Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

# Consensus Based Overlapping Decentralized Estimation With Missing Observations and Communication Faults

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**Abstract:** In this paper a new algorithm for discrete-time overlapping decentralized state estimation of large scale systems is proposed in the form of a multi-agent network based on a combination of local Kalman filters and a dynamic consensus strategy, assuming intermittent observations and communication faults. Conditions are derived for the algorithm to provide, under general conditions concerning the agent resources and the network topology, asymptotic stability in the sense of bounded mean-square estimation error. It is also demonstrated how the consensus gains can be chosen by minimizing the total steady-state mean-square estimation error. Numerical examples illustrate some properties of the proposed algorithm.

## 1. INTRODUCTION

A great deal of attention has been paid to the problem of *decentralized state estimation* of complex systems. Under this term one can consider different structures that are either totally decentralized, partially decentralized, or hierarchical. The key requirement is that the large scale system is modelled as an interconnection of subsystems, and that each subsystem has a decision maker (intelligent agent) associated with it. Depending on the available resources, an agent might have access to different information, such as the sensor characteristics, properties and models of the system and its environment and communication channels between the agents. Attempts to provide an insight into the principles and structures for decentralized estimation can be found in *e.g.* Sanders et al. [1974, 1978], Šiljak [1991], Speranzon et al. [2006], Tacker and Sanders [1980]. It should be noticed, however, that none of the existing methodologies is able to provide a systematic and general way of designing communication strategy between the agents without recurring to a strong fusion center. Also, the important problems of intermittent observations and lossy networks have not been treated in this context.

As early as in the 1980s, important results were obtained in the area of distributed asynchronous iterations in parallel computation and distributed optimization (*e.g.* Bertsekas and Tsitsiklis [1989], Tsitsiklis [1984], Tsitsiklis et al. [1986]). Also, a very intensive research has been carried out recently in the fields of multi-agent systems and sensor networks, including numerous applications (see, *e.g.* Fax and Murray [2004], Jadbabaie et al. [2003], Lin et al. [2005], Moreau [2005], Olfati-Saber and Murray [2004], Ren and Beard [2005], Ren et al. [2005]). The last references have a common methodology: they all use some kind of *agreement* or *consensus* strategy between the agents. The decentralized state estimation problem itself is deeply embedded in this line of thought either implicitly, through the very definition of the consensus algorithms (*e.g.*, see Ren et al. [2005]), or explicitly, where the dynamic consensus strategy between multiple agents is used for obtaining estimates (on the basis of averaging) of the quantities used subsequently for generating optimal parameter or state estimates (*e.g.*, see Olfati-Saber [2005], Xiao and Boyd [2004]). However, none of the mentioned schemes is aimed at establishing any type of real-time collaboration between the local estimators in the overlapping decentralized estimation problem.

In this paper a novel state estimation algorithm for complex linear discrete-time systems is proposed based on: (1) overlapping system decomposition and implementation of local state estimators by intelligent agents according to their sensing and computing resources; (2) application of a consensus strategy providing the global state estimates to all the agents in the network; (3) taking into account influence of intermittent observations and communication faults. The organization of the paper is as follows. The main definition of the problem is given in Section 2. In Section 3 the proposed estimation algorithm is described. The algorithm can be considered as a discrete-time version of the state estimation algorithm proposed in Stanković et al. [2007a, 2008], or an extension to the state estimation problem of the algorithm proposed in Stanković et al. [2007b], structurally resembling to the distributed computation algorithm proposed in Tsitsiklis [1984], Tsitsiklis et al. [1986]. In Section 4, stability of the proposed scheme is analyzed. Starting from a specially defined matrix norm, sufficient conditions for the convergence of the estimates in the mean and in the sense of preserving boundedness of the mean-square estimation error are derived for the general case of intermittent observations and lossy networks using the methodology of Sinopoli et al. [2004], Nilsson [1996]. The next section provides a strategy aimed at obtaining the gains of the consensus scheme on the basis of steady-state mean-square error minimization. Two simple numerical examples serve to illustrate some characteristic properties of the proposed estimation algorithm.

## 2. OVERLAPPING DECENTRALIZED ESTIMATION

Let a discrete-time stochastic system be represented by

$$\mathbf{S}: \quad x(t+1) = Fx(t) + Ge(t), \qquad y(t) = Hx(t) + v(t) \tag{1}$$

where $t$ is the discrete-time instant, $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_p)^T$, $e = (e_1, \ldots, e_m)^T$ and $v = (v_1, \ldots, v_p)^T$ are its state, output, input and measurement noise vectors, respectively, while $F$, $G$ and $H$ are constant $n \times n$, $n \times m$ and $p \times n$ matrices, respectively; $\{e(t)\}$ and $\{v(t)\}$ are white zero-mean sequences of independent vector random variables with covariance matrices $Q$ and $R$.

We shall consider the problem of *decentralized estimation* of the state $x$ of $\mathbf{S}$, in which $N$ *autonomous agents* generate the estimates on the basis of: (1) locally available measurements; (2) specific *a priori* knowledge they possess about the system model; and (3) real-time communication between the agents.

Formally, we shall assume that the $i$-th agent has a possibility to observe the $p_i$-dimensional vector $y^{(i)} = (y_{l_1^i}, \ldots, y_{l_{p_i}^i})^T$, composed of the set of components of $y$ with indices contained in the agent's *output index set* $I_i^y = \{l_1^i, \ldots, l_{p_i}^i\}$, $l_1^i, \ldots, l_{p_i}^i \in \{1, \ldots p\}$, $l_1^i < \ldots < l_{p_i}^i$, $p_i \leq p$. Vector $y^{(i)}$ can be represented by

$$y^{(i)}(t) = H^{(i)} x^{(i)}(t) + v^{(i)}(t), \tag{2}$$

where $x^{(i)}$ is an $n_i$-dimensional vector composed of the components of $x$ selected by the agent's *state index set* $I_i^x = \{j_1^i, \ldots, j_{n_i}^i\}$, $j_1^i, \ldots, j_{n_i}^i \in \{1, \ldots n\}$, $j_1^i < \ldots < j_{n_i}^i$, $n_i \leq n$, $H^{(i)}$ is a constant $p_i \times n_i$ matrix and $v^{(i)}$ a measurement noise vector containing the components of $v$ selected by $I_i^y$, having covariance matrix $R^{(i)}$ (which can be readily obtained from $R$). Notice that we have then $H^{(i)} x^{(i)} = H_i x$, where $H_i$ is a $p_i \times n$ matrix composed of the entire rows of $H$ selected by $I_i^y$. Definition of the vector $x^{(i)}$ leads further to the definition of an $n_i \times n_i$ matrix $F^{(i)}$ which contains the components of $F$ selected by the pairs of indices defined by $I_i^x \times I_i^x$. In an analogous way we can obtain matrix $G^{(i)}$, composed of $n_i$ rows of matrix $G$ selected by $I_i^x$. Consequently, the *local models* of $\mathbf{S}$ (or of its parts) available to the agents are defined by

$$\mathbf{S_i}: \quad x^{(i)}(t+1) = F^{(i)} x^{(i)}(t) + G^{(i)} e(t), \qquad y^{(i)}(t) = H^{(i)} x^{(i)}(t) + v^{(i)}(t), \tag{3}$$

$i = 1, \ldots, N$; dynamic systems $\mathbf{S_i}$ represent *overlapping subsystems* of $\mathbf{S}$ (Stanković and Šiljak [2001]).

Starting from the model $\mathbf{S_i}$ and the accessible measurements $y^{(i)}$, each agent is able to generate autonomously its own local estimate $\hat{x}^{(i)}$ of the vector $x^{(i)}$ using an estimator which can be designed on the basis of (3). Having in mind the nature of $\mathbf{S}$, the following local steady-state Kalman filters will be assumed to be implemented by each agent:

$$\bar{\mathbf{E}}_\mathbf{i}: \quad \hat{x}^{(i)}(t+1|t) = F^{(i)} \hat{x}^{(i)}(t|t-1) + \gamma_i(t) F^{(i)} L^{(i)} [y^{(i)}(t) - H^{(i)} \hat{x}^{(i)}(t|t-1)], \tag{4}$$

where $L^{(i)}$ is the steady state Kalman gain given by $L^{(i)} = P^{(i)} H^{(i)T} [H^{(i)} P^{(i)} H^{(i)T} + R^{(i)}]^{-1}$, $P^{(i)}$ is a solution of the algebraic Riccati equation

$$P^{(i)} = F^{(i)} [P^{(i)} - L^{(i)} H^{(i)} P^{(i)}] F^{(i)T} + G^{(i)} Q G^{(i)T}, \tag{5}$$

while $\gamma_i(t)$ is a scalar equal to 1 when the $i$-th agent receives measurements $y^{(i)}$, and 0 otherwise. We shall assume that the pairs $(F^{(i)}, G^{(i)} Q^{\frac{1}{2}})$ are stabilizable and the pairs $(F^{(i)}, H^{(i)})$ detectable, so that the matrices $F^{(i)} - L^{(i)} H^{(i)}$ are asymptotically stable and $P^{(i)} > 0$, $i = 1, \ldots, N$, (Anderson and Moore [1979], Sinopoli et al. [2004]).

Overlapping decentralized estimators defined by (4) provide a set of *overlapping estimates* $\hat{x}^{(i)}$. If the final goal is to get an estimate $\hat{x}$ of the whole state vector $x$ of $\mathbf{S}$, different strategies can be added to the local estimators (*e.g.*, see Sanders et al. [1974, 1978], Speranzon et al. [2006], Tacker and Sanders [1980], Šiljak [1991]). However, all such approaches require a kind of centralized strategy or special, model dependent communications for obtaining $\hat{x}$ by each agent.

## 3. CONSENSUS-BASED ESTIMATOR

Our task is to formulate a scheme which would provide to all the agents in the network reliable estimates of the whole state vector $x$ on the basis of the local estimates $\hat{x}^{(i)}$ and a decentralized communication strategy uniform for all the nodes, in spite of missing measurements and communication faults. We propose the following algorithm based on the introduction of a first-order *consensus scheme*:

$$\mathbf{E_i}: \quad \xi_i(t|t) = \xi_i(t|t-1) + \gamma_i(t) L_i [y^{(i)} - H_i \xi_i(t|t-1)], \qquad \xi_i(t+1|t) = \sum_{j=1}^{N} C_{ij}(t) F_j \xi_j(t|t) \tag{6}$$

$i = 1, \ldots, N$, where $\xi^i$ is an estimate of $x$ generated by the $i$-th agent, $F_i$ is an $n \times n$ matrix with $n_i \times n_i$ nonzero elements that are equal to those of $F^{(i)}$, but are placed at the indices defined by $I_i^x \times I_i^x$, $L_i$ is an $n \times p_i$ matrix obtained similarly as $F_i$, in such a way that its nonzero elements are those of $L^{(i)}$ placed row by row at row-indices defined by $I_i^x$. We shall assume that $C_{ij}(t)$ are $n \times n$ time-varying gain matrices defining communications between the nodes, given in the form $C_{ij}(t) = k_{ij}(t) K_{ij}$, where $k_{ij}(t) = 1$ when the directed communication from the node $j$ to the node $i$ exists, and $k_{ij}(t) = 0$ otherwise; $K_{ij}$ are diagonal matrices with nonnegative elements, giving appropriate weights to the communicated estimates. Further, we shall assume that $\{k_{ij}(t)\}$, $i, j = 1, \ldots, N, i \neq j$, are mutually independent scalar sequences of independent binary random variables, satisfying $P\{k_{ij}(t) = 1\} = p_{ij}$ and $P\{k_{ij}(t) = 0\} = 1 - p_{ij}$ for $i \neq j$, and that that $k_{ii}(t) = 1$, $i = 1, \ldots, N$; the

$N \times N$ matrix $K(t) = [k_{ij}(t)]$ determines connections between the agents at time $t$. Also, we shall assume that $\{\gamma_i(t)\}$ is a sequence of independent binary random variables independent of $\{k_{ij}(t)\}$, $i,j = 1,\dots,N, i \neq j$, such that $P\{\gamma_i(t) = 1\} = p_{ii}$ and $P\{\gamma_i(t) = 0\} = 1 - p_{ii}$. Furthermore, we shall introduce a random vector $\Xi_t$ composed of $N^2$ binary components: $N(N-1)$ elements $k_{ij}(t)$ $(i,j = 1,\dots,N, i \neq j)$ and $N$ elements $\gamma_i(t)$. This vector is, by assumption, generated on the basis of Bernoulli trials, *i.e.*, $\{\Xi_t\}$ represents a sequence of independent random vectors; let $\pi_r$ be the probabilities of all possible realizations $\Xi^{[r]}$ of $\Xi_t$, $r = 1,\dots,\nu$, $\nu = 2^{N^2}$. Define an $nN \times nN$ "consensus matrix" $\tilde{C}(t) = [C_{ij}(t)]$, $i,j = 1,\dots,N$, and assume that it is *row-stochastic* for all $t$. Introduce also $\Phi_i(t) = F_i - \gamma_i(t)L_iH_i$, $\tilde{F}_E = \mathrm{diag}\{F_1,\dots,F_N\}$ and $\tilde{\Phi}(t) = \mathrm{diag}\{\Phi_1(t),\dots,\Phi_N(t)\}$, as well as $\tilde{A}(t) = \tilde{C}(t)\tilde{\Phi}(t)$. Denote by $\tilde{A}_{[r]}$, $\tilde{C}_{[r]}$ and $\tilde{\Phi}_{[r]}$ realizations of $\tilde{A}(t)$, $\tilde{C}(t)$ and $\tilde{\Phi}(t)$ resulting from different realizations $\Xi^{[r]}$ of $\Xi_t$, $r = 1,\dots,\nu$ .

It is possible to observe that the algorithm is based on a combination of: a) *decentralized overlapping estimators* represented by (4), and b) a *the consensus scheme* tending to make the local estimates $\xi^i$ as close as possible (*e.g.*, see Tsitsiklis et al. [1986], Tsitsiklis [1984], Fax and Murray [2004], Jadbabaie et al. [2003], Lin et al. [2005], Moreau [2005], Olfati-Saber and Murray [2004], Ren and Beard [2005], Ren et al. [2005]). The algorithm reduces to the local estimators when the "consensus part" is eliminated ($K_{ij} = 0$). The "consensus part" alone asymptotically provides $\xi^i = \xi$ under a proper choice of the matrices $C_{ij}(t)$, where $\xi$ is a weighted sum of the *a priori* estimates $\xi(t_0)$, and $t_0$ the initial time instant (Ren and Beard [2005], Ren et al. [2005]). Notice that the estimator reminds structurally of the discrete-time distributed optimization algorithm proposed in Tsitsiklis et al. [1986], Tsitsiklis [1984], Bertsekas and Tsitsiklis [1989]: it performs "computation" by evaluating the term $L_i(y^{(i)} - C_i\xi^i)$ and forces the "agreement" between the agents by forming a linear combination of the available local predictions $F_j\xi_j(t|t)$ communicated between the agents. The proposed state estimator represents a discrete-time version of the estimator proposed in Stanković et al. [2007a, 2008] and a generalization of the parameter estimator based on stochastic approximation proposed in Stanković et al. [2007b].

Introducing $\hat{X}(t|t) = \mathrm{vec}\{\xi_1(t|t),\dots,\xi_N(t|t)\}$ and $\hat{X}(t+1|t) = \mathrm{vec}\{\xi_1(t+1|t),\dots,\xi_N(t+1|))\}$, we can obtain a compact formulation of the proposed algorithm

$$\begin{aligned}\hat{X}(t|t) &= \hat{X}(t|t-1) + \tilde{\Gamma}(t)\tilde{L}[Y(t) - \tilde{H}\hat{X}(t|t-1)]\\ \hat{X}(t+1|t) &= \tilde{C}(t)\tilde{F}_E\hat{X}(t|t),\end{aligned} \quad (7)$$

where $Y(t) = \mathrm{vec}\{y^{(1)}(t),\dots,y^{(N)}(t)\}$, $\tilde{L} = \mathrm{diag}\{L_1, \dots, L_N\}$, $\tilde{\Gamma}(t) = \mathrm{diag}\{\gamma_1(t),\dots,\gamma_N(t)\}$ and $\tilde{H} = \mathrm{diag}\{H_1,\dots,H_N\}$. Further, for the prediction error $\varepsilon(t+1|t) = \hat{X}(t+1|t) - X(t+1)$, we obtain

$$\begin{aligned}\varepsilon(t+1|t) = \tilde{A}(t)\varepsilon(t|t-1) + \tilde{C}(t)(\tilde{F}_E - \tilde{F})X(t) +\\ +\tilde{C}(t)\tilde{\Gamma}(t)\tilde{L}\tilde{H}V(t) - E(t),\end{aligned} \quad (8)$$

where $\tilde{F} = \mathrm{diag}\{F,\dots,F\}$, $V(t) = \mathrm{vec}\{v^{(1)}(t),\dots,v^{(N)}(t)\}$ and $E(t) = \mathrm{vec}\{e(t),\dots,e(t)\}$. Consequently, we obtain the following state space system-estimator model:

$$\begin{aligned}Z(t+1) = &\left[\begin{array}{c:c}\tilde{F} & 0\\ \hdashline \tilde{C}(t)(\tilde{F}_E - \tilde{F}) & \tilde{A}(t)\end{array}\right] Z(t) +\\ &+ \left[\begin{array}{c:c}\tilde{G} & 0\\ \hdashline -\tilde{G} & \tilde{C}(t)\tilde{\Gamma}(t)\tilde{L}\tilde{H}\end{array}\right] N(t),\end{aligned} \quad (9)$$

where $Z(t) = \mathrm{vec}\{X(t),\varepsilon(t|t-1)\}$ and $N(t) = \mathrm{vec}\{E(t), V(t)\}$. Applying the mathematical expectation on both sides of (9), we obtain for $\bar{Z}(t) = E\{Z(t)\}$ the recursion

$$\bar{Z}(t+1) = \sum_{r=1}^{\nu} \pi_i B_{[r]}\bar{Z}(t), \quad (10)$$

where $\tilde{B}_{[r]} = \left[\begin{array}{c:c}\tilde{F} & 0\\ \hdashline \tilde{C}_{[r]}(\tilde{F}_E - \tilde{F}) & \tilde{A}_{[r]}\end{array}\right]$ and $\tilde{C}_{[r]}$ is obtained from $\tilde{C}(t)$ by choosing $\Xi_t = \Xi^{[r]}$.

Similarly, we obtain the following recursion for the mean-square error matrix $P(t) = E\{Z(t)Z(t)^T\}$:

$$P(t+1) = \sum_{r=1}^{\nu} \pi_r[\tilde{B}_{[r]}P(t)\tilde{B}_{[r]}^T + \tilde{D}_{[r]}W\tilde{D}_{[r]}^T], \quad (11)$$

where $\tilde{D}_{[r]} = \left[\begin{array}{c:c}\tilde{G} & 0\\ \hdashline -\tilde{G} & \tilde{C}_{[r]}\tilde{\Gamma}_{[r]}]\tilde{L}\tilde{H}\end{array}\right]$ and $W = E\{N(t)N(t)^T\}$ $= \mathrm{diag}\{Q^*, R\}$, where $Q^* = \begin{bmatrix} Q & \cdots & Q\\ \vdots & & \\ Q & \cdots & Q\end{bmatrix}$ and $\tilde{R} = \mathrm{diag}\{R^{(1)}, \dots, R^{(N)}\}$. Relation (11) can be rewritten in the following way:

$$\begin{aligned}\mathrm{col}\{P(t+1)\} = \sum_{r=1}^{\nu} \pi_r[(\tilde{B}_{[r]} \otimes \tilde{B}_{[r]})\mathrm{col}\{P(t)\} +\\ +(\tilde{D}_{[r]} \otimes \tilde{D}_{[r]})\mathrm{col}\{W\}]\end{aligned} \quad (12)$$

where $\mathrm{col}\{.\}$ denotes a vector obtained by concatenating columns of an indicated matrix and the sign $\otimes$ denotes the Kronecker's product.

## 4. STABILITY

In the stability analysis of the proposed estimator, we shall use the following results from the matrix analysis.

*Lemma 1.* Let $f(.)$ be a matrix norm having the property $f(A) \leq f(B)$ for two $n \times n$ matrices $A$ and $B$ satisfying $A \leq B$ ($A \geq 0$ means that all the elements of $A$ are nonnegative). Let $g(.)$ be any matrix norm and let $A$ be partitioned into square blocks $A_{ii}$. Then, $h(A)$ is a matrix norm, where

$$h(A) = f\left(\begin{bmatrix} g(A_{11}) & \cdots & g(A_{1k})\\ \vdots & & \vdots\\ g(A_{k1}) & \cdots & g(A_{kk})\end{bmatrix}\right). \quad (13)$$

■

*Lemma 2.* Let $A$ be an $n \times n$ matrix and $\varepsilon > 0$. Then, there exists a matrix norm $\|A\|$ such that

$$\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon, \tag{14}$$

where $\rho(A)$ is the spectral radius of a matrix $A$ ($\rho(A) = \max_i |\lambda_i(A)|$, where $\lambda_i(A)$ are the eigenvalues of $A$). ■

A norm satisfying the requirement of Lemma 2 is the norm $\|A\|_t = \|D_t U^T A U D_t^{-1}\|_\infty$, where $U$ is an orthogonal matrix in $A = U \Delta U^T$, where $\Delta$ is an upper triangular matrix (Schur's theorem), $D_t = \mathrm{diag}\{t, t^2, t^3, \ldots, t^n\}$ and $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ for $A = [a_{ij}]$, $i, j = 1, \ldots, n$. Inequality (14) is satisfied for any given $\varepsilon > 0$ by choosing $t \geq 0$ large enough.

Lemma 1 can be found in Pierce [1974] as Conlisk observation, while Lemma 2 and the related statement can be found in Horn and Johnson [1985] (Lemma 5.6.10).

The following two theorems give sufficient conditions for stability of (10) and boundedness of the mean-square error (11). The applied methodology is based on Sinopoli et al. [2004], Nilsson [1996] and the definition of a specially constructed norm adapted to the partition of the consensus matrix.

*Theorem 1.* Let $\tilde{A}_{[r]}$ be partitioned into blocks $\tilde{A}_{jk}^{[r]} = C_{jk}^{[r]} \Phi_j^{[r]}$, where $C_{jk}^{[r]}$ and $\Phi_j^{[r]}$ are realizations of $C_{jk}(t)$ and $\Phi_j(t)$ obtained by choosing $\Xi_t = \Xi^{[r]}$, and let $\rho(\Phi_k^{[r]}) < R_k^{[r]}$, $k = 1, \ldots, N$, together with

$$\sum_{r=1}^{\nu} \pi_r \max_j \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) R_k^{[r]} < 1. \tag{15}$$

Then, the recursion (10) is stable if the system (1) is stable or if the system (1) is unstable and $\tilde{F}_E = \tilde{F}$.

*Proof:* Consider the matrix $\tilde{A}_{[r]}$ and define its norm, according to Lemma 1, in the following way:

$$\|\tilde{A}_{[r]}\|_c = \left\| \begin{bmatrix} \|C_{11}^{[r]} \Phi_1^{[r]}\|_t & \cdots & \|C_{1N}^{[r]} \Phi_N^{[r]}\|_t \\ \vdots & & \vdots \\ \|C_{N1}^{[r]} \Phi_1^{[r]}\|_t & \cdots & \|C_{NN}^{[r]} \Phi_N^{[r]}\|_t \end{bmatrix} \right\|_\infty, \tag{16}$$

having in mind properties of the norm $\|.\|_\infty$. For particular terms in (16) we have $\|C_{jk}^{[r]} \Phi_k^{[r]}\|_t \leq \rho(C_{jk}^{[r]}) \|\Phi_k^{[r]}\|_t$, having in mind that $\|C_{jk}^{[r]}\|_t = \rho(C_{jk}^{[r]})$ for $C_{jk}^{[r]}$ diagonal. Moreover, it is always possible to find such a $\bar{t} > 0$ that for any $t > \bar{t}$ we have $\|\Phi_k^{[r]}\|_t \leq \rho(\Phi_k^{[r]}) + \varepsilon$, for any given $\varepsilon > 0$. Making $\varepsilon$ small enough we always have $\rho(\Phi_k^{[r]}) + \varepsilon \leq R_k^{[r]}$ (having in mind the strict inequality in $\rho(\Phi_k^{[r]}) < R_k^{[r]}$), and, therefore, $\|\Phi_k^{[r]}\|_t \leq R_k^{[r]}$. Consequently,

$$\|\tilde{A}_{[r]}\|_c \leq \max_j \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) R_k^{[r]},$$

wherefrom the relation (15) directly follows. The second conclusion follows trivially form the definition of the matrix $\tilde{B}_{[r]}$. Thus the result. ■

*Theorem 2.* The proposed estimator is stable in the sense that $\|S(t)\| < \infty$ $\forall t \in \mathcal{I}$ ($\mathcal{I}$ is the set of all integers), where $S(t) = E\{\varepsilon(t|t-1)\varepsilon(t|t-1)^T\}$, if $\rho(\Phi_k^{[r]}) < R_k^{[r]}$, $k = 1, \ldots, N$,

$$\sum_{r=1}^{\nu} \pi_r (\max_j \sum_{k=1}^{N} \rho(C_{jk}^{[r]}) R_k^{[r]})^2 < 1. \tag{17}$$

and the system (1) is stable. If the system (1) is unstable, the estimator is stable if, additionally, $\tilde{F}_E = \tilde{F}$.

*Proof:* If $\tilde{A}_{[r]}$ is partitioned into $n \times n$ blocks $A_{ij}^{[r]}$, $i, j = 1, \ldots, N$, then the matrix $\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}$ is cogredient to $\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P$ defined by

$$\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P = T(\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}) T^T =$$

$$\begin{bmatrix} A_{11}^{[r]} \otimes A_{11}^{[r]} & \ldots & A_{11}^{[r]} \otimes A_{1N}^{[r]} & \ldots & A_{1N}^{[r]} \otimes A_{1N}^{[r]} \\ \vdots & & & & \\ A_{21}^{[r]} \otimes A_{11}^{[r]} & \ldots & A_{21}^{[r]} \otimes A_{1N}^{[r]} & \ldots & A_{2N}^{[r]} \otimes A_{1N}^{[r]} \\ \vdots & & & & \\ A_{N1}^{[r]} \otimes A_{N1}^{[r]} & \ldots & A_{N1}^{[r]} \otimes A_{nN}^{[r]} & \ldots & A_{NN}^{[r]} \otimes A_{NN}^{[r]} \end{bmatrix},$$

where $T$ is a permutation transformation. Therefore, the norm $\|\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P\|_c$ is a norm $\|.\|_o$ of $\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}$, *i.e.*

$$\|\tilde{A}_{[r]} \otimes \tilde{A}_{[r]}\|_o = \|\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P\|_c =$$

$$= \left\| \begin{bmatrix} \|A_{11}^{[r]} \otimes A_{11}^{[r]}\|_t & \ldots & \|A_{1N}^{[r]} \otimes A_{1N}^{[r]}\|_t \\ \vdots & & \vdots \\ \|A_{N1}^{[r]} \otimes A_{1N}^{[r]}\|_t & \ldots & \|A_{NN}^{[r]} \otimes A_{NN}^{[r]}\|_t \end{bmatrix} \right\|_\infty.$$

Majorizing the last expression similarly as in Theorem 1, one obtains that

$$\|\tilde{A}_{[r]}^P \otimes \tilde{A}_{[r]}^P\|_c \leq \max_{j,l} (\sum_{k=1}^{N} \rho(C_{jk}^{[r]}) R_k^{[r]}) (\sum_{m=1}^{N} \rho(C_{lm}^{[r]}) R_m^{[r]}).$$

As (17) implies (15), we also have that both $\sum_{r=1}^{\nu} \pi_r (\tilde{F} \otimes \tilde{A}_{[r]})$ and $\sum_{r=1}^{\nu} \pi_r (\tilde{A}_{[r]} \otimes \tilde{F})$ are stable if $\tilde{F}$ is stable, and the result directly follows, similarly as in Theorem 1. ■

*Remark 1.* When all the agents have the exact information about the system model, the off-block-diagonal term in $\tilde{B}_{[r]}$ reduces to zero, and stability of the estimator depends entirely on $\sum_{r=1}^{\nu} \pi_i \tilde{A}_{[r]}$. According to (15) and (17), unstable elements in the last sum can be compensated by the remaining terms, and the system remains stable. When the agents do not have the exact information about the system model, the state $X(t)$ propagates to the prediction error part of the model (9). ■

*Remark 2.* A comparison with the results obtained in relation with the continuous-time estimator based on consensus described in Stanković et al. [2007a, 2008] shows basic similarity of the main ideas, but also some differences. Namely, the main point of the stability analysis in Stanković et al. [2007a, 2008] has been to show the existence of stabilizing consensus gains, while the above stability conditions give a deeper insight into the influence of particular terms in the case of intermittent observations and communication faults. The assumption that the consensus matrix $\tilde{C}(t)$ is row-stochastic for all $t$ does not have its direct counterpart in the continuous-time case (notice that eigenvalues at zero in the first case correspond to the eigenvalues at $-\infty$ in the second). ■

*Example 1.* Intercommunications between the agents introduced by the consensus matrix increase, in principle, robustness of the local estimators in the case of intermittent measurements. An insight can be obtained by analyzing a simple example with two estimators. Assume that the system is of first order and unstable, with $F = 1.1$; assume also that $F_1 = 1.1$, $L_1 = 0.2$ and $H_1 = 1$ for the first estimator, and $F_2 = 1.1$, $L_2 = 0.3$ and $H_2 = 1$ for the second; both estimators are stable when the measurements are available (when $\gamma_i = 1$). According to the analysis given in Sinopoli et al. [2004], Nilsson [1996], the local steady state estimators alone are mean-square stable (in the sense of Theorem 2) if the probabilities $p_{11}$ and $p_{22}$ for getting measurements are less than $\bar{p}_{11} = 0.475$ and $\bar{p}_{22} = 0.632$. Assume now that a multi-agent network is implemented with the fixed matrix gain $\tilde{C} = 0.5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, according to the proposed algorithm. Then, it is possible, according to Theorem 2, to construct regions in the plane $p_{11} - p_{22}$ which guarantee the mean-square stability of the whole estimator for different values of the communication probability $p = p_{12} = p_{21}$ (Fig. 1). The obtained boundaries are conservative, as expected; however, the beneficial effect of the consensus scheme is obvious. ■

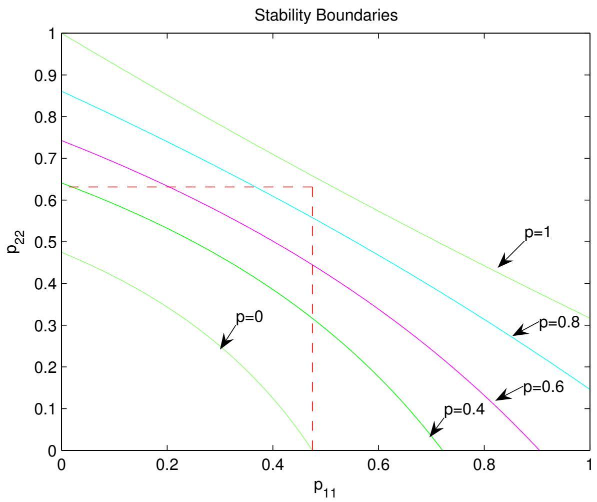


Fig. 1. Stability boundaries

*Remark 3. Remark 3.* It could seem somewhat surprising that the above analysis does not take into account the network topology more explicitly, like in Stanković et al. [2007a,b, 2008], for example. This is a consequence of the adopted focus placed on the conditions for mean-square stability, rather than on the estimation quality or the agreement between the agents (notice that the local estimators alone can perform their job more or less satisfactorily). The proposed scheme represents a simple and general tool for increasing both quality and robustness of distributed estimation, providing reliable estimates of the whole state vector using estimates based on subsystem models. In this context, connectedness of the network contributes, in general, to the overall performance of the algorithm, but does not represent a necessary condition for the estimator to exist. ■

## 5. OPTIMIZATION

In the previous section general conditions for stability of the proposed algorithm are given: there has been, however, no indication of how to choose the parameters of the consensus scheme in practice. It should be recalled that our general formulation encompasses diagonal $n \times n$ block matrices $K_{ij}$ with nonnegative elements, which should be chosen in accordance with the local estimation quality: in the case of higher uncertainty of the estimates, the corresponding weights should be taken to be lower. Then, the general strategy could be such that the elements of $K_{ij}$ are taken to be proportional to the diagonal elements of the inverse of the covariance matrices of the local estimators, including zeros at the places which correspond to the components of the state vector that are not estimated by a particular agent (see also Stanković et al. [2007a, 2008]). However, the problem remains of how to define relative weights for communication between the agents, taking into account the constraint that the overall consensus matrix has to be row-stochastic. Obviously, this problem can be treated pragmatically, starting from the equal neighbor rules, etc. (see *e.g.* Jadbabaie et al. [2003]). However, the given problem formulation allows the application of a more general strategy based on optimization. If the optimization criterion is taken to be the steady-state mean-square prediction error of the estimator defined as $J = \mathrm{Tr}\, S = \mathrm{Tr}\ \lim_{i\to\infty} S(t)$, then, if we collect all the unknown parameters in a vector $\theta$, we can pose the following problem: minimize $J$ with respect to $\theta$, where $J$ is calculated from the solution of the following Lyapunov-like algebraic equation

$$P = \sum_{r=1}^{\nu} \pi_r [\tilde{B}_{[r]} P \tilde{B}_{[r]}^T + \tilde{D}_{[r]} W \tilde{D}_{[r]}^T], \tag{18}$$

having in mind that $S$ is a block of $P$. This equation has a solution under the conditions formulated within Theorem 2. It is to be noticed that the incorporation of intermittent measurements and communication losses makes this optimization problem numerically more complex than the optimization problem formulated in Stanković et al. [2007a] for continuous-time estimators with similar properties.

*Example 2.* The following example gives the results of an application of the proposed optimization procedure in the case of an unstable system. The system is supposed to be given by (1) with $F = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, Q=1; the eigenvalues of $F$ are at $1.5 \pm j0.866$. There are two Kalman filters, the first using $H_1 = [\,1\ 0\,]$ with $R = 0.1$, and the second $H_1 = [\,0\ 1\,]$ with $R = 1$, so that $L_1 = \begin{bmatrix} 2.0750 \\ -0.7807 \end{bmatrix}$ and $L_2 = \begin{bmatrix} -1.5448 \\ -2.1632 \end{bmatrix}$, supposing that both estimators possess the information about the system model. Optimization is done by using (18) with respect to the scalar parameters $\alpha_1$ and $\alpha_2$ in $C_{11}(t) = \alpha_1 I$ and $C_{22}(t) = \alpha_2 I$. The results have been found to be sensitive to the initial conditions, having in mind system instability. Fig. 2 depicts the dependence of the obtained parameters on the communication probability $p = p_{12} = p_{21}$. As it can be seen, the quality of the first estimator dominates in the case of high communication reliability; when the communication reliability deteriorates, the relative importance of the second local estimator increases, as expected.

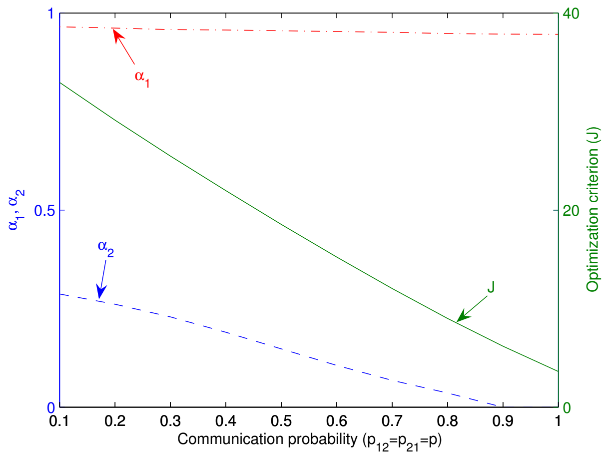


Fig. 2. Optimal consensus parameters

## 6. CONCLUSION

In this paper a new algorithm for overlapping decentralized state estimation is proposed on the basis of a combination of local Kalman filters with intermittent observations and a consensus strategy connecting the local estimators with communication faults. Sufficient conditions for stability of the algorithm in the sense of preserving boundedness of the mean-square estimation error are derived using a specially defined matrix norm. It is also shown that the algorithm can be optimized with respect to the consensus parameters.

Immediate continuation of the research can be carried out in the sense of determining the influence of the consensus scheme to the measurement noise suppression.

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