic modelling (see Li et al (2007b) for details).

3. THE BOOTSTRAP IDEA

Bootstrap is a computational statistical method. It was originally introduced (see e.g. Efron and Tibshiran (1993)) to assess the accuracy of a statistic, where no standard methods could be applied. With advances in the technology of affordable and high-speed computers, the use of the bootstrap has been extended to more and more complicated situations. Introductions to the bootstrap can be found in Efron and Tibshiran (1993), Politis (1998). A survey of bootstrap applications in signal processing is given in Zoubir and Boashash (1998). In recent years, the bootstrap techniques have been introduced into system identification (see Bittanti and Lovera (2000), Tjarnstrom and Ljung (2002), Lopes et al (2006)) for solving the problem of evaluating the uncertainty of the identified model.

To illustrate the basic idea behind bootstrap, let us consider the following problem: given an independent, identically distributed (i.i.d.) sample $\mathbf{x} = (x_1, \cdots, x_N)$ of size N observations from a stochastic variable X with distribution function F, one calculates an estimate $\hat{\theta}$ of some parameter θ associated with F with a statistic $T(\mathbf{x})$, i.e. $\hat{\theta} = T(\mathbf{x})$

based on the available data and one would like to evaluate the uncertainty of the obtained estimate in terms of its standard deviation or its variance $Var_F(T)$.

If F was known, $Var_F(T)$ could be calculated exactly by analytical methods or approximately by Monte Carlo simulations in the case where the analytical computation is difficult. In the latter case, the calculation is done as follows: since F is known, one can generate any number of i.i.d. samples from it. Suppose that B new "samples" $\mathbf{x}_1, \cdots, \mathbf{x}_B$ are generated from F with each containing N i.i.d. observations. Putting these into the estimator T, then one obtains B observations $\hat{\theta}_1 = T(\mathbf{x}_1), \cdots, \hat{\theta}_B = T(\mathbf{x}_B)$ of the statistic $T(\mathbf{x})$. If B is large enough, one can then approximate $Var_F(T)$ with the sample variance of the above obtained observations, that is:

$$Var_F(T) \approx \frac{1}{B-1}\sum_{b=1}^{B}(T(\mathbf{x}_b) - \bar{T})^2 \tag{14}$$

where

$$\bar{T} = \frac{1}{B}\sum_{b=1}^{B} T(\mathbf{x}_b) \tag{15}$$

Following the above idea, if the distribution function F is unknown, a simple and natural idea is to replace it with its estimate $\hat{F}$. Such direct non-parametric estimate is the empirical discrete distribution derived from the available observed sample $\mathbf{x} = (x_1, \cdots, x_N)$:

$$\hat{F}(x) = \frac{\#\{x_i \le x\}}{N} \tag{16}$$

where $\#\{x_i \le x\}$ means the number of the x_is among $x_1, \cdots, x_N$ that are observed to be less than or equal to x. To sum up, the above empirical discrete distribution $\hat{F}$ can be obtained by associating a probability mass of $1/N$ to each of the observations in sample $\mathbf{x}$. In this fashion, the aforementioned Monte Carlo procedure can be used to estimate $Var_F(T)$, but in this case, the B new "samples" $\mathbf{x}_1^*, \cdots, \mathbf{x}_B^*$ (each of size N) are generated from $\hat{F}$ defined by (16) instead of F. These new "samples" are called bootstrap resamples and the bootstrap estimate of variance is then given by:

$$\hat{Var}^*(T) = \frac{1}{B-1}\sum_{b=1}^{B}(T(\mathbf{x}_b^*) - \bar{T})^2 \tag{17}$$

where

$$\bar{T} = \frac{1}{B}\sum_{b=1}^{B} T(\mathbf{x}_b^*) \tag{18}$$

As can be seen, the bootstrap principle is based on the idea of replacing the (unknown) distribution F of the data with an estimate $\hat{F}$ of it, where $\hat{F}$ is estimated from the available data sample $\mathbf{x} = (x_1, \cdots, x_N)$. It needs to be pointed out that the estimate $\hat{F}$ of F can be either parametric or non-parametric depending on the prior information one has on F. When F is completely unknown, it is approximated by the empirical distribution defined by (16) and this results in the non-parametric bootstrap setup as described above. In such a case, the bootstrap resample $\mathbf{x}^* = (x_1^*, \cdots, x_N^*)$ is an unordered collection of N observations drawn randomly from $\mathbf{x} = (x_1, \cdots, x_N)$ with replacement, so that each x_j^* in $\mathbf{x}^*$ has probability $1/N$ of being equal to any one of the x_i's in $\mathbf{x}$. Whenever prior information on F is available, it can be incorporated with a parametric approach for the estimation of F. In general, the parametric bootstrap setup assumes that the distribution F is known up to some parameter θ; that is, $F = F_\theta$ belongs to a known class of functions parameterized by θ. To determine F one just needs to determine the corresponding θ-value. Therefore, the estimate $\hat{F}$ of F is simply $F_{\hat{\theta}}$, where $\hat{\theta}$ is the estimated (from the available sample of observations $\mathbf{x} = (x_1, \cdots, x_N)$) value of the parameter θ.

The bootstrap procedure discussed above relies on the assumption that the original data $\mathbf{x} = (x_1, \cdots, x_N)$ is an i.i.d. sample from an (unknown) distribution F underlying the problem considered. This assumption can break down, either because the data are not independent or because they are not identically distributed, or both. For example, the response data to a control input generated from a dynamic system are not i.i.d. data, because the data are correlated in time and also depend on the system's input (thus not identically distributed). To obtain the correct results from the bootstrap procedure described previously, some measure has to be taken to circumvent this difficulty.

The residual bootstrap (see e.g. Bittanti and Lovera (2000), Tjarnstrom and Ljung (2002) and the references therein) is a method proposed to extend the use of the aforementioned bootstrap procedure for more complicated data structures such as the data generated from a dynamic system. The key idea of the residual bootstrap is to reduce the non-i.i.d. situation to an i.i.d. situation by looking at the residuals. This can briefly be illustrated with the regression example as follows. Suppose that the data $\mathbf{y} = (y_1, \cdots, y_N)$ are generated from a regression model $y_k = f(x_k, \theta) + \varepsilon_k (k = 1, \cdots, N)$, where f is the regression function (which can be either linear or nonlinear), x_ks are known and nonrandom, θ is the vector of unknown parameters to be estimated and the ε_ks are i.i.d. with distribution F of zero mean. Note that, although the y_ks are independent they are not identically distributed as they depend on nonrandom variables x_ks, thus here $y_k (k = 1, \cdots, N)$ is not i.i.d but the error $\varepsilon_k (k = 1, \cdots, N)$ is. Although $\varepsilon_k (k = 1, \cdots, N)$ is not directly observable, it may be approximated. If a parameter estimator $\hat{\theta}$ of θ can be constructed from the data such that the residuals $\hat{\varepsilon}_k = y_k - f(x_k, \hat{\theta}) (k = 1, \cdots, N)$ are zero mean i.i.d, these residuals can be considered as good approximations to the unobservable i.i.d errors $\varepsilon_k (k = 1, \cdots, N)$ from which one can draw B i.i.d. residual resamples $\boldsymbol{\varepsilon}_1^*, \cdots, \boldsymbol{\varepsilon}_B^*$ (each of size N), where $\boldsymbol{\varepsilon}_b^* = (\hat{\varepsilon}_1^{*(b)}, \cdots, \hat{\varepsilon}_N^{*(b)}) (b = 1, \cdots, B)$. These bootstrap residual resamples are then fed into the estimated regression model to generate B replications of pseudo-data $\mathbf{y}_1^*, \cdots, \mathbf{y}_B^*$ (each of size N) where $\mathbf{y}_b^* = (y_1^{*(b)}, \cdots, y_N^{*(b)}) (b = 1, \cdots, B)$ and $y_k^{*(b)} = f(x_k, \hat{\theta}) + \hat{\varepsilon}_k^{*(b)} (k = 1, \cdots, N)$. B replications of bootstrap parameter estimates can be obtained by repeating the estimation calculation for each of the B replications of the above pseudo-data. The bootstrap estimate of variance $\hat{Var}^*(\hat{\theta})$ of the parameter estimate $\hat{\theta} = T(\mathbf{y})$ can then be calcu-

lated by (17). The above residual bootstrap technique for the regression model can readily be extended to the case of estimating uncertainty in the identified input/output dynamic models.

4. BOOTSTRAP-BASED METHOD FOR ESTIMATING MODEL UNCERTAINTY IN PO-MOESP

The variance, or standard deviation, of the model parameter estimates mentioned in Section 1 is not applicable for quantifying the uncertainty of the model identified by the subspace-based method because of the absence of a specified parametrization of the model and the arbitrary choice of the state-space basis for the obtained models in the subspace identification framework. This directs the uncertainty analysis towards confidence intervals for input/output characteristics of the model such as its frequency response or towards realization independent features such as the locations of its poles and zeros. In such a case, one is actually looking for an estimate of the standard deviation of a prescribed statistic that is a function of the identified model.

Based on the idea of bootstrapping residuals described in the last section, the bootstrap-based procedure for evaluating the standard deviation of the frequency response of the state-space model (1) and (2) identified by a subspace-based identification method can be summarized as follows:

(1) Identify the model by estimating $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ from the available original I/O data $\mathcal{D}_N = \{\mathbf{u}_k, \mathbf{y}_k\}_{k=1}^N$ using a subspace identification method and compute the estimate of its frequency response $\hat{G}(e^{j\omega_i})(i = 1, \cdots, n)$ for the frequency points of interest.

(2) Compute the prediction error for the identified state-space model: $\hat{\mathbf{e}}_k = \mathbf{y}_k - \hat{\mathbf{y}}_k (k = 1, \cdots, N)$ where $\hat{\mathbf{y}}_k$ is the predicted output based on the estimate $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$.

(3) Draw B i.i.d. resamples $\mathbf{E}_1^*, \cdots, \mathbf{E}_B^*$ (each of size N) from the empirical distribution derived from the computed prediction error in the last step, where $\mathbf{E}_b^* = (\hat{\mathbf{e}}_1^{*(b)}, \cdots, \hat{\mathbf{e}}_N^{*(b)})(b = 1, \cdots, B)$ and generate B bootstrap replications $\mathcal{D}_N^{*(1)}, \cdots, \mathcal{D}_N^{*(B)}$ (each of size N) of the original data set $\mathcal{D}_N$, where $\mathcal{D}_N^{*(b)} = \{\mathbf{u}_k^{*(b)}, \mathbf{y}_k^{*(b)}\}_{k=1}^N (b = 1, \cdots, B)$ with $\mathbf{u}_k^{*(b)} = \mathbf{u}_k$ and $\mathbf{y}_k^{*(b)}$ obtained by feeding the identified state-space model $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ with the deterministic input $\mathbf{u}_k^{*(b)} = \mathbf{u}_k$ and the bootstrap residual resample $\hat{\mathbf{e}}_k^{*(b)} (k = 1, \cdots, N)$.

(4) Identify B replications $[\hat{\mathbf{A}}^{*(b)}, \hat{\mathbf{B}}^{*(b)}, \hat{\mathbf{C}}^{*(b)}, \hat{\mathbf{D}}^{*(b)}]$ of the model using respectively the B set of bootstrap data $\mathcal{D}_N^{*(b)} (b = 1, \cdots, B)$ obtained in the last step and compute the corresponding B replications $\hat{G}^{*(b)}(e^{j\omega_i})(b = 1, \cdots, B;\ i = 1, \cdots, n)$ of the frequency response for the frequency points of interest.

(5) The estimate of the standard deviation for the frequency response of the identified state-space model is given by

$$\hat{\sigma}^*(\hat{G}(e^{j\omega_i})) = \frac{1}{\sqrt{B-1}} \Big(\sum_{b=1}^{B} (\hat{G}^{*(b)}(e^{j\omega_i}) - \bar{G}^*(e^{j\omega_i}))^2\Big)^{\frac{1}{2}} \tag{19}$$

where

$$\bar{G}^*(e^{j\omega_i}) = \frac{1}{B} \sum_{b=1}^{B} \hat{G}^{*(b)}(e^{j\omega_i}) \tag{20}$$

In a similar way, the estimates of the standard deviation for the poles and zeros or the singular values of the frequency response (for MIMO system) of the identified state-space model can be obtained.

As can be seen, the bootstrap is a computer-intensive method. Replicating the identification process B times can be extremely time-consuming. Therefore, reducing the computational cost is a major issue for the practical application of the above bootstrap standard deviation estimation method.

With subspace-based system identification methods, it turns out that the replication of the identification process can be speeded up significantly by taking advantage of the structure of the algorithms. In Bittanti and Lovera (2000), a computationally efficient algorithm for implementing the above bootstrap standard deviation estimation method for the model identified by the ordinary MOESP subspace identification method proposed in Verhaegen and Dewilde (1992) was developed. The algorithm exploits the following fact (see Bittanti and Lovera (2000)): when working with the bootstrap replications $\mathcal{D}_N^{*(b)} (b = 1, \cdots, B)$ of the original data, the noise (residual) processes acting on the system are known a priori. This fact implies that the generation of a bootstrap replica $[\hat{\mathbf{A}}^{*(b)}, \hat{\mathbf{B}}^{*(b)}, \hat{\mathbf{C}}^{*(b)}, \hat{\mathbf{D}}^{*(b)}]$ of the identified model in step 4 of the above bootstrap procedure can be achieved with two steps: (1) obtain a bootstrap-noise free replica of the identified model, or in other words, identify a model $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ with the original data set $\mathcal{D}_N$; (2) perturb the obtained bootstrap-noise free replica with the bootstrap noise $\mathbf{E}_b^* = (\hat{\mathbf{e}}_1^{*(b)}, \cdots, \hat{\mathbf{e}}_N^{*(b)})(b = 1, \cdots, B)$ so as to take into account the effect of noise in the identification process. In this two-step procedure, the bootstrap-noise free replica only has to be computed once, while the effect of noise in the identification process can be taken into account by computing the perturbations to the bootstrap-noise free replica. Within the subspace-based system identification, such computations can be performed in a much cheaper way.

Following the idea presented in Bittanti and Lovera (2000), an algorithm for implementing the above two-step procedure for the model identified by the PO-MOESP will be outlined next. To simplify the presentation, it is assumed that the steps 1 to 3 in the above bootstrap procedure have been performed. That is, an estimate $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ of the model has been obtained with the original data set $\mathcal{D}_N$ and the corresponding sequence of the prediction error $\hat{\mathbf{e}}_k = \mathbf{y}_k - \hat{\mathbf{y}}_k (k = 1, \cdots, N)$ has been computed. Now, a bootstrap replica $\mathbf{E}_b^* = (\hat{\mathbf{e}}_1^{*(b)}, \cdots, \hat{\mathbf{e}}_N^{*(b)})$ of the prediction error sequence is drawn as described in step 3 and the bootstrap replica $[\hat{\mathbf{A}}^{*(b)}, \hat{\mathbf{B}}^{*(b)}, \hat{\mathbf{C}}^{*(b)}, \hat{\mathbf{D}}^{*(b)}]$

of the identified model has to be computed from the bootstrap data set $\mathcal{D}_N^{*(b)}$ as stated in step 4.

The bootstrap replica of output data $\mathbf{y}_k^{*(b)}$ is obtained by feeding the identified state-space model $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ with the deterministic input $\mathbf{u}_k^{*(b)} = \mathbf{u}_k$ and the bootstrap residual resample $\hat{\mathbf{e}}_k^{*(b)} (k = 1, \cdots, N)$. Therefore, the bootstrap version of the structured input-output data equation (5) can be written as

$$\mathbf{Y}_{k,s,j}^{*(b)} = \hat{\mathbf{\Gamma}}_s \mathbf{X}_{k,j} + \hat{\mathbf{H}}_s \mathbf{U}_{k,s,j} + \mathbf{N}^{*(b)} = \mathbf{Y}_{k,s,j}^{nf} + \mathbf{N}^{*(b)} \quad (21)$$

where matrices $\hat{\mathbf{\Gamma}}_s$ and $\hat{\mathbf{H}}_s$ are formed from $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ of the identified model, $\mathbf{Y}_{k,s,j}^{*(b)}$ denotes the Hankel matrix constructed with bootstrap output data $\mathbf{y}_k^{*(b)}$, $\mathbf{N}^{*(b)}$ is formed by the bootstrap residual $\hat{\mathbf{e}}_k^{*(b)}$ and $\mathbf{Y}_{k,s,j}^{nf}$ (where superscript nf stands for bootstrap-noise free) denotes the Hankel matrix formed by the output data obtained by simulating the identified state-space model $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$ with the input $\mathbf{u}_k^{*(b)} = \mathbf{u}_k$ and neglecting the effect of all noises. With the bootstrap output data, the first and also the most computationally expensive step in PO-MOESP, i.e. the RQ decomposition (12), can be expressed as:

$$\begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \\ \mathbf{Y}_p^{*(b)} \\ \mathbf{Y}_f^{*(b)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11}^* & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{21}^* & \mathbf{R}_{22}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{31}^* & \mathbf{R}_{32}^* & \mathbf{R}_{33}^* & \mathbf{0} \\ \mathbf{R}_{41}^* & \mathbf{R}_{42}^* & \mathbf{R}_{43}^* & \mathbf{R}_{44}^* \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^* \\ \mathbf{Q}_2^* \\ \mathbf{Q}_3^* \\ \mathbf{Q}_4^* \end{bmatrix} \quad (22)$$

Based on (21), the factorization (22) can be rewritten in terms of the bootstrap-noise free data matrix $\mathbf{Y}_{t,s,j}^{nf}$ as follows:

$$\begin{aligned} \begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \\ \mathbf{Y}_p^{*(b)} \\ \mathbf{Y}_f^{*(b)} \end{bmatrix} &= \begin{bmatrix} \mathbf{U}_f \\ \mathbf{U}_p \\ \mathbf{Y}_p^{nf} \\ \mathbf{Y}_f^{nf} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{N}_p^{*(b)} \\ \mathbf{N}_f^{*(b)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{21} & \mathbf{R}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} & \mathbf{0} \\ \mathbf{R}_{41} & \mathbf{R}_{42} & \mathbf{R}_{43} & \mathbf{R}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{R}_{11}^{n*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{21}^{n*} & \mathbf{R}_{22}^{n*} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{31}^{n*} & \mathbf{R}_{32}^{n*} & \mathbf{R}_{33}^{n*} & \mathbf{0} \\ \mathbf{R}_{41}^{n*} & \mathbf{R}_{42}^{n*} & \mathbf{R}_{43}^{n*} & \mathbf{R}_{44}^{n*} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{bmatrix} \end{aligned} \quad (23)$$

The first term in (23) is obtained by RQ-factorizing the bootstrap-noise free (that is, purely simulating) data Hankel matrices. The bootstrap-noise dependent term is then expressed in terms of the Q-matrix as in the second term in (23),where we have:

$$\mathbf{R}_{11}^{n*} = \mathbf{R}_{21}^{n*} = \mathbf{R}_{22}^{n*} = \mathbf{0}$$
$$\mathbf{R}_{42}^{n*} = \mathbf{N}_f^{*(b)} \mathbf{Q}_2^T \quad \text{and} \quad \mathbf{R}_{43}^{n*} = \mathbf{N}_f^{*(b)} \mathbf{Q}_3^T$$

Therefore:

$$[\mathbf{R}_{42}^* \ \mathbf{R}_{43}^*] = \left[\mathbf{R}_{42} + \mathbf{N}_f^{*(b)} \mathbf{Q}_2^T \ \ \mathbf{R}_{43} + \mathbf{N}_f^{*(b)} \mathbf{Q}_3^T \right] \quad (24)$$

Equation (24) shows that it is possible to determine the bootstrap-noise perturbed terms $\mathbf{R}_{42}^*$ and $\mathbf{R}_{43}^*$ in the factorization (22) by adding the required perturbations or corrections to the corresponding bootstrap-noise free terms $\mathbf{R}_{42}$ and $\mathbf{R}_{43}$. The required corrections can be determined by projecting the known "future" bootstrap noise matrix $\mathbf{N}_f^{*(b)}$ onto the row space of $\mathbf{Q}_2$ and $\mathbf{Q}_3$ respectively. The corrected matrix $[\mathbf{R}_{42}^* \ \ \mathbf{R}_{43}^*]$ will then be used to estimate the extended observability matrix $\mathbf{\Gamma}_s$ and subsequently to estimate the $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$ matrices of the model.

As can be seen from the above discussion, once the RQ factorization for noise free data (i.e. the first term in (23)) is obtained (this RQ factorization needs to be performed *only once*), the subsequent RQ factorizations (which would be performed B times) for bootstrap-noise perturbed data can be replaced by two matrix products. From a practical point of view, this leads to a significant reduction in the number of computations. This is particularly so when the number B of bootstrap resamples is large. See Bittanti and Lovera (2000) for a detailed analysis of a similar case.

5. APPLICABILITY STUDIES VIA SIMULATIONS

To study the applicability and to illustrate the operation of the bootstrap method discussed in the last section for the helicopter dynamic modelling application, simulation studies are performed and the results are presented in this section. Our attention has concentrated on the application of the method to the estimation of the uncertainty in the frequency response of the identified models, but the method presented in this paper applies equally well to the estimation of the uncertainty in other statistics.

The problem of estimating the standard deviation of the frequency response for an identified extended helicopter dynamic model, involving coupling both between longitudinal and lateral motions and between rotor and fuselage, is considered. The data is generated from a 13th order unmanned rotorcraft (Yamaha R-50) model taken from Mettler et al (2002). The model describes the dynamics of the perturbed motion about the hover condition of the vehicle and is extended to include the additional dynamics from the rotor and control augmentation such as the active yaw damping system and the stabilizer bar. The model is of the state-space form as specified in (1) and (2) where the 13-dimensional state vector is defined as (see Mettler et al (2002) for details):

$$\mathbf{x} = [u \ v \ p \ q \ \phi \ \theta \ a \ b \ w \ r \ r_{fb} \ c \ d]^T \quad (25)$$

where u, v, w are the translational velocities along the three orthogonal directions of the fuselage fixed axes (x, y and z) and p, q, r are the roll, pitch and yaw rates about the $x-$, $y-$ and $z-$axes; ϕ and θ are the roll and pitch angle of the fuselage; a and b denote the longitudinal and lateral rotor flapping angles; c and d denote the longitudinal and lateral stabilizer bar flapping angles and r_{fb} is a state variable for the active yaw damping dynamics. There are, in general, four stick inputs which can be used by a pilot to cope with six degrees of freedom, i.e. lateral cyclic δ_{lat}, longitudinal cyclic δ_{lon}, tail rotor collective δ_{ped} and main rotor collective δ_{col} and they are represented in the control input vector $\mathbf{u} = [\delta_{lat} \ \delta_{lon} \ \delta_{ped} \ \delta_{col}]^T$.

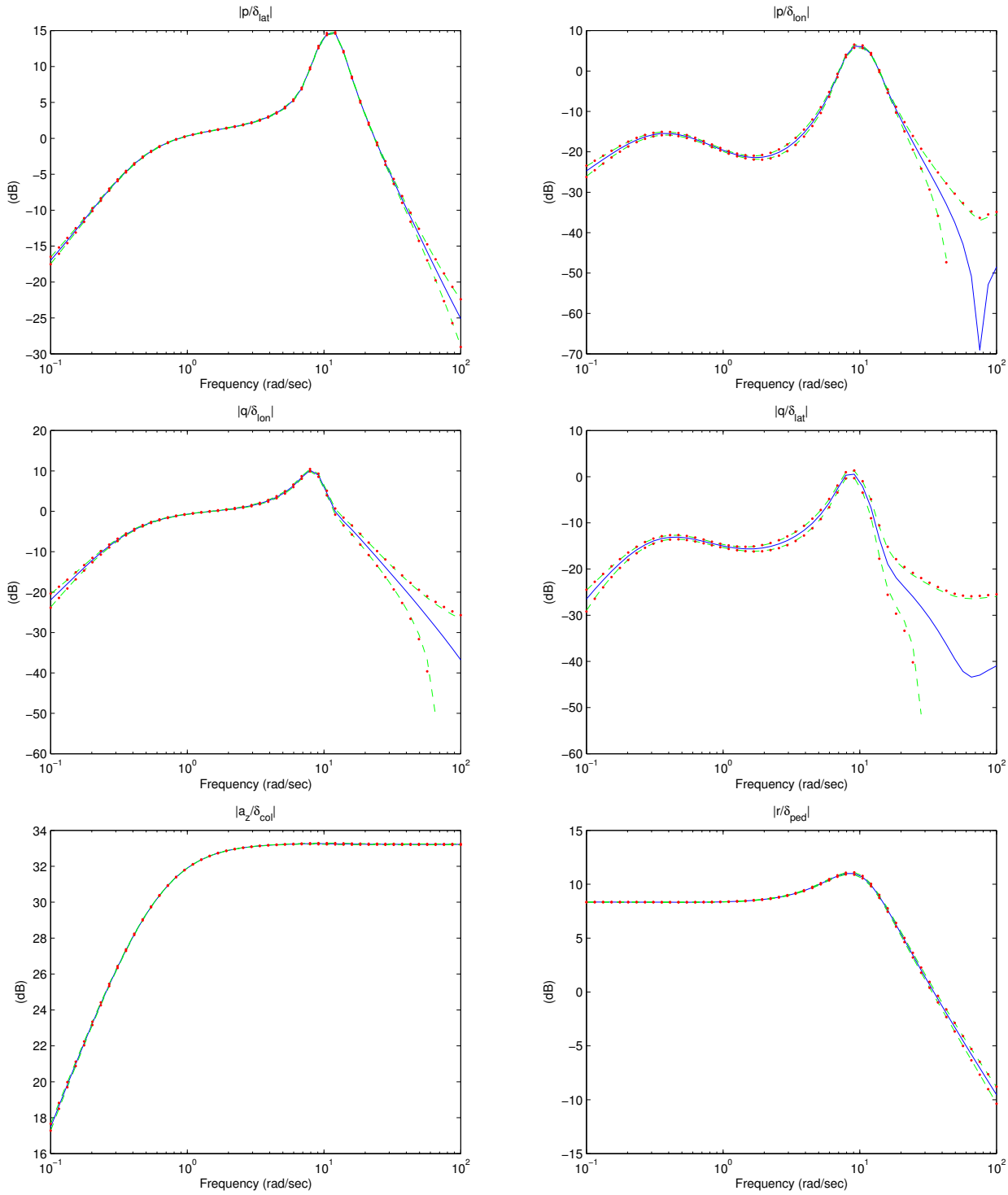

Fig. 1. Comparison of the Monte Carlo (dashed line) and bootstrap (dotted line) $3-\sigma$ confidence intervals for the estimated frequency responses

The measured outputs available for identification are the rigid-body fuselage states including the translational velocities u, v, w and rotational rates p, q, r; roll and pitch angles ϕ, θ and the accelerations $\dot{u}$, $\dot{v}$, $\dot{w}$. Therefore the measurement vector is defined as:

$$\mathbf{y} = [\, u \; v \; p \; q \; \phi \; \theta \; w \; r \; \dot{u} \; \dot{v} \; \dot{w} \,]^T \tag{26}$$

In the present simulation studies, a doublet signal is applied to each one of the four control channels in turn, and four sets of data are obtained with each containing the response generated with one control input. Then, four sets of input-output data Hankel matrices, as defined by (6), are constructed using these four sets of data respectively. To identify a fully coupled MIMO state-space helicopter model, these Hankel matrices are compounded for the RQ decomposition defined by (12) as follows:

$$\begin{bmatrix} \mathbf{U}_f^{lat} & \mathbf{U}_f^{lon} & \mathbf{U}_f^{ped} & \mathbf{U}_f^{col} \\ \mathbf{U}_p^{lat} & \mathbf{U}_p^{lon} & \mathbf{U}_p^{ped} & \mathbf{U}_p^{col} \\ \mathbf{Y}_p^{lat} & \mathbf{Y}_p^{lon} & \mathbf{Y}_p^{ped} & \mathbf{Y}_p^{col} \\ \mathbf{Y}_f^{lat} & \mathbf{Y}_f^{lon} & \mathbf{Y}_f^{ped} & \mathbf{Y}_f^{col} \end{bmatrix} \tag{27}$$

The PO-MOESP scheme outlined in Section 2 is then used for model identification and a 13th order state-space model is identified (see Li et al (2007b) for details).

Once the identified model is obtained, the bootstrap-based procedure developed in this paper is used for estimation of the standard deviations of the frequency response derived from the identified model. The bootstrap estimates of the standard deviations are compared with those obtained by Monte Carlo simulations and the results are shown in Figures 1 and 2, where Figure 1 shows the comparisons of the Monte Carlo (dashed-line) and bootstrap (dotted-line) 3–σ confidence intervals for the magnitudes of the estimated frequency responses and Figure 2 gives the values of the ratios σ_{MC}/σ_{bs} between the standard deviation computed by Monte Carlo method and that by bootstrap for the 50 considered frequency points. These results are obtained on the basis of 500 replications of the identification process. Specifically, for the Monte Carlo method, 500 data sets are generated by simulating the true model as described previously 500 times, and the Monte Carlo results are then obtained by repeating the identification procedure 500 times with these 500 data sets respectively. For the bootstrap method, the results are obtained using non-parametric bootstrap as described in Section 3. That is, we assume that the distribution F of the prediction error $\hat{\mathbf{e}}_k (k = 1, \cdots, N)$ is completely unknown and it is approximated by the empirical distribution defined by (16). The bootstrap resamples $\hat{\mathbf{e}}_k^* (k = 1, \cdots, N)$ of the prediction error are obtained using the following systematic resampling algorithm (see e.g. Li et al (2004)):

- Generate a uniformly distributed random point $u_1 \in [0, N^{-1}]$ and let $i = 1, \alpha_k(0) = 0$.
- For $j = 1 : N$
  (1) Let $u_j = u_1 + N^{-1}(j-1)$.
  (2) If $\sum_{l=0}^{i-1} \alpha_k(l) < u_j \leq \sum_{l=0}^{i} \alpha_k(l)$, set $\hat{\mathbf{e}}_j^* = \hat{\mathbf{e}}_i$.
  (3) Otherwise, $i = i + 1$ then goto step (2)
- End For

It can be seen from these figures that the estimated (by bootstrap) $3-\sigma$ confidence intervals for the frequency responses of the identified model agree quite well with the "true" (Monte Carlo) ones. The $3-\sigma$ confidence intervals at the high frequency parts of the off-axis angular responses (p to δ_{lon} and q to δ_{lat}) are relatively large which indicates that these identified high frequency responses are relatively unreliable. This is in agreement with the results obtained in Li et al (2007b).

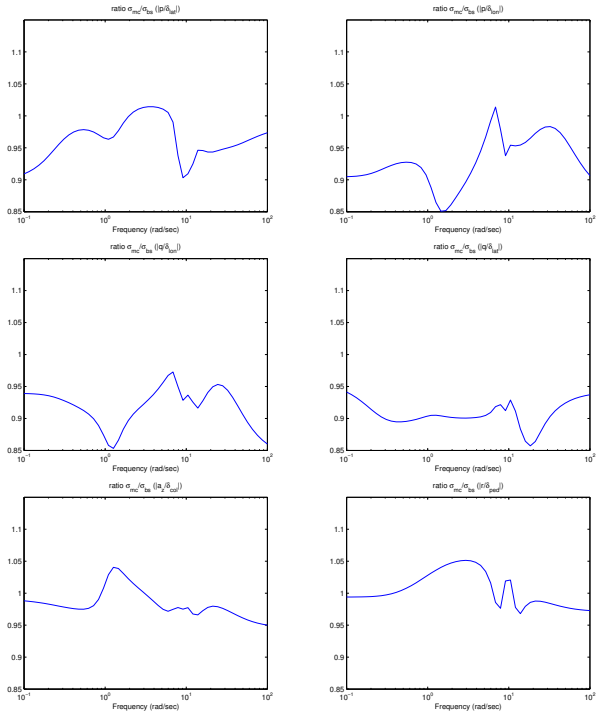

Fig. 2. Ratio between the "true" (Monte Carlo) and the estimated (bootstrap) standard deviations for the estimated frequency responses

6. CONCLUDING REMARKS

We have studied the problem of evaluating model uncertainty in the framework of PO-MOESP subspace identification for helicopter modelling and a bootstrap-based method has been presented. The operation and performance of the method was illustrated by a realistic example taken from the literature and the results show that the developed method is very promising. Future work will apply the method to real flight test data for the purposes of robust controller design.

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