Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Primal-Dual Power Control of Optical Networks with Time-Delay ⋆

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Abstract:
We study the effects of time-delay on the stability of a primal-dual controller applied to an optical communication network. Signal powers are adjusted at the sources while the links return dynamic pricing information. The objective of the source algorithm is to adjust the signal powers such that predefined OSNR targets are achieved according to the OSNR channel optimization problem. We incorporate time-delay into the closed loop system for the multi-source single-link case. Sufficient conditions for stability are derived based on a tuning parameter in the control algorithm. The work utilizes singular perturbation theory and Lyapunov-Razumikhin time-delay techniques. We also include simulations based on realistic network parameters.

1. INTRODUCTION

Optical networks are essential to providing high bandwidth for the Internet. They have more bandwidth than any other communication medium. Optical networks are distributed across oceans and continents.

The transmission of error-free data in optical communication networks depends on the optical signal to noise ratio (OSNR) at reception (Rx). In Pavel (2004) an OSNR optimization problem is formulated as a non-cooperative game between signal channels (players). As power in one signal increases, thereby increasing its OSNR, the noise in the other channels increases, thus decreasing their OSNRs. The work in Pavel (2004) also devises a network level power control algorithm at the signal sources to converge to a unique Nash equilibrium. In addition, Pavel (2007) augments the system with a link pricing algorithm resulting in a primal-dual algorithm. We take the signal powers at the sources and the pricing parameter as the inputs to the optical network system, and the OSNR outputs as the feedback signal. For simplicity, the work discussed herein assumes only one link, but multiple inputs and outputs.

Due to the fiber optical cable being spread out over a vast surface area, the signal propagation delay can not be considered negligible. In practical terms, for a signal to propagate across an optical span and return a distance of 100km, the propagation delay is approximately 1ms. When such spans are cascaded, it is not uncommon to see delays up in the tens of milliseconds. Despite having a convergent control algorithm for the OSNR optimization problem, the practical implementation of such a control law would have to take time-delay into account. In fact, we can no longer assume the control algorithm would even be stable given a sufficiently large time-delay.

⋆ The authors gratefully acknowledge the support of the Natural Science and Engineering Research Council of Canada.

The effects of time-delay in primal-dual control strategies are only beginning to be studied in literature. A series of papers Paganini et al. (2001); Wang and Paganini (2002, 2003, 2004); Paganini et al. (2005) provide the best starting point for the work presented herein. The main body of work by Paganini et. al. is extended in Wang and Paganini (2002, 2003, 2004) to utilize Lyapunov analysis techniques. These papers also touch upon timescale decoupling which can simplify time-delay analysis for nonlinear systems. However, the network is restricted to a single source single link case, and the utility function is decoupled with respect to the state. Another paper, Paganini et al. (2003), nicely summarizes previous time-delay work by Paganini et. al. A paper by Wen and Arcak (2004) uses a passivity approach and Lyapunov techniques to present a unifying framework for several flow control schemes. In addition, the control schemes show similarities to the optical network control schemes due to gradient-like positive projection dynamics and the use of primal-dual algorithms. However, the utility function is decoupled. The paper Liu et al. (2003) summarizes past work very well, and analyzes a positive projection gradient algorithm which is similar to the algorithm used herein. The work of Mazenc and Niculescu (2003) is a good reference for the application of Lyapunov analysis to time-delayed systems. Finally, the work Alpcan and Basar (2003) considers fixed heterogeneous delays in the network using Lyapunov stability theory. The work herein will focus on the Lyapunov-Razumikhin approach, as Gu et al. (2003) to studying time-delays. We couple this technique with a singular-perturbation approach as in Khalil (2002). Optical networks are multi-input multi-output systems characterized by a coupled utility function, and with decentralized control laws.

In this paper, we study the effects of time-delay in optical communication networks, based on the OSNR model and the network level control algorithm presented in Pavel (2004) and Pavel (2007). The algorithm of Pavel (2004) is analyzed for stability in the presence of time-delays in Stefanovic and Pavel (2007). We extend those results to include dynamic pricing at the links. We simplify the time-delay primal control algorithm from Stefanovic and Pavel (2007) by only considering the single link case and performing the link pricing calculations at the sources. We utilize singular perturbation theory enhanced to handle time-delayed differential equations. We obtain an upper bound for time-delays to ensure stability in the system using Lyapunov-Razumikhin theory. The compensating factor is a tunable gain parameter, ρ_i, at the sources.

The paper is organized as follows. Section 2 reviews the OSNR model, control algorithms and link algorithms. Section 3 modifies the resulting model Stefanovic and Pavel (2007) and presents a continuous-time closed loop system. The following section introduces time-delays into the closed loop system. Section 5 presents the main stability result and corresponding proof. Section 6 shows the ensuing simulations. The last section provides conclusions and future work.

2. REVIEW OF OSNR MODEL AND CONTROL ALGORITHMS

We begin by reviewing the OSNR model. The discrete-time control algorithm from Pavel (2004) is presented next. We then introduce the link control law from Pavel (2007).

2.1 OSNR Model

Consider an optical network that is defined by a single optical link. Channels in the network can be added or dropped. The link is composed of N spans that include one optical amplifier per span. A set of channels, $M = \{1, ..., m\}$, (intensity modulated wavelengths) are multiplexed together and transmitted across the link. We denote u_i, s_i, and n_i, the optical input power for channel i at the transmitter (Tx), the output signal at the receiver (Rx), and the output noise at Rx, respectively. The Optical Signal-to-Noise Ratio (OSNR) for any channel, $i \in M$, is defined as

$$OSNR_i = \frac{s_i}{n_i} \tag{1}$$

The following provides the framework for modeling OSNR in a single link optical network. An optical span is composed of an optical amplifier (OA) with channel dependent gain, G_i and optical fiber with wavelength independent loss coefficient, L_k. The amplifiers have the same spectral shape and are operated in automatic power control (APC) mode with total power targets P_0. The OA introduces amplified spontaneous emission (ASE) noise power, denoted $ASE_{k,i}$.

The following lemma from Pavel (2004) describes the OSNR model for optical networks.

Lemma 1. The OSNR for the i^{th} channel is given as

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in M} \Gamma_{i,j} u_j} \tag{2}$$

where $\Gamma_{i,j}$, elements of the full (n x n) system matrix Γ, are defined as

$$\Gamma_{i,j} = \sum_{k=1}^{N} \frac{G_j^k}{G_i^k} \frac{ASE_{k,i}}{P_0}$$

and $n_{0,i}$ is the noise optical power at transmitter (Tx) for the i^{th} channel.

With Lemma 1 presented, we can next introduce the primal-dual control law presented in Pavel (2007). The control law is introduced in two parts, the channel algorithm and the link algorithm. The channel algorithm is the same game-theoretic algorithm derived in Pavel (2004) with the exception that the constant cost parameter, α_i is substituted for an adjustable variable $\mu(\bar{k})$. The link algorithm is a computed every K iterations of the channel algorithm at the network links. Its purpose is to feedback the channel price parameter $\mu(\bar{k})$ to the channel sources.

2.2 Channel Algorithm

Given the OSNR model in Lemma 1, a non-cooperative game between channels was defined in Pavel (2004). The objective of each channel (player) is to maximize its utility related to OSNR in the presence of other channels. We use the utility function Pavel (2004)

$$U_i = ln\left(1 + a_i \frac{OSNR_i}{1 - \Gamma_{i,i} OSNR_i}\right)$$

where a_i is a channel dependent design parameter. A greater utility implies greater OSNR values which further implies a lower bit error rate in the optical network. Each channel adjusts its power towards this goal in the presence of other channels. The game settles at an equilibrium when no channel can improve its utility unilaterally; the equilibrium of the game being a Nash equilibrium. By converging to the Nash equilibrium the distributed system will achieve optimal OSNR values for its channels without centralized control. A full Nash game solution was presented in Pavel (2004). This work was then modified in Pavel (2007) to include an adjustable cost parameter, μ, that reflects the network channel utilization. The OSNR game admits a unique Nash equilibrium, u^*, which is the solution of

$$a_i u_i^* + \sum_{j \neq i} \Gamma_{i,j} u_j^* = \frac{a_i \beta_i}{\mu(\bar{k})} - n_{0,i} \quad \forall i \tag{3}$$

Based on this solution an iterative network level control algorithm was proposed in Pavel (2004) to control the network (2) at the sources

$$u_i(k+1) = \frac{\beta_i}{\mu(\bar{k})} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(k)} - \Gamma_{i,i}\right) u_i(k) \tag{4}$$

where β_i and a_i are design parameters and k is the iteration time step. The parameter $\mu(\bar{k})$ is the channel price. The control algorithm (4) converges to the Nash equilibrium (3) if

$$a_i > \sum_{j \neq i} \Gamma_{i,j} \tag{5}$$

2.3 Link Algorithm

The link algorithm is computed every K iterations of the control algorithm (4). The value K is a large number such that the link algorithm is updated infrequently with respect to the control algorithm (ie. K=100). The link algorithm Pavel (2007) is:

$$\mu(\bar{k}+1) = [\mu(\bar{k}) + \nu(\sum_{j=1} u_j(K) - P_0)]^+$$

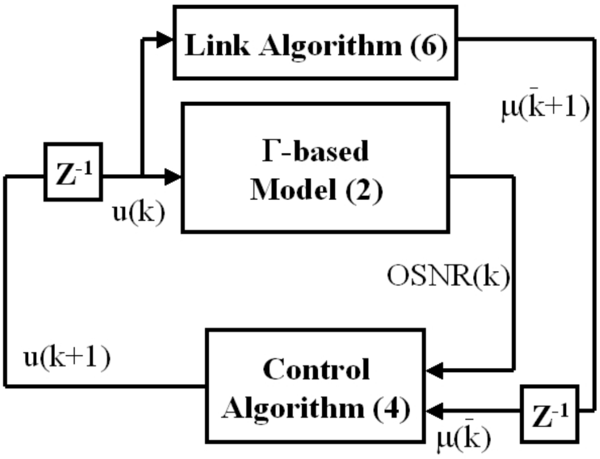

Fig. 1. Discrete-time Control Algorithm (4) and Link Algorithm (6) applied to OSNR system (2)

where ν is the step-size and $[z]^+ = max\{z, 0\}$. The variable $\bar{k}$ represents the discrete-time variable which is on a different time-scale than the control algorithm (4). Under normal operations, $\mu(\bar{k})$ must be greater than zero because it enters (4) inverted. Thus, similarly to Liu et al. (2003), we drop $[\cdot]^+$ in the ensuing analysis to obtain

$$\mu(\bar{k}+1) = \mu(\bar{k}) + \nu(\sum_{j=1} u_j(K) - P_0) \tag{6}$$

Notice that $\sum_{j=1} u_j(K)$ is the received total power for all channels in the link. This is easily measured in practice as a real-time value. Moreover, under the assumption of stationary channel powers, which can be made given that the link algorithm updates very slowly, the link algorithm corresponds to a gradient descent technique. The combined channel-link algorithm converges to the optimal NE channel power and price (u^*, μ^*) Pavel (2007).

Figure 1 depicts the control algorithm (4) and the link algorithm (6) acting on the OSNR system (2). Despite having a convergent control algorithm for the OSNR optimization problem, the practical implementation of such a control law has to take time-delay into account.

3. CONTINUOUS-TIME CLOSED LOOP SYSTEM

We first generalize the control algorithm (4) by introducing a control gain, ρ_i, $0 < \rho_i \leq 1$, for each channel i. We convert the control algorithm (4) and the link algorithm (6) into the continuous-time domain. Then we formulate the closed loop system by applying the primal-dual controller Pavel (2007) to the new model.

We introduce a tuning parameter at each source, ρ_i, into (4) to act as a weighting between the RHS of (4) and $u_i(k)$

$$u_i(k+1) - u_i(k) = \rho_i \left\{ \frac{\beta_i}{\mu(\bar{k})} - \frac{1}{a_i} \left(\frac{1}{OSNR_i(k)} - \Gamma_{i,i} + a_i \right) u_i(k) \right\} \tag{7}$$

Note that $\rho_i = 1$ gives (4). Approximating the LHS of (7) by $\frac{du_i(t)}{dt_f}$, where t_f denotes the "fast" time variable, we can write a continuous-time version of the control algorithm (7) as

$$\frac{du_i(t)}{dt_f} = \rho_i \left\{ \frac{\beta_i}{\mu(\bar{k})} - \frac{1}{a_i} \left(\frac{1}{OSNR_i(t)} - \Gamma_{i,i} + a_i \right) u_i(t) \right\} \tag{8}$$

Substituting (2) into (8), yields the closed loop system

$$\frac{du_i(t)}{dt_f} = \rho_i \left\{ \frac{\beta_i}{\mu(\bar{k})} - \frac{n_{0,i} + \Sigma_j \Gamma_{i,j} u_j}{a_i} + \left(\frac{\Gamma_{i,i}}{a_i} - 1 \right) u_i \right\} \tag{9}$$

Similarly, the dual control law at the link (6) can be rewritten in the continuous-time domain as

$$\dot{\mu}(t) = \nu(\sum_{j=1} u_j(t) - P_0) \tag{10}$$

We can relate the link algorithm time variable, t, to t_f according to the relation $t_f = tK$. Let $\epsilon = \frac{1}{K}$. Then from (9) and (10), along with the relation

$$\frac{du_i}{dt_f} = \frac{du_i}{dt} \frac{dt}{dt_f} = \epsilon \frac{du_i}{dt} \tag{11}$$

the closed loop, time-scale decoupled system in the standard singular perturbation form is given as

$$\dot{\mu}(t) = \nu(\sum_{j=1} u_j(t) - P_0) \tag{12}$$

$$\epsilon \dot{u}_i = \rho_i \left\{ \frac{\beta_i}{\mu(t)} - \frac{n_{0,i} + \Sigma_j \Gamma_{i,j} u_j}{a_i} + \left(\frac{\Gamma_{i,i}}{a_i} - 1 \right) u_i \right\} \tag{13}$$

The equilibrium point, (u^*, μ^*), of the closed loop system (13) and (12) is obtained by setting the dynamics equal to 0. The equation (13) gives the unique Nash equilibrium u^* with components u_i^* as in (3) and $\mu(k) = \mu^*$. We have the added expression from (12) that gives

$$\sum_j u_j^* = P_0 \tag{14}$$

Shifting the system around the equilibrium point (3) and (14), using the change of variables $\tilde{u}_i = u_i - u_i^*$ and $\tilde{\mu} = \mu - \mu^*$, we obtain the closed loop system

$$\begin{aligned} \dot{\mu}(t) &= \nu \sum_j u_j \\ \epsilon \dot{u}_i(t) &= -\frac{\rho_i}{a_i} \left(\sum_j \tilde{\Gamma}_{i,j} u_j(t) \right) + \rho_i \beta_i \left(\frac{1}{\mu + \mu^*} - \frac{1}{\mu^*} \right) \end{aligned} \tag{15}$$

where for simplicity we denote by (u_i, μ) the shifted variable $(\tilde{u}_i, \tilde{\mu})$. The variable $\tilde{\Gamma}_{i,j}$ is defined as

$$\tilde{\Gamma}_{i,j} = \begin{cases} a_i, & i = j \\ \Gamma_{i,j}, & i \neq j \end{cases} \tag{16}$$

4. TIME-DELAY SYSTEM

The closed loop system (15) does not take time-delay into account. Forward time-delay occurs from the channel sources u_j to the OSNR outputs $OSNR_i$ at the end of the link. We adopt the notation that $\tau_f \geq 0$ represents the forward time-delay. Similarly, the backward time-delay occurs from the OSNR outputs $OSNR_i$ back to its associated source u_i. We denote this time-delay as τ_b. We denote $\tau = \tau_f + \tau_b$ as the total round trip delay. We modify (2), (8) and (10) to include time-delay.

We explicitly introduce forward time-delay into (2) as

$$OSNR_i(t) = \frac{u_i(t-\tau_f)}{n_{0,i} + \Sigma_{j\in M}\Gamma_{i,j}u_j(t-\tau_f)} \tag{17}$$

so that $OSNR_i(t)$ depends on delayed input signals.

The algorithm (8) applied at the sources (Tx) will use a delayed version of (17), $OSNR_i'(t)$, due to the backward time-delay, $OSNR_i'(t) = OSNR_i(t-\tau_b)$, i.e.,

$$OSNR_i'(t) = \frac{u_i(t-\tau)}{n_{0,i} + \Sigma_{j\in M}\Gamma_{i,j}u_j(t-\tau)} \tag{18}$$

Substituting (18) into (8), and using (11), gives

$$\epsilon\frac{du_i(t)}{dt} = \rho_i\left\{\frac{\beta_i}{\mu(t)} - \frac{1}{a_i}\left(\frac{n_{0,i} + \sum_j \Gamma_{i,j}u_j(t-\tau)}{u_i(t-\tau)} - \Gamma_{i,i} + a_i\right)u_i(t)\right\} \tag{19}$$

Since (19) derives from our control algorithm, we can modify (19) by noticing that if we keep a record of past power inputs, $u_i(t)$, we can eliminate $u_i(t-\tau)$ in the denominator by design. Then appropriately, we obtain the modified algorithm

$$\epsilon\frac{du_i(t)}{dt} = \rho_i\left\{\frac{\beta_i}{\mu(t)} - \frac{1}{a_i}\left(n_{0,i} + \sum_j \Gamma_{i,j}u_j(t-\tau)\right) + \left(\frac{\Gamma_{i,i}}{a_i} - 1\right)u_i(t)\right\} \tag{20}$$

We must still add the link dynamics (10) along with any associated time-delays. We can directly represent the time-delay in (10) as a forward time-delay on the input powers

$$\dot{\mu}(t) = \nu(\sum_{j=1} u_j(t-\tau_f) - P_0) \tag{21}$$

Finally, since the value μ from (21) must propagate from the links to the sources, a backward time-delayed value is considered in (20) as

$$\epsilon\dot{u}_i(t) = \rho_i\left\{\frac{\beta_i}{\mu(t-\tau_b)} - \frac{1}{a_i}\left(n_{0,i} + \sum_j \Gamma_{i,j}u_j(t-\tau)\right) + \left(\frac{\Gamma_{i,i}}{a_i} - 1\right)u_i(t)\right\} \tag{22}$$

Thus, (21) and (22) represent the closed loop, time-delayed interconnected model that is used to study the stability of the network system.

Shifting (21) and (22) about the equilibrium point (u^*, μ^*) as defined in (14) and (3) results in

$$\dot{x}(t) = \nu(\sum_{j=1} z_j(t-\tau_f)) \tag{23}$$

$$\epsilon\frac{dz_i(t)}{dt} = \rho_i\left\{\frac{\beta_i}{x(t-\tau_b)+\mu^*} - \frac{\beta_i}{\mu^*} - \sum_j \frac{\Gamma_{i,j}}{a_i}z_j(t-\tau) + \left(\frac{\Gamma_{i,i}}{a_i} - 1\right)z_i(t)\right\} \tag{24}$$

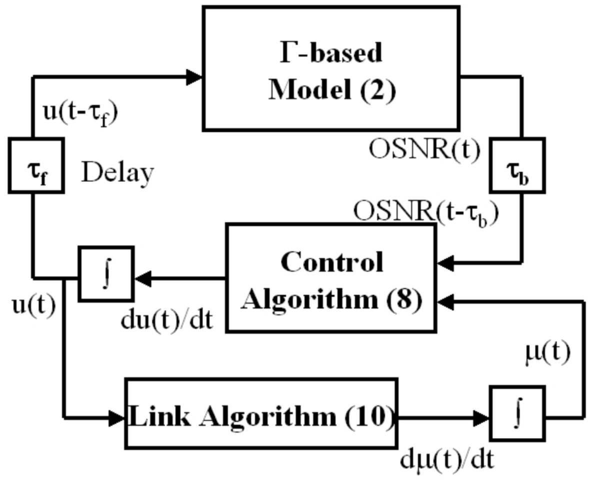

Fig. 2. Continuous-time Control Algorithm (8) and Link Algorithm (10) at the sources

where the shifted variables are denoted z and x.

Notice that if the time-delays τ, τ_f and τ_b in (24) and (23) were all equal to zero, we would get the no delay system (15). Recall that the system condition for stability in the no delay case is (5), where a_i are design parameters.

An interesting observation is made by examining the equilibrium point of (24). We obtain the non-causal functional equation, in matrix form

$$z(t) = \tilde{\Gamma}_a^{-1}\beta\left(\frac{1}{x(t+\tau_f)+\mu^*} - \frac{1}{\mu^*}\right) \tag{25}$$

where $\tilde{\Gamma}_a$ is a matrix with elements $\frac{\Gamma_{i,j}}{a_i}$, $\Gamma_{i,j}$ defined as (16), β is a column matrix with elements β_i and z is the column matrix with z_i elements. Notice that $\tilde{\Gamma}_a$ is invertible because of the diagonal dominance condition (5).

To avoid the non-causality of (25) as well as to simplify the time-delay analysis of the system, we rewrite (24) and (23) in an equivalent form. This involves shifting the link control algorithm to the sources, rather than at the links, as shown in Figure 2. This is equivalent to assuming that all propagation delay is concentrated on the feedback path of the closed loop system.

The resultant closed loop, time-delayed system is

$$\dot{x}(t) = \nu(\sum_{j=1} z_j(t)) \tag{26}$$

$$\epsilon\frac{dz_i(t)}{dt} = \rho_i\left\{\frac{\beta_i}{x(t-\tau)+\mu^*} - \frac{\beta_i}{\mu^*} - \frac{1}{a_i}\sum_j \tilde{\Gamma}_{i,j}z_j(t-\tau)\right\} \tag{27}$$

By moving the channel price algorithm to the sources, we considerably simplify the problem since (27) has one time-delay now and (26) has no delay whatsoever. Equations (26) and (27) are in a form suitable to apply singular perturbation theory. Figure 2 shows the block diagram with the link dynamics at the sources.

5. MAIN RESULT

We use Lyapunov-Razumikhin theory and singular perturbation approach to study the stability of the delayed closed loop system (26) and (27). The main result gives sufficient conditions for stability.

We first review the notions of reduced and boundary-layer systems. Re-write (26) and (27) as

$$\dot{x} = f(z) \tag{28}$$

$$\epsilon \dot{z} = g(x(t-\tau), z(t-\tau)) \tag{29}$$

***Definition** 1.* Consider the system (28) and (29). Let the reduced system be defined as

$$\dot{x} = f(h(x))$$

and the boundary system be defined as

$$\frac{dy}{dt_f} = g(x(t-\tau), y(t-\tau) + h(x(t-\tau)))$$

where $h(x)$ is the isolated root from the RHS of (29), t_f is defined as in (11), and $y = z - h(x)$ is a co-ordinate shift.

Using Definition 1, the reduced and boundary-layer systems of (26) and (27) are

$$\dot{x} = \nu 1_{row} \tilde{\Gamma}_a^{-1} \beta \left(\frac{1}{x+\mu^*} - \frac{1}{\mu^*} \right) \tag{30}$$

$$\frac{dy}{dt_f} = -\rho \tilde{\Gamma}_a y(t-\tau) \tag{31}$$

where 1_{row} is a row vector of 1 elements, and

$$h(x) = \tilde{\Gamma}_a^{-1} \beta \left(\frac{1}{x(t)+\mu^*} - \frac{1}{\mu^*} \right) \tag{32}$$

Notice that the reduced system (30) is a nonlinear, single input, scalar system with no time-delay. Also, the boundary-layer system (31) is a linear, time-delayed system. This will greatly reduce the ensuing analysis.

The following lemmas are used to prove the main theorem.

***Lemma** 2.* The boundary-layer system (31) is exponentially stable given that we satisfy

$$\tau < \frac{\underline{\sigma}\left(\rho\tilde{\Gamma}_a + \tilde{\Gamma}_a^T \rho\right)}{2\sqrt{\bar{\sigma}\left((\rho\tilde{\Gamma}_a)^2((\rho\tilde{\Gamma}_a)^2)^T\right)}} \tag{33}$$

where $\tilde{\Gamma}_a$ is defined as in (25), $\rho = diag(\rho_i)$ and $\bar{\sigma}(\cdot)$,$\underline{\sigma}(\cdot)$ denote the largest and smallest singular value Zhou and Doyle (1998).

***Lemma** 3.* The reduced system (30) is exponentially stable.

***Theorem** 1.* For the singularly perturbed system (26) and (27), there exists $\epsilon^* > 0$ such that for $0 < \epsilon < \epsilon^*$, the origin is exponentially stable given that (33) is satisfied.

Proof We present here a short version of the proof due to space limitations. The full proof can be found in Stefanovic and Pavel. We use the singular perturbation approach in Theorem 11.4 Khalil (2002), but modified to handle time-delays. It is easy to show that $f(0) = 0$, $g(0,0) = 0$, and $h(0) = 0$. Also, f, g and h are sufficiently smooth.

The basic approach to prove that the origin of (26) and (27) is exponentially stable, is to simplify the problem by considering only the reduced (30) and boundary-layer (31) systems. By lemmas 3 and 2 we know that the reduced (30) and boundary-layer (31) systems are exponentially stable. We attempt to find a composite Lyapunov function for (26) and (27) based on the Lyapunov functions for the reduced and boundary-layer systems.

Since (30) is exponentially stable, by the converse Lyapunov theorem Khalil (2002), the Lyapunov function $V(x) = \frac{1}{2}x^2$ applies to the reduced system. For the boundary-layer system, we pick the Lyapunov function $W(y) = y^T y$. Notice from Lemma 2, this choice of Lyapunov function is used to prove time-delay stability for the boundary-layer system, given that (33) is satisfied. We can then define the composite Lyapunov function as $\eta(x,y) = V(x) + W(y) = \frac{1}{2}x^2 + y^T y$.

Since f and g are not functions of ϵ, they are immediately Lipschitz in ϵ linearly. Also, we satisfy

$$||f(y+h(x)) - f(h(x))|| = ||\nu 1_{row} y|| \le \nu\sqrt{m}||y||_2 \tag{34}$$

$$||f(h(x))|| \le \frac{\nu|1_{row}\tilde{\Gamma}_a^{-1}\beta|}{\mu^*(r+\mu^*)}||x||_2 \tag{35}$$

$$\begin{aligned} ||\frac{\partial h(x)}{\partial x}|| &= || - \tilde{\Gamma}_a^{-1}\beta \frac{1}{(x+\mu^*)^2}|| \\ &\le ||\tilde{\Gamma}_a^{-1}\beta||_2 \frac{1}{(\mu^*-r)^2} \le k_2 \end{aligned} \tag{36}$$

for $x \in [-r, r]$, where $r \in (-\mu^*, \mu^*)$, m is the size of y, and $k_2 > 0$ a constant.

By applying the composite Lyapunov function, $\eta(x,y)$, to (27) and (26), with co-ordinate shift $y = z - h(x)$, and exploiting Lyapunov inequalities, Lipschitz properties, and (34)-(36), we can prove exponential stability.

End of Proof

6. SIMULATIONS

In the following we validate the results of Theorem 1 through simulations. For the upcoming simulations, we will use the following design. For the a_i parameters, we choose $a_i = \Gamma_{i,i}$ if $\Gamma_{i,i} > \sum_{j\neq i}\Gamma_{i,j}$, otherwise, we pick $a_i = 1.2 * \sum_{j\neq i}\Gamma_{i,j}$, where 1.2 is an arbitrary multiplying factor. We choose a parabolic gain shape for the optical amplifiers according to the formula $G = -4e16 \times (\lambda - 1555 \times 10^{-9})^2 + 15$ decibels, where λ denotes a channel wavelength. Normally, a game-theoretic controller would achieve the best possible OSNR values for the channels without any guarantees. Here, the β_i values are chosen by a centralized algorithm to achieve a minimum OSNR target of $23dB$. This allows easier verification for the simulations.

Finally, we simulate time-delay as realistically as follows. Since light in the fiber optic cable propagates at approximately $v = 200,000km/s$, we can compute the round-trip propagation time across one optical span of $100km$ to be $1ms$. In addition, we can assume for the purpose of these simulations that the channel algorithms are updated every $10ms$, to be specified. We assume ten optical spans for a total of $10ms$ delay.

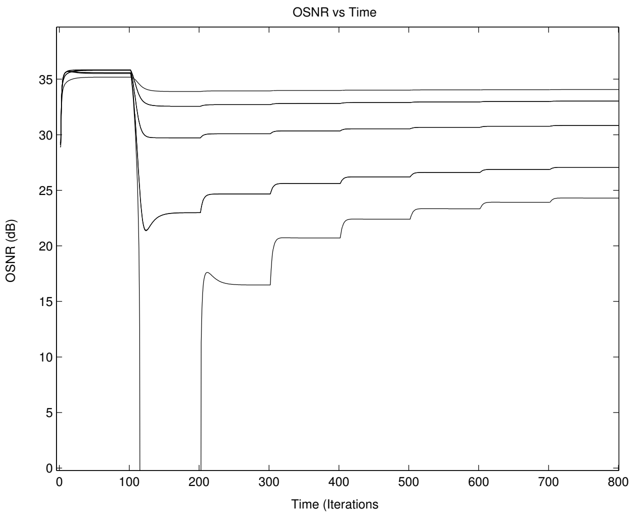

Fig. 3. $\mu = 0.25 \times \mu^*$ with $\rho_i = 0.1$

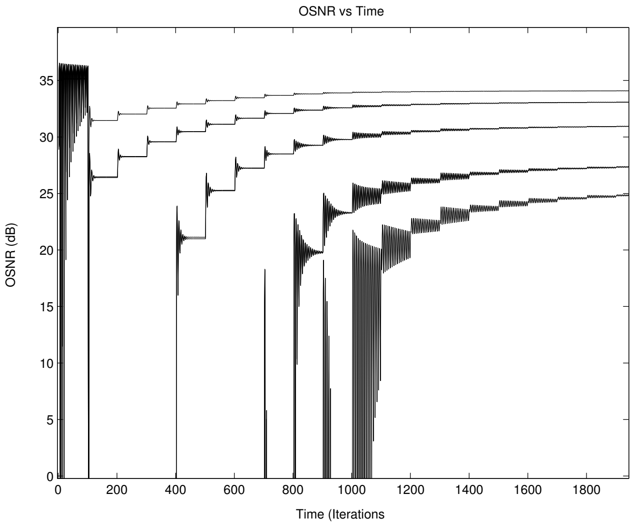

Fig. 4. $\mu = 0.25 \times \mu^*$ with $\rho_i = 0.54$

We pick $\mu = 0.25 \times \mu^* = 66.25$ as an initial condition for simulation. Figures 3-4 depict the cases for $\rho_i = 0.1$ and $\rho_i = 0.54$ for all i. The case $\rho_i = 0.55$ generates an unstable system response and is omitted. For the cases $\rho_i = 0.1$, $\rho_i = 0.54$ and $\rho_i = 0.55$, the calculated time bounds are $\tau < 25ms$, $\tau < 4.6ms$ and $\tau < 4.5ms$, respectively. Notice that $\rho_i = 0.54$ calculates a time bound of $4.6ms$, yet the system is still stable for $10ms$. This implies the time bound is conservative.

7. CONCLUSIONS AND FUTURE WORK

In this paper, we have studied the effects of time-delay on a primal-dual control algorithm for optical communications. We derived sufficient conditions for stability under any time-delays. We reconfigured the control algorithm and the link update law from Stefanovic and Pavel (2007) and Pavel (2007) to update at the sources. We also found an upper bound for the time-delay that closely predicted when instability would occur. A set of simulations validated the resultant time-delayed model and its stability predictions.

Future work would involve extending the multi-source single link network above to include any general network configuration with multiple time-delays. We can also improve the time-delay theory to utilize the more general Lyapunov-Krasovskii functionals rather than the simplified Lyapunov-Razumikhin theory.

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