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Robust H_∞ Filtering for Uncertain Discrete-Time Singular Systems ⋆

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Abstract: This paper concerns the robust H_∞ filtering problem for discrete-time singular systems with norm-bounded uncertainties in all system matrices of state equations. On the basis of the admissibility assumption of the uncertain singular systems, a set of necessary and sufficient conditions for the existence of the desired filters is established, and a normal filter design method under the linear matrix inequality framework is proposed. A numerical example is given to illustrate the application of the proposed method.

1. INTRODUCTION

In the past decades, the H_∞ filtering problem for singular systems has been an important research topic. This is due not only to the theoretical interests but also to the relevance of the topic in various engineering applications. Many works, such as Xu et al. [2003], Yue and Han [2004], Sun and Packard [2005], Zhang et al. [2006], Xu and Lam [2007], consider the filters for continuous-time singular systems, in which the filter design criteria are mainly based on the generalized Lyapunov theorem for singular systems Takaba et al. [1995], and the formulations are under the linear matrix inequality (LMI) framework for easier applications. Unlike in the discrete-time singular system stabilization problem Xu and Yang [2000], Xu and Lam [2004], in the filtering problem for discrete-time singular systems, applications of the approaches parallel to those for the continuous-time systems are not often adopted. One possible reason is the difficulty to manage the resultant constraints related to the singular matrix in the difference term of the state-space model, especially when the constraints need to be represented as LMIs. In Xu and Lam [2007], a necessary and sufficient condition for the solvability of filtering problem for nominal discrete-time singular systems is contributed, which is difficult to apply to the problem for singular systems with uncertainties.

In this paper, the robust H_∞ filtering problem is discussed for discrete-time singular systems with norm-bounded uncertainties. The goal of the filter is to satisfy the H_∞ performance level requirement on the filtering error dynamics. The proposed filter design method is formulated under the LMI framework. Different from Xu et al. [2003], Yue and Han [2004], Xu and Lam [2007], which directly handle singular systems by using the generalized Lyapunov theorem, here a "normal transformation" to get normal system models (i.e., those with the system matrix for the difference term being the identity matrix) Dai [1989] from singular system models is applied first, and normal filters are found directly. Then, instead of using criteria such those in Xu and Yang [2000], Xu and Lam [2004], the easier-to-use criterion which is based on the direct Lyapunov theorem for normal systems is applied. It is believed that the consideration of normal filters is beneficial, since sometimes the physical realizations of singular filters are not easy Dai [1988, 1989].

In practical applications, the structure and behavior of a singular system are also directly related to the system matrix for the difference term. More flexibility will be gained if all system matrices in state equations of dynamical models are allowed to contain uncertainties. To handle systems with uncertainties in the system matrix for the difference term, it is assumed that the uncertain systems are admissible, and the concept of the restricted system equivalence (r.s.e.) Dai [1989] is applied. In addition, some preliminary results in Lin et al. [2000], which considers the stabilization problem for singular systems using the algebraic Riccati equation method, provide the further assumptions for the theoretical development of this paper.

Some notations to be used subsequently are introduced here. The inequality $\mathbf{X} \geq \mathbf{0}$ means that the matrix $\mathbf{X}$ is symmetric and positive semi-definite, and $\mathbf{X} \geq \mathbf{Y}$ means $\mathbf{X} - \mathbf{Y} \geq \mathbf{0}$. Similar definitions apply to symmetric positive/negative definite matrices. $\lambda(\mathbf{X})$ represents the eigenvalues of a square matrix $\mathbf{X}$. For a matrix $\mathbf{M}$, $\|\mathbf{M}\|$ denotes its spectral norm, and for a stable discrete-time transfer function matrix $\mathbf{G}(z)$, $\|\mathbf{G}\|_\infty = \sup_{\omega\in[0,2\pi)} \|\mathbf{G}(e^{j\omega})\|$ is its H_∞ norm. $\mathbf{I}_r$ is the identity matrix with dimension r, the superscript $^{\mathrm{T}}$ represents the transpose of a matrix, and $diag(\mathbf{X},\ \mathbf{Y}, \ldots, \mathbf{Z})$ is the block diagonal matrix with diagonal elements $\mathbf{X}$, $\mathbf{Y}$, …, $\mathbf{Z}$. Finally, $*$ is used to simplify the presentation of symmetric matrices.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

Consider the following nominal singular system,

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$$\Sigma_0 : \begin{cases} \mathbf{E}_0\mathbf{x}(k+1) = \mathbf{A}_0\mathbf{x}(k) + \mathbf{B}_0\mathbf{u}(k) \\ \mathbf{z}(k) = \mathbf{L}_0\mathbf{x}(k), \end{cases} \tag{1}$$

where $\mathbf{x}(k) \in \mathcal{R}^n$ and $rank\mathbf{E}_0 = r < n$. The unforced singular system pair $(\mathbf{E}_0, \mathbf{A}_0)$ of (1) with $\mathbf{u}(k) \equiv \mathbf{0}$ is *regular*, if $det(z\mathbf{E}_0 - \mathbf{A}_0)$ is not identically zero. If $deg(det(z\mathbf{E}_0 - \mathbf{A}_0)) = rank\mathbf{E}_0$, then $(\mathbf{E}_0, \mathbf{A}_0)$ is said to be *causal*. The pair $(\mathbf{E}_0, \mathbf{A}_0)$ is stable if all the roots of $det(z\mathbf{E}_0 - \mathbf{A}_0) = 0$ have magnitudes less than unity. Finally $(\mathbf{E}_0, \mathbf{A}_0)$ is *admissible* if it is regular, causal and stable. For Σ_0, its transfer function matrix from $\mathbf{u}(k)$ to $\mathbf{z}(k)$ is $\mathbf{G}(z) = \mathbf{L}_0(z\mathbf{E}_0 - \mathbf{A}_0)^{-1}\mathbf{B}_0$.

Definition 1. Dai [1989] Suppose Σ_0 in (1) is regular. Let $\mathbf{P}_0$ and $\mathbf{Q}_0$ be two $n \times n$ nonsingular matrices, and $\mathbf{E}_{0r} = \mathbf{P}_0\mathbf{E}_0\mathbf{Q}_0$, $\mathbf{A}_{0r} = \mathbf{P}_0\mathbf{A}_0\mathbf{Q}_0$, $\mathbf{B}_{0r} = \mathbf{P}_0\mathbf{B}_0$, $\mathbf{L}_{0r} = \mathbf{L}_0\mathbf{Q}_0$. The system

$$\Sigma_{0r} : \begin{cases} \mathbf{E}_{0r}\mathbf{x}_r(k+1) = \mathbf{A}_{0r}\mathbf{x}_r(k) + \mathbf{B}_{0r}\mathbf{u}(k) \\ \mathbf{z}(k) = \mathbf{L}_{0r}\mathbf{x}_r(k), \end{cases} \tag{2}$$

with $\mathbf{x}_r(k) = \mathbf{Q}_0^{-1}\mathbf{x}(k)$ is *restricted system equivalent* (r.s.e.) to Σ_0.

For any given regular Σ_0, there exist nonsingular matrices $\mathbf{P}_0$ and $\mathbf{Q}_0$ such that

$$\mathbf{E}_{0r} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathbf{A}_{0r} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}. \tag{3}$$

Lemma 2. Suppose Σ_{0r} in (2) is regular and has the system matrices in (3). Then the pair $(\mathbf{E}_{0r}, \mathbf{A}_{0r})$ is causal and stable if and only if $\mathbf{A}_4 \in \mathcal{R}^{(n-r)\times(n-r)}$ is invertible, and all the roots of $det(z\mathbf{E}_{0r} - \mathbf{A}_{0r}) = 0$ have magnitudes less than unity.

Lemma 2 is the discrete-time version of the corresponding Lemma in Xu and Yang [1999], and can be proved similarly.

Lemma 3. Lee and Fong [2006] Suppose Σ_{0r} in (2) is r.s.e. to Σ_0 in (1). The pair $(\mathbf{E}_0, \mathbf{A}_0)$ in (1) is admissible if and only if the pair $(\mathbf{E}_{0r}, \mathbf{A}_{0r})$ in (2) is admissible.

The next three lemmas are useful for formulating the filtering problem stated in the next Section within the LMI framework.

Lemma 4. Xie [1996] Let $\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta} > \mathbf{0}$, and define the set

$$\Upsilon = \{\mathbf{\Delta}(\mathbf{I} - \mathbf{\Theta}\mathbf{\Delta})^{-1}, \ \mathbf{\Delta}^{\mathrm{T}}\mathbf{\Delta} \leq \mathbf{I}\}.$$

Then, $\Upsilon = \{\mathbf{\Theta}^{\mathrm{T}}(\mathbf{I} - \mathbf{\Theta}\mathbf{\Theta}^{\mathrm{T}})^{-1} + \mathbf{\Pi}^{\mathrm{T}}(\mathbf{I} - \mathbf{\Theta}\mathbf{\Theta}^{\mathrm{T}})^{-1/2}, \ \mathbf{\Pi}^{\mathrm{T}}\mathbf{\Pi} \leq (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1}\}$.

Lemma 5. Suppose $\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta} > \mathbf{0}$. Let $\mathbf{I} - (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1} + \mathbf{I} - \bar{\mathbf{\Theta}}^{\mathrm{T}}\bar{\mathbf{\Theta}} > \mathbf{0}$, and define the set

$$\bar{\Upsilon} = \left\{ (\mathbf{I} - \mathbf{\Pi}^{\mathrm{T}}\bar{\mathbf{\Theta}})^{-1}\mathbf{\Pi}^{\mathrm{T}}, \ \mathbf{\Pi}^{\mathrm{T}}\mathbf{\Pi} \leq (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1} \right\}.$$

Then,

$$\bar{\Upsilon} = \left\{ \begin{array}{r} (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1}\bar{\mathbf{\Theta}}^{\mathrm{T}}(\mathbf{I} - \bar{\mathbf{\Theta}}(\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1}\bar{\mathbf{\Theta}}^{\mathrm{T}})^{-1} \\ +\bar{\mathbf{\Pi}}^{\mathrm{T}}(\mathbf{I} - \bar{\mathbf{\Theta}}(\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1}\bar{\mathbf{\Theta}}^{\mathrm{T}})^{-1/2}, \\ \bar{\mathbf{\Pi}}^{\mathrm{T}}\bar{\mathbf{\Pi}} \leq (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta} - \bar{\mathbf{\Theta}}^{\mathrm{T}}\bar{\mathbf{\Theta}})^{-1} \end{array} \right\}.$$

In Lemma 5, $\mathbf{I} - (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1} + \mathbf{I} - \bar{\mathbf{\Theta}}^{\mathrm{T}}\bar{\mathbf{\Theta}} > \mathbf{0}$ not only implies $\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta} - \bar{\mathbf{\Theta}}^{\mathrm{T}}\bar{\mathbf{\Theta}} > \mathbf{0}$, but also guarantees that the term $(\mathbf{I} - \mathbf{\Pi}^{\mathrm{T}}\bar{\mathbf{\Theta}})^{-1}$ in $\bar{\Upsilon}$ is well defined Xie [1996] with $\mathbf{\Pi}^{\mathrm{T}}\mathbf{\Pi} \leq (\mathbf{I} - \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta})^{-1}$. It is not difficult to verify that Lemma 5 is an extension to Lemma 4.

Lemma 6. Luo et al. [2004] Let $\mathbf{\Omega}$, $\mathbf{H}_0$, $\mathbf{F}_0$, and $\mathbf{R}_0 > \mathbf{0}$ be real matrices with appropriate dimensions, and the matrix $\bar{\mathbf{\Pi}}$ satisfy $\bar{\mathbf{\Pi}}^{\mathrm{T}}\bar{\mathbf{\Pi}} \leq \mathbf{R}_0$. Then for all $\bar{\mathbf{\Pi}}^{\mathrm{T}}\bar{\mathbf{\Pi}} \leq \mathbf{R}_0$ the matrix inequality

$$\mathbf{\Omega} + \mathbf{H}_0\bar{\mathbf{\Pi}}\mathbf{F}_0 + \mathbf{F}_0^{\mathrm{T}}\bar{\mathbf{\Pi}}^{\mathrm{T}}\mathbf{H}_0^{\mathrm{T}} < \mathbf{0}$$

holds if and only if there exists a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \mathbf{\Omega} & \mathbf{H}_0 \\ \mathbf{H}_0^{\mathrm{T}} & \mathbf{0} \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{F}_0^{\mathrm{T}}\mathbf{R}_0\mathbf{F}_0 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} < \mathbf{0}.$$

2.2 System Transformation

The uncertain singular system to be discussed is

$$\Sigma : \begin{cases} (\mathbf{E} + \delta\mathbf{E})\mathbf{x}(k+1) = (\mathbf{A} + \delta\mathbf{A})\mathbf{x}(k) + (\mathbf{B} + \delta\mathbf{B})\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \\ \mathbf{z}(k) = \mathbf{F}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k), \end{cases} \tag{4}$$

where $\mathbf{x}(k) \in \mathcal{R}^n$ is the state vector, $\mathbf{y}(k) \in \mathcal{R}^p$ is the measured output vector, $\mathbf{z}(k) \in \mathcal{R}^q$ is the vector to be estimated, and $\mathbf{u}(k) \in \mathcal{R}^m$ is the disturbance input vector. The matrices $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{F}$, and $\mathbf{H}$ are known real constant matrices with appropriate dimensions. The constant uncertainty matrices satisfy

$$[\, \delta\mathbf{E} \ \delta\mathbf{A} \ \delta\mathbf{B} \,] = \mathbf{M}_x\mathbf{\Delta} \, [\, \mathbf{N} \ \mathbf{N}_x \ \mathbf{N}_u \,] \tag{5}$$

where $\mathbf{\Delta}^{\mathrm{T}}\mathbf{\Delta} \leq \mathbf{I}$ and $\mathbf{\Delta} \in \mathcal{R}^{d_1\times d_2}$. We shall restrict our attention to all $\mathbf{\Delta}$ in (5) for which the pair $(\mathbf{E} + \mathbf{\Delta}\mathbf{E}, \mathbf{A} + \mathbf{\Delta}\mathbf{A})$ is admissible and $rank(\mathbf{E} + \mathbf{\Delta}\mathbf{E}) = rank(\mathbf{E})$. Let $\mathbf{P}$ and $\mathbf{Q}$ be two $n \times n$ nonsingular matrices and be such that

$$\mathbf{PEQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathbf{PM}_x = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix}, \ \ \mathbf{NQ} = [\, \mathbf{N}_1 \ \ \mathbf{N}_2 \,]. \tag{6}$$

By Lin et al. [2000], it can be assumed without loss of generality that $\mathbf{N}_2 = \mathbf{0}$ in (6), which is not difficult to prove similarly. Here, it is further assumed that $\|\mathbf{N}_1\mathbf{M}_1\| < 1$.

Under the assumption that the pair $(\mathbf{E} + \delta\mathbf{E}, \mathbf{A} + \delta\mathbf{A})$ is admissible, there exist nonsingular matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{P}_\triangle$, and $\mathbf{Q}_\triangle$ Lin et al. [2000] such that the system Σ in (4) is r.s.e. to the system

$$\tilde{\Sigma} : \begin{cases} \tilde{\mathbf{E}}\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{u}(k) \\ \mathbf{y}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k) \\ \mathbf{z}(k) = \tilde{\mathbf{F}}\tilde{\mathbf{x}}(k) + \mathbf{H}\mathbf{u}(k), \end{cases} \tag{7}$$

where $\tilde{\mathbf{x}}(k) = \mathbf{Q}_\triangle^{-1}\mathbf{Q}^{-1}\mathbf{x}(k) = \left[\, \tilde{\mathbf{x}}_1^{\mathrm{T}}(k) \ \ \tilde{\mathbf{x}}_2^{\mathrm{T}}(k) \,\right]^{\mathrm{T}}$, $\tilde{\mathbf{x}}_1(k) \in \mathcal{R}^r$, $\tilde{\mathbf{x}}_2(k) \in \mathcal{R}^{n-r}$,

$$\begin{array}{ll} \tilde{\mathbf{E}} = \mathbf{P}_\triangle(\mathbf{E}_r + \mathbf{M}_{xr}\mathbf{\Delta}\mathbf{N}_r)\mathbf{Q}_\triangle, & \\ \tilde{\mathbf{A}} = \mathbf{P}_\triangle(\mathbf{A}_r + \mathbf{M}_{xr}\mathbf{\Delta}\mathbf{N}_{xr})\mathbf{Q}_\triangle, & \tilde{\mathbf{C}} = \mathbf{C}_r\mathbf{Q}_\triangle, \\ \tilde{\mathbf{B}} = \mathbf{P}_\triangle(\mathbf{B}_r + \mathbf{M}_{xr}\mathbf{\Delta}\mathbf{N}_u), & \tilde{\mathbf{F}} = \mathbf{F}_r\mathbf{Q}_\triangle, \end{array} \tag{8}$$

and

$$\begin{array}{llll} \mathbf{E}_r = \mathbf{PEQ}, & \mathbf{A}_r = \mathbf{PAQ}, & \mathbf{B}_r = \mathbf{PB}, & \mathbf{C}_r = \mathbf{CQ}, \\ \mathbf{F}_r = \mathbf{FQ}, & \mathbf{M}_{xr} = \mathbf{PM}_x, & \mathbf{N}_r = \mathbf{NQ}, & \mathbf{N}_{xr} = \mathbf{N}_x\mathbf{Q}. \end{array} \tag{9}$$

For $\mathbf{N}_2 = \mathbf{0}$ in (6), $\mathbf{P}_\triangle = \mathbf{I}_n - \mathbf{M}_{xr}\tilde{\mathbf{\Delta}}\mathbf{N}_r$ and $\mathbf{Q}_\triangle = \mathbf{I}_n$, where $\tilde{\mathbf{\Delta}} = \mathbf{\Delta}(\mathbf{I} - \mathbf{J}\mathbf{\Delta})^{-1}$ and $\mathbf{J} = -\mathbf{N}_1\mathbf{M}_1$, result in

$$\begin{aligned}&\tilde{\mathbf{E}} = \mathbf{E}_r,\\&\tilde{\mathbf{A}} = \mathbf{A}_r + \mathbf{M}_{xr}\tilde{\mathbf{\Delta}}\tilde{\mathbf{N}}_x,\ \tilde{\mathbf{C}} = \mathbf{C}_r,\\&\tilde{\mathbf{B}} = \mathbf{B}_r + \mathbf{M}_{xr}\tilde{\mathbf{\Delta}}\tilde{\mathbf{N}}_u,\ \tilde{\mathbf{F}} = \mathbf{F}_r\end{aligned} \tag{10}$$

in (8) with $\tilde{\mathbf{N}}_x = \mathbf{N}_{xr} - \mathbf{N}_r\mathbf{A}_r$ and $\tilde{\mathbf{N}}_u = \mathbf{N}_u - \mathbf{N}_r\mathbf{B}_r$.

By Lemma 4, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ in (10) may be more explicitly written as

$$\begin{aligned}\tilde{\mathbf{A}} &= \tilde{\mathbf{A}}_0 + \mathbf{M}_{xr}\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{xp},\\ \tilde{\mathbf{B}} &= \tilde{\mathbf{B}}_0 + \mathbf{M}_{xr}\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{up}\end{aligned} \tag{11}$$

with $\mathbf{\Pi}^{\mathrm{T}}\mathbf{\Pi} \le (\mathbf{I} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}$, and

$$\begin{aligned}\tilde{\mathbf{A}}_0 &= \mathbf{A}_r + \mathbf{M}_{xr}\mathbf{J}^{\mathrm{T}}(\mathbf{I} - \mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1}\tilde{\mathbf{N}}_x,\\ \tilde{\mathbf{B}}_0 &= \mathbf{B}_r + \mathbf{M}_{xr}\mathbf{J}^{\mathrm{T}}(\mathbf{I} - \mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1}\tilde{\mathbf{N}}_u,\\ \tilde{\mathbf{N}}_{xp} &= (\mathbf{I} - \mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1/2}\tilde{\mathbf{N}}_x,\\ \tilde{\mathbf{N}}_{up} &= (\mathbf{I} - \mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1/2}\tilde{\mathbf{N}}_u.\end{aligned} \tag{12}$$

Note that $\mathbf{C}_r$, $\mathbf{F}_r$ in (9) and $\tilde{\mathbf{A}}_0$, $\tilde{\mathbf{B}}_0$, and $\tilde{\mathbf{N}}_{xp}$ in (12) can be partitioned as

$$\begin{aligned}&\mathbf{C}_r = [\mathbf{C}_1\ \mathbf{C}_2], \qquad \mathbf{F}_r = [\mathbf{F}_1\ \mathbf{F}_2],\\ &\tilde{\mathbf{A}}_0 = \begin{bmatrix}\tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12}\\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22}\end{bmatrix},\ \tilde{\mathbf{B}}_0 = \begin{bmatrix}\tilde{\mathbf{B}}_1\\ \tilde{\mathbf{B}}_2\end{bmatrix},\ \tilde{\mathbf{N}}_{xp} = [\tilde{\mathbf{N}}_{p1}\ \tilde{\mathbf{N}}_{p2}],\end{aligned} \tag{13}$$

respectively, in accordance with the partition of the state vector $\tilde{\mathbf{x}}(k)$. The system $\tilde{\Sigma}$ in (7) may be more explicitly written as

$$\begin{aligned}\tilde{\mathbf{x}}_1(k+1) =& (\tilde{\mathbf{A}}_{11} + \mathbf{M}_1\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p1})\tilde{\mathbf{x}}_1(k)\\ &+(\tilde{\mathbf{A}}_{12}+\mathbf{M}_1\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p2})\tilde{\mathbf{x}}_2(k)+(\tilde{\mathbf{B}}_1+\mathbf{M}_1\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{up})\mathbf{u}(k)\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{0} =& (\tilde{\mathbf{A}}_{21} + \mathbf{M}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p1})\tilde{\mathbf{x}}_1(k)\\ &+(\tilde{\mathbf{A}}_{22}+\mathbf{M}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p2})\tilde{\mathbf{x}}_2(k)+(\tilde{\mathbf{B}}_2+\mathbf{M}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{up})\mathbf{u}(k)\end{aligned} \tag{15}$$

$$\mathbf{y}(k) = \mathbf{C}_1\tilde{\mathbf{x}}_1(k) + \mathbf{C}_2\tilde{\mathbf{x}}_2(k) + \mathbf{D}\mathbf{u}(k) \tag{16}$$

$$\mathbf{z}(k) = \mathbf{F}_1\tilde{\mathbf{x}}_1(k) + \mathbf{F}_2\tilde{\mathbf{x}}_2(k) + \mathbf{H}\mathbf{u}(k). \tag{17}$$

By Lemma 3, the pair $(\mathbf{E}_r, \tilde{\mathbf{A}})$ of Σ_r with parameter matrices in (11) and (12) is admissible. In addition, by Lemma 2 the term $(\tilde{\mathbf{A}}_{22} + \mathbf{M}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p2})$ in (15) is nonsingular for all $\mathbf{\Pi}^{\mathrm{T}}\mathbf{\Pi} \le (\mathbf{I} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}$, including $\mathbf{\Pi} = \mathbf{0}$, which implies that $\tilde{\mathbf{A}}_{22}$ is nonsingular. Let the nonsingular matrices $\bar{\mathbf{P}} = diag(\mathbf{I}_r, \tilde{\mathbf{A}}_{22}^{-1})$ and $\bar{\mathbf{Q}} = \mathbf{I}_n$. Then $\tilde{\Sigma}$ in (14)–(17) is, via $\bar{\mathbf{P}}$ and $\bar{\mathbf{Q}}$, r.s.e. to

$$\tilde{\Sigma}_{\mathsf{r}}: \begin{cases}\mathbf{E}_r\tilde{\mathbf{x}}(k+1) = \bar{\mathbf{P}}\tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \bar{\mathbf{P}}\tilde{\mathbf{B}}\mathbf{u}(k)\\ \mathbf{y}(k) = \mathbf{C}_r\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k)\\ \mathbf{z}(k) = \mathbf{F}_r\tilde{\mathbf{x}}(k) + \mathbf{H}\mathbf{u}(k),\end{cases} \tag{18}$$

which can be represented more explicitly by (14), (16), (17), and

$$\begin{aligned}\mathbf{0} =& (\bar{\mathbf{A}}_{21}+\bar{\mathbf{M}}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p1})\tilde{\mathbf{x}}_1(k)\\ &+(\mathbf{I}_{n-r}+\bar{\mathbf{M}}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p2})\tilde{\mathbf{x}}_2(k) + (\bar{\mathbf{B}}_2+\bar{\mathbf{M}}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{up})\mathbf{u}(k)\end{aligned} \tag{19}$$

with $\bar{\mathbf{A}}_{21} = \tilde{\mathbf{A}}_{22}^{-1}\tilde{\mathbf{A}}_{21}$, $\bar{\mathbf{B}}_2 = \tilde{\mathbf{A}}_{22}^{-1}\tilde{\mathbf{B}}_2$, and $\bar{\mathbf{M}}_2 = \tilde{\mathbf{A}}_{22}^{-1}\mathbf{M}_2$.

By Lemma 2, the term $(\mathbf{I}_{n-r}+\bar{\mathbf{M}}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p2})$ in (19) is also nonsingular, because of the admissibility of $\tilde{\Sigma}_{\mathsf{r}}$ maintained by Lemma 3. Using the identity

$$(\mathbf{I} + \mathbf{M}_o\mathbf{N}_o)^{-1} = \mathbf{I} - \mathbf{M}_o(\mathbf{I} + \mathbf{N}_o\mathbf{M}_o)^{-1}\mathbf{N}_o \tag{20}$$

for any real matrices $\mathbf{M}_o$ and $\mathbf{N}_o$ with appropriate dimensions, one has

$$(\mathbf{I}_{n-r} + \bar{\mathbf{M}}_2\mathbf{\Pi}^{\mathrm{T}}\tilde{\mathbf{N}}_{p2})^{-1} = \mathbf{I}_{n-r} - \bar{\mathbf{M}}_2\hat{\mathbf{\Pi}}\tilde{\mathbf{N}}_{p2}, \tag{21}$$

where $\hat{\mathbf{\Pi}} = (\mathbf{I}_{d1} - \mathbf{\Pi}^{\mathrm{T}}\bar{\mathbf{J}})^{-1}\mathbf{\Pi}^{\mathrm{T}}$ is well defined, and $\bar{\mathbf{J}} = -\tilde{\mathbf{N}}_{p2}\bar{\mathbf{M}}_2$. Therefore, (19) may be re-arranged as

$$\tilde{\mathbf{x}}_2(k)=-(\bar{\mathbf{A}}_{21}+\bar{\mathbf{M}}_2\hat{\mathbf{\Pi}}\bar{\mathbf{N}}_{p1})\tilde{\mathbf{x}}_1(k)-(\bar{\mathbf{B}}_2+\bar{\mathbf{M}}_2\hat{\mathbf{\Pi}}\bar{\mathbf{N}}_{up})\mathbf{u}(k), \tag{22}$$

where $\bar{\mathbf{N}}_{p1} = \tilde{\mathbf{N}}_{p1} - \tilde{\mathbf{N}}_{p2}\bar{\mathbf{A}}_{21}$ and $\bar{\mathbf{N}}_{up} = \tilde{\mathbf{N}}_{up} - \tilde{\mathbf{N}}_{p2}\bar{\mathbf{B}}_2$.

By substituting (22) into (14), (16), and (17), the system $\tilde{\Sigma}_{\mathsf{r}}$ is reduced to

$$\tilde{\Sigma}_{\mathsf{r}}: \begin{cases}\tilde{\mathbf{x}}_1(k+1) = \hat{\mathbf{A}}\tilde{\mathbf{x}}_1(k) + \hat{\mathbf{B}}\mathbf{u}(k)\\ \mathbf{y}(k) = \hat{\mathbf{C}}\tilde{\mathbf{x}}_1(k) + \hat{\mathbf{D}}\mathbf{u}(k)\\ \mathbf{z}(k) = \hat{\mathbf{F}}\tilde{\mathbf{x}}_1(k) + \hat{\mathbf{H}}\mathbf{u}(k),\end{cases} \tag{23}$$

where

$$\begin{bmatrix}\hat{\mathbf{A}} & \hat{\mathbf{B}}\\ \hat{\mathbf{C}} & \hat{\mathbf{D}}\\ \hat{\mathbf{F}} & \hat{\mathbf{H}}\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{A}}_1 & \hat{\mathbf{B}}_1\\ \hat{\mathbf{C}}_1 & \hat{\mathbf{D}}_1\\ \hat{\mathbf{F}}_1 & \hat{\mathbf{H}}_1\end{bmatrix} + \begin{bmatrix}\hat{\mathbf{M}}_1\\ \hat{\mathbf{M}}_y\\ \hat{\mathbf{M}}_z\end{bmatrix}\hat{\mathbf{\Pi}}\left[\bar{\mathbf{N}}_{p1}\ \bar{\mathbf{N}}_{up}\right] \tag{24}$$

and

$$\begin{aligned}&\hat{\mathbf{A}}_1 = \tilde{\mathbf{A}}_{11} - \tilde{\mathbf{A}}_{12}\bar{\mathbf{A}}_{21},\ \hat{\mathbf{B}}_1 = \tilde{\mathbf{B}}_1 - \tilde{\mathbf{A}}_{12}\bar{\mathbf{B}}_2,\\ &\hat{\mathbf{C}}_1 = \mathbf{C}_1 - \mathbf{C}_2\bar{\mathbf{A}}_{21},\quad \hat{\mathbf{D}}_1 = \mathbf{D} - \mathbf{C}_2\bar{\mathbf{B}}_2,\\ &\hat{\mathbf{F}}_1 = \mathbf{F}_1 - \mathbf{F}_2\bar{\mathbf{A}}_{21},\quad \hat{\mathbf{H}}_1 = \mathbf{H} - \mathbf{F}_2\bar{\mathbf{B}}_2,\\ &\hat{\mathbf{M}}_1 = \mathbf{M}_1 - \tilde{\mathbf{A}}_{12}\bar{\mathbf{M}}_2,\ \hat{\mathbf{M}}_y = -\mathbf{C}_2\bar{\mathbf{M}}_2,\\ &\hat{\mathbf{M}}_z = -\mathbf{F}_2\bar{\mathbf{M}}_2.\end{aligned} \tag{25}$$

Note that $\tilde{\Sigma}_{\mathsf{r}}$ in (23) is a normal system, and its stability is guaranteed by Lemma 3 with the r.s.e. relationship.

The transformation from singular to normal system models enables one to handle the robust filtering problem for uncertain singular systems more easily, since many existing filter design methods for normal systems can be applied. Besides, filters designed this way have less number of states than singular filters designed directly from the singular system models.

2.3 Problem Statement

Consider the normal stable system $\tilde{\Sigma}_{\mathsf{r}}$ in (23) subject to $\hat{\mathbf{\Pi}} = (\mathbf{I}_{d1} - \mathbf{\Pi}^{\mathrm{T}}\bar{\mathbf{J}})^{-1}\mathbf{\Pi}^{\mathrm{T}}$ and $\mathbf{\Pi}^{\mathrm{T}}\mathbf{\Pi} \le (\mathbf{I} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}$. To estimate $\mathbf{z}(k)$, the following filter

$$\Sigma_{\mathsf{f}}: \begin{cases}\mathbf{x}_f(k+1) = \mathbf{A}_f\mathbf{x}_f(k) + \mathbf{B}_f\mathbf{y}(k)\\ \mathbf{z}_f(k) = \mathbf{C}_f\mathbf{x}_f(k) + \mathbf{D}_f\mathbf{y}(k)\end{cases} \tag{26}$$

is adopted, where $\mathbf{x}_f(k)\in\mathcal{R}^r$ and $\mathbf{z}_f(k)\in\mathcal{R}^q$. The matrices $\mathbf{A}_f, \mathbf{B}_f, \mathbf{C}_f$, and $\mathbf{D}_f$ are to be determined. From $\tilde{\Sigma}_r$ in (23) and Σ_{f} in (26), the filtering error dynamics may be written as

$$\Sigma_{\mathsf{e}}: \begin{cases}\mathbf{x}_e(k+1) = \mathbf{A}_e\mathbf{x}_e(k) + \mathbf{B}_e\mathbf{u}(k)\\ \mathbf{e}(k) = \mathbf{C}_e\mathbf{x}_e(k) + \mathbf{D}_e\mathbf{u}(k),\end{cases} \tag{27}$$

where $\mathbf{e}(k) = \mathbf{z}(k) - \mathbf{z}_f(k)$, $\mathbf{x}_e^{\mathrm{T}}(k) = [\,\tilde{\mathbf{x}}_1^{\mathrm{T}}(k)\ \ \mathbf{x}_f^{\mathrm{T}}(k)\,]$,

$$\mathbf{A}_e = \begin{bmatrix} \hat{\mathbf{A}} & \mathbf{0} \\ \mathbf{B}_f\hat{\mathbf{C}} & \mathbf{A}_f \end{bmatrix}, \qquad \mathbf{B}_e = \begin{bmatrix} \hat{\mathbf{B}} \\ \mathbf{B}_f\hat{\mathbf{D}} \end{bmatrix},$$
$$\mathbf{C}_e = \begin{bmatrix} \hat{\mathbf{F}} - \mathbf{D}_f\hat{\mathbf{C}} & -\mathbf{C}_f \end{bmatrix}, \; \mathbf{D}_e = \hat{\mathbf{H}} - \mathbf{D}_f\hat{\mathbf{D}}. \tag{28}$$

The purpose here is to design a stable filter Σ_{f} such that

$$\sup_{\mathbf{\Pi}} \|\mathbf{C}_e(z\mathbf{I}_{2r} - \mathbf{A}_e)^{-1}\mathbf{B}_e + \mathbf{D}_e\|_\infty < \mu_e \tag{29}$$

for a prescribed H_∞-norm bound $\mu_e > 0$.

At this point an extra assumption $\mathbf{I}–(\mathbf{I}–\mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}+\mathbf{I}–\bar{\mathbf{J}}^{\mathrm{T}}\bar{\mathbf{J}} > \mathbf{0}$ is added, which is solely for enabling the LMI formulation in Theorem 9 to be developed in the next Section.

The following is a well-known lemma extended from the Bounded Real Lemma in Gahinet and Apkarian [1994] for characterizing the H_∞-norm constraint. See Grigoriadis and Waston [1997] and Yang and Hung [2002].

Lemma 7. The error dynamic system Σ_{e} in (27) is quadratically stable Amato et al. [1998] and satisfies (29) for a given $\mu_e > 0$, if and only if there exists a $\mathbf{P}_e > \mathbf{0}$ such that

$$\begin{bmatrix} -\mathbf{P}_e & \mathbf{0} & \mathbf{A}_e^{\mathrm{T}}\mathbf{P}_e & \mathbf{C}_e^{\mathrm{T}} \\ \mathbf{0} & -\mu_e^2\mathbf{I} & \mathbf{B}_e^{\mathrm{T}}\mathbf{P}_e & \mathbf{D}_e^{\mathrm{T}} \\ \mathbf{P}_e\mathbf{A}_e & \mathbf{P}_e\mathbf{B}_e & -\mathbf{P}_e & \mathbf{0} \\ \mathbf{C}_e & \mathbf{D}_e & \mathbf{0} & -\mathbf{I} \end{bmatrix} < \mathbf{0}. \tag{30}$$

In Amato et al. [1998], it is known that the quadratic stability of a system implies its asymptotic stability. Since $\tilde{\Sigma}_{\mathsf{r}}$ in (23) is stable, the quadratic stability of Σ_{e} in (27) implies that the filter Σ_{f} in (26) is asymptotically stable.

3. ROBUST FILTER DESIGN

In the literatures, many authors have discussed the normal robust filtering problems with various specifications, mainly based on the Lemma 7. See Geromel et al. [2000], Palhares and Peres [2001], and Yang and Hung [2002] etc.. Here the method for proving Theorem 1 of Palhares and Peres [2001] is modified to treat a different kind of uncertainty, and to derive the following preliminary theorem, which is the first step toward developing an LMI solution to problem stated in the previous Section.

Theorem 8. The filtering error dynamics Σ_{e} in (27) is quadratically stable and satisfies (29) for all admissible uncertainties, if and only if there exist $\mathbf{\Phi} \in \mathcal{R}^{r\times r}$, $\mathbf{X} \in \mathcal{R}^{r\times r}$, $\mathbf{Y} \in \mathcal{R}^{q\times r}$, $\mathbf{Z} \in \mathcal{R}^{r\times q}$, $\mathbf{W} \in \mathcal{R}^{r\times r}$, and $\mathbf{D}_f \in \mathcal{R}^{q\times p}$ such that

$$\begin{bmatrix} -\mathbf{\Phi} & * & * & * & * & * \\ -\mathbf{\Phi} & -\mathbf{X} & * & * & * & * \\ \mathbf{0} & \mathbf{0} & -\mu_e^2\mathbf{I} & * & * & * \\ \mathbf{\Phi}\hat{\mathbf{A}} & \mathbf{\Phi}\hat{\mathbf{A}} & \mathbf{\Phi}\hat{\mathbf{B}} & -\mathbf{\Phi} & * & * \\ \mathbf{\Psi}_{51} & \mathbf{\Psi}_{52} & \mathbf{\Psi}_{53} & -\mathbf{\Phi} & -\mathbf{X} & * \\ \mathbf{\Psi}_{61} & \mathbf{\Psi}_{62} & \mathbf{\Psi}_{63} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{31}$$

$$\begin{bmatrix} \mathbf{\Phi} & \mathbf{\Phi} \\ \mathbf{\Phi} & \mathbf{X} \end{bmatrix} > \mathbf{0}, \tag{32}$$

where

$$\begin{array}{ll} \mathbf{\Psi}_{51} = \mathbf{X}\hat{\mathbf{A}} + \mathbf{Z}\hat{\mathbf{C}} + \mathbf{W}, & \mathbf{\Psi}_{52} = \mathbf{X}\hat{\mathbf{A}} + \mathbf{Z}\hat{\mathbf{C}}, \\ \mathbf{\Psi}_{53} = \mathbf{X}\hat{\mathbf{B}} + \mathbf{Z}\hat{\mathbf{D}}, & \mathbf{\Psi}_{61} = \hat{\mathbf{F}} - \mathbf{D}_f\hat{\mathbf{C}} - \mathbf{Y}, \\ \mathbf{\Psi}_{62} = \hat{\mathbf{F}} - \mathbf{D}_f\hat{\mathbf{C}}, & \mathbf{\Psi}_{63} = \hat{\mathbf{H}} - \mathbf{D}_f\hat{\mathbf{D}}, \end{array} \tag{33}$$

$\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$, $\hat{\mathbf{F}}$, and $\hat{\mathbf{H}}$ are defined in (24). When the above inequalities hold, the filter Σ_{f} in (26) with filter gains

$$\begin{array}{ll} \mathbf{A}_f = -\mathbf{U}^{-1}\mathbf{W}\mathbf{U}^{-\mathrm{T}}, & \mathbf{B}_f = \mathbf{U}^{-1}\mathbf{Z}, \\ \mathbf{C}_f = -\mathbf{Y}\mathbf{U}^{-\mathrm{T}}, & \mathbf{D}_f \end{array} \tag{34}$$

is a solution to the considered robust filtering problem, where $\mathbf{U}$ is nonsingular and satisfies $\mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{X} - \mathbf{\Phi}$.

Proof. The proof is similar to the one in Palhares and Peres [2001] and is omitted for brevity.

Note that in addition to the filter gain matrices shown in (34), the following filter gains

$$\begin{array}{ll} \mathbf{A}_f = (\mathbf{\Phi} - \mathbf{X})^{-1}\mathbf{W}, & \mathbf{B}_f = (\mathbf{X} - \mathbf{\Phi})^{-1}\mathbf{Z}, \\ \mathbf{C}_f = -\mathbf{Y}, & \mathbf{D}_f \end{array} \tag{35}$$

are also usable, because the transfer function matrix $\mathbf{G}_f(z)$ of the filter from $\mathbf{y}(k)$ to $\mathbf{z}_f(k)$ satisfies

$$\begin{aligned} \mathbf{G}_f(z) &= -\mathbf{Y}\mathbf{U}^{-\mathrm{T}}(z\mathbf{I} + \mathbf{U}^{-1}\mathbf{W}\mathbf{U}^{-\mathrm{T}})^{-1}\mathbf{U}^{-1}\mathbf{Z} + \mathbf{D}_f \\ &= -\mathbf{Y}[z\mathbf{I} + (\mathbf{U}\mathbf{U}^{\mathrm{T}})\mathbf{W}]^{-1}(\mathbf{U}\mathbf{U}^{\mathrm{T}})^{-1}\mathbf{Z} + \mathbf{D}_f \\ &= -\mathbf{Y}[z\mathbf{I} - (\mathbf{\Phi} - \mathbf{X})^{-1}\mathbf{W}]^{-1}(\mathbf{X} - \mathbf{\Phi})^{-1}\mathbf{Z} + \mathbf{D}_f. \end{aligned} \tag{36}$$

Next, in order to put the results of Theorem 8 under the LMI framework, by Lemma 5 the uncertainty $\hat{\mathbf{\Pi}}$ is reformulated by the equivalent description

$$\hat{\mathbf{\Pi}} = \hat{\mathbf{J}}_1 + \bar{\mathbf{\Pi}}^{\mathrm{T}}\hat{\mathbf{J}}_2, \tag{37}$$

where

$$\begin{aligned} \hat{\mathbf{J}}_1 &= (\mathbf{I}_{d1} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}\bar{\mathbf{J}}^{\mathrm{T}}(\mathbf{I}_{d2} - \bar{\mathbf{J}}(\mathbf{I}_{d1} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}\bar{\mathbf{J}}^{\mathrm{T}})^{-1}, \\ \hat{\mathbf{J}}_2 &= (\mathbf{I}_{d2} - \bar{\mathbf{J}}(\mathbf{I}_{d1} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1}\bar{\mathbf{J}}^{\mathrm{T}})^{-1/2}, \end{aligned} \tag{38}$$

and $\bar{\mathbf{\Pi}}^{\mathrm{T}}\bar{\mathbf{\Pi}} \leq (\mathbf{I}_{d1} - \mathbf{J}^{\mathrm{T}}\mathbf{J} - \bar{\mathbf{J}}^{\mathrm{T}}\bar{\mathbf{J}})^{-1}$. Correspondingly, the matrices in (24) may be represented as

$$\begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \\ \hat{\mathbf{F}} & \hat{\mathbf{H}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \\ \mathbf{F}_t & \mathbf{H}_t \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{M}}_1 \\ \hat{\mathbf{M}}_y \\ \hat{\mathbf{M}}_z \end{bmatrix} \bar{\mathbf{\Pi}}^{\mathrm{T}} \begin{bmatrix} \hat{\mathbf{N}}_{x1} & \hat{\mathbf{N}}_{up} \end{bmatrix}, \tag{39}$$

where

$$\begin{bmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \\ \mathbf{F}_t & \mathbf{H}_t \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}_1 & \hat{\mathbf{B}}_1 \\ \hat{\mathbf{C}}_1 & \hat{\mathbf{D}}_1 \\ \hat{\mathbf{F}}_1 & \hat{\mathbf{H}}_1 \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{M}}_1 \\ \hat{\mathbf{M}}_y \\ \hat{\mathbf{M}}_z \end{bmatrix} \hat{\mathbf{J}}_1 \begin{bmatrix} \bar{\mathbf{N}}_{p1} & \bar{\mathbf{N}}_{up} \end{bmatrix},$$
$$\begin{bmatrix} \hat{\mathbf{N}}_{x1} & \hat{\mathbf{N}}_{up} \end{bmatrix} = \hat{\mathbf{J}}_2 \begin{bmatrix} \bar{\mathbf{N}}_{p1} & \bar{\mathbf{N}}_{up} \end{bmatrix}. \tag{40}$$

Then Theorem 9 below is an LMI version of Theorem 8.

Theorem 9. Under the assumption of $\mathbf{I} - (\mathbf{I} - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1} + \mathbf{I} - \bar{\mathbf{J}}^{\mathrm{T}}\bar{\mathbf{J}} > \mathbf{0}$, the filtering error dynamics Σ_{e} in (27) is quadratically stable and satisfies (29) for a given $\mu_e > 0$ with all considered uncertainties, if and only if there exist $\mathbf{\Phi} \in \mathcal{R}^{r\times r}$, $\mathbf{X} \in \mathcal{R}^{r\times r}$, $\mathbf{Y} \in \mathcal{R}^{q\times r}$, $\mathbf{Z} \in \mathcal{R}^{r\times q}$, $\mathbf{W} \in \mathcal{R}^{r\times r}$, $\mathbf{D}_f \in \mathcal{R}^{q\times p}$, and $\varepsilon^{-1} > 0$ such that the LMIs in (32) and

$$\begin{bmatrix} \mathbf{M}_{11} & * & * & * & * & * & * \\ \mathbf{M}_{11} & \mathbf{M}_{22} & * & * & * & * & * \\ \mathbf{M}_{31} & \mathbf{M}_{31} & \mathbf{M}_{33} & * & * & * & * \\ \mathbf{\Phi A}_t & \mathbf{\Phi A}_t & \mathbf{\Phi B}_t & -\mathbf{\Phi} & * & * & * \\ \mathbf{M}_{51} & \mathbf{M}_{52} & \mathbf{M}_{53} & -\mathbf{\Phi} & -\mathbf{X} & * & * \\ \mathbf{M}_{61} & \mathbf{M}_{62} & \mathbf{M}_{63} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_q & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{M}}_1^{\mathrm{T}}\mathbf{\Phi} & \mathbf{M}_{75} & \mathbf{M}_{76} & \mathbf{M}_{77} \end{bmatrix} < \mathbf{0} \quad (41)$$

are satisfied, where

$$\begin{array}{ll} \mathbf{M}_{11} = -\mathbf{\Phi} + \varepsilon^{-1}\hat{\mathbf{N}}_{x1}^{\mathrm{T}}\hat{\mathbf{N}}_{x1} & \mathbf{M}_{22} = -\mathbf{X} + \varepsilon^{-1}\hat{\mathbf{N}}_{x1}^{\mathrm{T}}\hat{\mathbf{N}}_{x1} \\ \mathbf{M}_{31} = \varepsilon^{-1}\hat{\mathbf{N}}_{up}^{\mathrm{T}}\hat{\mathbf{N}}_{x1} & \mathbf{M}_{33} = -\mu_e^2\mathbf{I}_m + \varepsilon^{-1}\hat{\mathbf{N}}_{up}^{\mathrm{T}}\hat{\mathbf{N}}_{up} \\ \mathbf{M}_{51} = \mathbf{XA}_t + \mathbf{ZC}_t + \mathbf{W} & \mathbf{M}_{52} = \mathbf{XA}_t + \mathbf{ZC}_t, \\ \mathbf{M}_{53} = \mathbf{XB}_t + \mathbf{ZD}_t, & \mathbf{M}_{61} = \mathbf{F}_t - \mathbf{D}_f\mathbf{C}_t - \mathbf{Y}, \\ \mathbf{M}_{62} = \mathbf{F}_t - \mathbf{D}_f\mathbf{C}_t, & \mathbf{M}_{63} = \mathbf{H}_t - \mathbf{D}_f\mathbf{D}_t, \\ \mathbf{M}_{75} = \hat{\mathbf{M}}_1^{\mathrm{T}}\mathbf{X} + \hat{\mathbf{M}}_y^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}, & \mathbf{M}_{76} = \hat{\mathbf{M}}_z^{\mathrm{T}} - \hat{\mathbf{M}}_y^{\mathrm{T}}\mathbf{D}_f^{\mathrm{T}}, \\ \mathbf{M}_{77} = -\varepsilon^{-1}(\mathbf{I}_{d_1} - \bar{\mathbf{J}}^{\mathrm{T}}\mathbf{J} - \bar{\mathbf{J}}^{\mathrm{T}}\bar{\mathbf{J}}). & \end{array} \quad (42)$$

When the above inequalities hold, the filter Σ_f in (26) with filter gains (34) or (35) is a solution to the considered robust filtering problem.

Proof. It is enough to establish the equivalence of (31) and (41) with an $\varepsilon^{-1} > 0$. By (37), (31) may be re-written as

$$\tilde{\mathbf{\Omega}} + \tilde{\mathbf{H}}_0\bar{\mathbf{\Pi}}\tilde{\mathbf{F}}_0 + \tilde{\mathbf{F}}_0^{\mathrm{T}}\bar{\mathbf{\Pi}}^{\mathrm{T}}\tilde{\mathbf{H}}_0^{\mathrm{T}} < \mathbf{0}, \quad (43)$$

with $\bar{\mathbf{\Pi}}^{\mathrm{T}}\bar{\mathbf{\Pi}} \le (\mathbf{I}_{d1} - \mathbf{J}^{\mathrm{T}}\mathbf{J} - \bar{\mathbf{J}}^{\mathrm{T}}\bar{\mathbf{J}})^{-1}$, where

$$\tilde{\mathbf{\Omega}} = \begin{bmatrix} -\mathbf{\Phi} & * & * & * & * & * \\ -\mathbf{\Phi} & -\mathbf{X} & * & * & * & * \\ \mathbf{0} & \mathbf{0} & -\mu_e^2\mathbf{I}_m & * & * & * \\ \mathbf{\Phi A}_t & \mathbf{\Phi A}_t & \mathbf{\Phi B}_t & -\mathbf{\Phi} & * & * \\ \mathbf{M}_{51} & \mathbf{M}_{52} & \mathbf{M}_{53} & -\mathbf{\Phi} & -\mathbf{X} & * \\ \mathbf{M}_{61} & \mathbf{M}_{62} & \mathbf{M}_{63} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_q \end{bmatrix},$$

$$\tilde{\mathbf{F}}_0 = \left[\, \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{M}_1^{\mathrm{T}}\mathbf{\Phi}\ \mathbf{M}_{75}\ \mathbf{M}_{76} \,\right], \quad (44)$$

$$\tilde{\mathbf{H}}_0^{\mathrm{T}} = \left[\, \hat{\mathbf{N}}_{x1}\ \hat{\mathbf{N}}_{x1}\ \hat{\mathbf{N}}_{up}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \,\right].$$

By Lemma 6 and the Schur complement, it is seen that (43) is equivalent to (41) with an $\varepsilon^{-1} > 0$.

Remark 10. Based on Theorem 9, the following convex optimization problem may be formulated with respect to a chosen pair $\{\mathbf{P}, \mathbf{Q}\}$ in (9) to find the H_∞ optimal filter of the form (26) such that (29) is satisfied with the minimal μ_e:

$$\min_{\mu_e^2,\, \varepsilon^{-1}, \mathbf{\Phi},\, \mathbf{W},\, \mathbf{X},\, \mathbf{Y},\, \mathbf{Z},\, \mathbf{D}_f} \mu_e^2, \quad (45)$$

subject to the LMIs (32), (41), $\varepsilon^{-1} > 0$ and $\mu_e^2 > 0$.

4. A NUMERICAL EXAMPLE

In this section, an example is worked out to illustrate the proposed filter design method. Suppose that the system matrices of the system Σ in (4) are as follows:

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -0.102 & -0.030 & -0.046 \\ -0.104 & -0.168 & -0.104 \\ -0.090 & 0.114 & 0.424 \end{bmatrix},$$

$$\mathbf{B}^{\mathrm{T}} = [\,1\ 1\ 0.2\,],\ \mathbf{C} = [\,0.1\ 0\ 0.5\,], \quad (46)$$

$$\mathbf{D} = -0.5, \qquad \mathbf{F} = [\,-1\ 0.3\ -0.5\,], \qquad \mathbf{H} = 0.$$

The uncertainty matrices in (5) are assumed to be

$$\mathbf{M}_x = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \begin{array}{l} \mathbf{N} = [\,0.03\ 0\ 0\,], \\ \mathbf{N}_x = [\,0.1\ 0.2\ 0.1\,], \\ \mathbf{N}_u = 1, \end{array} \quad (47)$$

and $|\Delta| \le 1$. The prescribed H_∞-norm bound μ_e in (29) is 3. It is easy to verify that $(\mathbf{E}+\delta\mathbf{E}, \mathbf{A}+\delta\mathbf{A})$ is an admissible pair, and $rank(\mathbf{E}+\delta\mathbf{E}) = rank\mathbf{E} = 2$. By applying singular value decomposition to $\mathbf{E}$, one may choose

$$\mathbf{P} = \begin{bmatrix} 0.2283 & 0.2045 & 0.0238 \\ 0.2850 & -0.3977 & 0.6827 \\ -0.5774 & 0.5774 & 0.5774 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0.2521 & 0.9677 & 0 \\ 0.8655 & -0.2255 & 0.4472 \\ 0.4328 & -0.1128 & -0.8944 \end{bmatrix}. \quad (48)$$

Since $\|\mathbf{J}\| = 0.02 < 1$ and $\lambda(\mathbf{I}_1 - (\mathbf{I}_1 - \mathbf{J}^{\mathrm{T}}\mathbf{J})^{-1} + \mathbf{I}_1 - \bar{\mathbf{J}}^{\mathrm{T}}\bar{\mathbf{J}}) = 0.9938 > 0$, the assumption of Theorem 9 is satisfied. The filter Σ_f in (26) is designed by solving the LMIs of Theorem 9, and the filter gains (35) are found to be

$$\mathbf{A}_f = \begin{bmatrix} -0.0116 & 0.0195 \\ 0.1430 & -0.1175 \end{bmatrix}, \ \mathbf{B}_f = \begin{bmatrix} 0.2667 \\ -0.4196 \end{bmatrix}, \quad (49)$$

$$\mathbf{C}_f = [\,-0.1832\ 1.3640\,], \qquad \mathbf{D}_f = 1.2646,$$

which is a second-order normal stable filter as desired. With respect to the chosen $\{\mathbf{P}, \mathbf{Q}\}$ in (48), the corresponding H_∞ optimal filter is also designed by solving the convex optimization problem mentioned in *Remark 10*, which is implemented by the *MATLAB LMI Control Toolbox* Gahinet et al. [1995]. The resulting optimal μ_e is 2.6116, and the filter gains (35) are found to be

$$\mathbf{A}_f = \begin{bmatrix} -0.0272 & 0.0190 \\ 0.1319 & -0.1191 \end{bmatrix}, \ \mathbf{B}_f = \begin{bmatrix} 0.2640 \\ -0.4236 \end{bmatrix}, \quad (50)$$

$$\mathbf{C}_f = [\,-0.0612\ 1.4202\,], \qquad \mathbf{D}_f = 1.2788.$$

5. CONCLUSION

The H_∞ filter design problem has been considered for uncertain discrete-time singular systems, in which norm-bounded uncertainties appear in all system matrices of the state equations. The algebraic equations in the singular system model are eliminated, and a normal dynamic system model is constructed with uncertainties in the linear fractional transformation form. For the H_∞ filter design problem, the normal system model allows one to utilize many existing methods to design normal filters directly, but how to utilize the degrees of freedom in the choices of normal system models is worthy of further investigations. In this paper a set of necessary and sufficient conditions is provided in terms of LMIs for the normal filter design.

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