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Filtering with nonrandom noise: invariant ellipsoids technique

B. T. Polyak M. V. Topunov

Institute of Control Science,
Russian Academy of Sciences, Moscow, Russia
(e-mail: {boris,mtopunov}@ ipu.ru)

Abstract: Linear time-invariant filter is presented for state estimation in LTI systems with bounded noise. The filter is optimal in the sense that it guarantees minimal error bounds (minimal invariant ellipsoid for errors of filtering). Both continuous-time and discrete-time cases are covered. The key role plays LMI technique and new version of S-theorem. Double pendulum velocity estimation is considered as an example.

Keywords: Filtering, bounded noise, linear systems, state estimation, invariant ellipsoids, LMI

1. INTRODUCTION

Filtering problem with random Gaussian noises admits complete solution via Kalman filter. However randomness assumption is doubtful in many applications. More typical is the situation with unknown-but-bounded noises. Then one can design estimators with *guaranteed* (versus probabilistic) error bounds. This approach has been initiated in 1960-70th by Witsenhausen, Bertsekas and Rhodes, Schweppe, see Schweppe (1973). Simultaneously similar research has been developed by Russian participants of N.N.Krasovski's seminar such as Kurzhanski, Subbotin, Osipov and others, see references in Kurzhanski (1977). Other important contributions can be found in Chernousko (1994). In particular, works Schweppe (1973); Kurzhanski (1977); Chernousko (1994) provide *ellipsoidal technique* for filtering with bounded noises. New results in this field can be found in Kurzhanski and Valyi (1997); Chernousko and Polyak (2005); Nazin et al. (2007) and references therein.

We also address the problem of filtering with bounded nonrandom noise. We restrict our research with *linear time-invariant models* and *linear time-invariant filters*. The goal is ensure bounded error of filtering which belongs to an *invariant ellipsoid*, that is the error bound is uniform in time. For this restricted problem formulation the *optimal* filter and state estimate can be found. This is the main difference with above mentioned approach (Schweppe (1973); Kurzhanski (1977); Chernousko (1994)) which covers more general models (for instance time-varying) and more general estimators (also time-varying), but the solutions obtained are suboptimal.

The concept of invariant ellipsoids combined with LMI technique is widely used for analysis and design of control systems Boyd et al. (1994); Blanchini (1999); Abedor et al. (1996); Nazin et al. (2007). However it found less applications in filtering problems, one of the few exceptions being the paper Abedor et al. (1996). We extend the results of this paper in several directions. First, we provide more simple and precise error bounds; second, we treat the discrete-time case as well; third, we study the behavior of the estimates for arbitrary initial values (more details can be found below). From technical point of view we exploit the extension of S-theorem for two quadratic forms, see Polyak (1998).

2. CONTINUOUS-TIME CASE

Consider LTI continuous-time system

$$\begin{cases} \dot{x} = Ax + D_1 w, \\ y = Cx + D_2 w, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$ are given matrices, $x(t) \in \mathbb{R}^n$ is state vector, $y(t) \in \mathbb{R}^l$ is the measured output, $w(t) \in \mathbb{R}^m$ is bounded noise: [1]

$$\|w(t)\| \leqslant 1, \quad \forall t \geqslant 0.$$

Thus we deal with L_∞-bounded noises, no other assumptions on $w(t)$ are imposed. We suppose that (A, D) is controllable and $D_1 D_2^{\mathrm{T}} = 0$.

The state x is unavailable and the only information is provided by the output y. Construct the state estimate $\widehat{x}$ as linear time-invariant filter exploiting the output y and its forecast $C\widehat{x}$:

$$\dot{\widehat{x}} = A\widehat{x} + F(y - C\widehat{x}), \quad F \in \mathbb{R}^{n\times l}. \tag{2}$$

Thus the structure of the filter is fixed in advance, the only design variable is the gain matrix F. The structure is the same as in *Luenberger observer*, compare Luenberger (1971). Introduce *the error* $e(t) = x(t) - \widehat{x}(t)$, it characterizes the accuracy of filtering. Due to (1), (2), it satisfies the ordinary differential equation

$$\dot{e} = (A - FC)e + (D_1 - FD_2)w. \tag{3}$$

Our goal is to choose F to make e small. For this purpose we use the framework of *invariant* (sometimes they are called *holdable* or *inescapable*) ellipsoids Blanchini (1999); Boyd et al. (1994); Abedor et al. (1996); Nazin et al. (2007). We modify this notion, including *large deviation* case.

[1] Here and after $\|\cdot\|$ is Euclidean norm, I is the unit matrix, and $A \geq 0$ denotes nonnegative definite matrix .

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Definition. The ellipsoid

$$\mathcal{E} = \{e \in \mathbb{R}^n : \quad e^{\mathrm{T}} P^{-1} e \leqslant 1\}, \quad P > 0, \tag{4}$$

is invariant for system (3), if two conditions hold:

1. (Small deviation case) $e(0) \in \mathcal{E}$ implies $e(t) \in \mathcal{E}$ for all $t \geqslant 0$.

2. (Large deviation case) $e(0) \notin \mathcal{E}$ implies $e(t) \to \mathcal{E}, t \to \infty$ (in particular it can be $e(t) \in \mathcal{E}$ for $t \geqslant T$ with some $T > 0$).

Notice that in the above mentioned works Blanchini (1999); Boyd et al. (1994); Abedor et al. (1996); Nazin et al. (2007) just the first property was the basis for definition of invariant ellipsoids. Moreover in some of them (e.g. Abedor et al. (1996)) zero initial value $e(0) = 0$ was the only option. However it is unnatural to assume that we know $x(0)$ precisely, thus our definition seems to be much more robust and tailored to real-life situations.

Size of invariant ellipsoid is a natural measure of filtering accuracy. For small deviations it provides uniform on t error bound, for large deviations — asymptotic error bound. In this section we say that an ellipsoid $\mathcal{E}$ is *minimal* if it possess the minimal sum of squares of its halfaxes, i.e. if $\operatorname{tr} P$ is minimal.

Theorem 1. The solution $\widehat{Q}$ and $\widehat{Y}$ of minimization problem

$$\operatorname{tr} H \to \min$$

subject to

$$\begin{pmatrix} A^{\mathrm{T}}Q + QA - YC - C^{\mathrm{T}}Y^{\mathrm{T}} + \alpha Q & QD - YD_2 \\ (QD - YD_2)^{\mathrm{T}} & -\alpha I \end{pmatrix} \leqslant 0,$$

$$\begin{pmatrix} H & I \\ I & Q \end{pmatrix} \geqslant 0,$$

on matrix variables $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times l}$, $H = H^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and scalar parameter $\alpha > 0$, defines the matrix $\widehat{P} = \widehat{Q}^{-1}$ of the minimal invariant ellipsoid and corresponding filter gain

$$\widehat{F} = \widehat{Q}^{-1}\widehat{Y}.$$

Proof. Construct quadratic Lyapunov function

$$V(e) = e^{\mathrm{T}}Qe, \quad Q \in \mathbb{R}^{n\times n}, \quad Q > 0.$$

Then for solutions of (3)

$$\begin{aligned} \dot{V}(e) &= \big((A - FC)e + (D_1 - FD_2)w\big)^{\mathrm{T}}Qe + \\ &\quad + e^{\mathrm{T}}Q\big((A - FC)e + (D_1 - FD_2)w\big) = \\ &= e^{\mathrm{T}}\big((A - FC)^{\mathrm{T}}Q + Q(A - FC)\big)e + 2e^{\mathrm{T}}Q(D_1 - FD_2)w. \end{aligned}$$

To satisfy both properties of invariant ellipsoid for (3) it is necessary and sufficient to suppose that $V(e) \geqslant 1$ and $w^{\mathrm{T}}w \leqslant 1$ imply $\dot{V}(e) \leqslant 0$. Hence

$$\begin{aligned} &e^{\mathrm{T}}\big((A - FC)^{\mathrm{T}}Q + Q(A - FC)\big)e + 2w^{\mathrm{T}}(D_1 - FD_2)^{\mathrm{T}} \times \\ &\quad Qe \leqslant 0, \quad \forall (e, w): \quad e^{\mathrm{T}}Qe \geqslant 1, \quad w^{\mathrm{T}}w \leqslant 1. \end{aligned} \tag{5}$$

Let $s = \begin{pmatrix} e \\ w \end{pmatrix}$,

$$M_0 = \begin{pmatrix} (A - FC)^{\mathrm{T}}Q + Q(A - FC) & Q(D_1 - FD_2) \\ (D_1 - FD_2)^{\mathrm{T}}Q & 0 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -Q & 0 \\ 0 & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and $\widetilde{f}_i(s) = s^{\mathrm{T}}M_i s, \ i = 0, 1, 2$. Then (5) reads

$$\widetilde{f}_0(s) \leqslant 0, \quad \forall s: \quad \widetilde{f}_1(s) \leqslant -1, \quad \widetilde{f}_2(s) \leqslant 1.$$

Applying S-theorem for two quadratic forms Polyak (1998) we conclude that (5) is equivalent to LMI

$$M_0 \leqslant \alpha M_1 + \beta M_2$$

with some α, β such that $\alpha \geqslant \beta \geqslant 0$ or

$$\begin{pmatrix} (A - FC)^{\mathrm{T}}Q + Q(A - FC) + \alpha Q & Q(D_1 - FD_2) \\ (D_1 - FD_2)^{\mathrm{T}}Q & -\beta I \end{pmatrix} \leqslant 0.$$

Thus invariant ellipsoid with matrix $P = Q^{-1} > 0$ exists if and only if the last LMI holds with some $\alpha \geqslant \beta > 0$. We are interested in minimal ellipsoids, this implies $\beta = \beta_{\max} = \alpha$. We can eliminate F defining new variable $Y = QF$:

$$\begin{pmatrix} A^{\mathrm{T}}Q + QA - YC - C^{\mathrm{T}}Y^{\mathrm{T}} + \alpha Q & QD_1 - YD_2 \\ (QD_1 - YD_2)^{\mathrm{T}} & -\alpha I \end{pmatrix} \leqslant 0. \tag{6}$$

To reduce minimization of $\operatorname{tr} Q^{-1}$ to linear problem introduce matrix $H = H^{\mathrm{T}}$ such that $Q^{-1} \leqslant H$. Due to Schur lemma this inequality is equivalent to LMI

$$\begin{pmatrix} H & I \\ I & Q \end{pmatrix} \geqslant 0. \tag{7}$$

Finally we arrive to minimization

$$\operatorname{tr} H \to \min$$

subject to (6), (7). □

For α fixed this is *Semi-Definite Programming, SDP* problem, a convex optimization one. Numerous software is available for its solution such as SeDuMi Toolbox, YALMIP Toolbox, and LMI Toolbox for Matlab.

3. DISCRETE-TIME CASE

Consider discrete-time LTI system

$$\begin{cases} x_{k+1} = Ax_k + D_1 w_k, \\ \quad y_k = Cx_k + D_2 w_k, \end{cases} \tag{8}$$

with $A \in \mathbb{R}^{n\times n}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$, state $x_k \in \mathbb{R}^n$, measured output $y_k \in \mathbb{R}^l$ and l_∞-bounded noise $w_k \in \mathbb{R}^m$

$$\|w_k\| \leqslant 1, \quad k = 0, 1, 2, \ldots$$

Suppose (A, D) controllable and $D_1 D_2^{\mathrm{T}} = 0$.

Proposed filtering state estimator has the form:

$$\widehat{x}_{k+1} = A\widehat{x}_k + F(y_k - C\widehat{x}_k), \quad F \in \mathbb{R}^{n\times l}$$

with constant gain matrix F. The error $e_k = x_k - \widehat{x}_k$ satisfies difference equation

$$e_{k+1} = (A - FC)e_k + (D_1 - FD_2)w_k. \tag{9}$$

The definition of an invariant ellipsoid is close to continuous-time case: ellipsoid

$$\mathcal{E} = \{e_k \in \mathbb{R}^n : \quad e_k^{\mathrm{T}} P^{-1} e_k \leqslant 1\}, \quad P > 0, \tag{10}$$

is *invariant* for system (9), if 1) $e_0 \in \mathcal{E}$ (small deviations) implies $e_k \in \mathcal{E}$ for all $k = 1, 2, \ldots,$ 2) $e_0 \notin \mathcal{E}$ (large deviations) implies $e_k \to \mathcal{E}, k \to \infty$.

Our goal is to choose gain F to find the minimal (with minimal $\operatorname{tr} P$) invariant ellipsoid for error bound.

Theorem 2. *Solution $\widehat{Q}$ and $\widehat{Y}$ of minimization problem*

$$\operatorname{tr} H \to \min$$

subject to

$$\begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^{\mathrm{T}} & \Psi_3 \end{pmatrix} \leqslant 0, \qquad \begin{pmatrix} Z & Y^{\mathrm{T}} \\ Y & Q \end{pmatrix} \geqslant 0, \qquad \begin{pmatrix} H & I \\ I & Q \end{pmatrix} \geqslant 0,$$

with

$$\Psi_1 = A^{\mathrm{T}}QA - A^{\mathrm{T}}YC - C^{\mathrm{T}}Y^{\mathrm{T}}A + C^{\mathrm{T}}ZC - \alpha Q,$$
$$\Psi_2 = A^{\mathrm{T}}QD_1 - C^{\mathrm{T}}Y^{\mathrm{T}}D_1 - A^{\mathrm{T}}YD_2 + C^{\mathrm{T}}ZD_2,$$
$$\Psi_3 = D_1^{\mathrm{T}}QD_1 - D_2^{\mathrm{T}}Y^{\mathrm{T}}D_1 - D_1^{\mathrm{T}}YD_2 + D_2^{\mathrm{T}}ZD_2 - (1-\alpha)I,$$

and minimization is over matrix variables $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Z = Z^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times l}$, $H = H^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and scalar $\alpha > 0$, defines $\widehat{P} = \widehat{Q}^{-1}$ — the minimal ellipsoid for (8) *and corresponding filter*

$$\widehat{F} = \widehat{Q}^{-1}\widehat{Y}.$$

The proof remains mainly the same as for Theorem 1, we focus on the differences. For Lyapunov function

$$V(e_k) = e_k^{\mathrm{T}}Qe_k, \quad Q = P^{-1} \in \mathbb{R}^{n\times n}, \quad Q > 0,$$

we obtain

$$\begin{aligned} V(e_{k+1}) &= \big((A-FC)e_k + (D_1 - FD_2)w_k\big)^{\mathrm{T}}Q\times \\ &\quad \big((A-FC)e_k + (D_1 - FD_2)w_k\big) = \\ &= e_k^{\mathrm{T}}(A-FC)^{\mathrm{T}}Q(A-FC)e_k + w_k^{\mathrm{T}}(D_1-FD_2)^{\mathrm{T}}Q\times \\ &\quad (D_1-FD_2)w_k + 2w_k^{\mathrm{T}}(D_1-FD_2)^{\mathrm{T}}Q(A-FC)e_k. \end{aligned}$$

To guarantee the first property of the invariant ellipsoid we require $V(e_k) \leqslant 1$ implies $V(e_{k+1}) \leqslant 1$.

After technical manipulations with application of Schur lemma and S-theorem we get

$$\begin{aligned} &(A-FC)^{\mathrm{T}}Q(A-FC) - \alpha Q \leqslant \\ &\leqslant (A-FC)^{\mathrm{T}}Q(D_1-FD_2)\big((D_1-FD_2)^{\mathrm{T}}Q\times \\ &\quad (D_1-FD_2) - \beta I\big)^{-1}(D_1-FD_2)^{\mathrm{T}}Q(A-FC). \end{aligned}$$

We arrive to the same inequality if we start from the second property of invariant ellipsoids and require $V(e_{k+1}) \leqslant V(e_k)$ for $V(e_k) \geqslant 1$. Further considerations are the same as for continuous-time case.

4. SOME EXTENSIONS

Sometimes a priori information relating initial value is available: $x(0) \in E_0$, for some ellipsoid $E_0 = \{x : x^{\mathrm{T}}P_0^{-1}x \leqslant 1\}$. Then choosing $\widehat{x}(0) = 0$ we can guarantee $e(0) \in E_0$. If $E_0 \subset \mathcal{E}$ we deal with small deviation case and $e(t) \in \mathcal{E}$ for all $t \geq 0$. Thus additional LMI constraint $Q \leqslant P_0^{-1}$ possess uniform filtering error bound.

Instead of estimation of full state vector x, often the problem is estimation of a regulated output $y_1 = C_1x$ and we wish to make its error $e_1 = y_1 - \widehat{y}_1 = C_1(x - \widehat{x})$ as small as possible. Then minimization of $\operatorname{tr} C_1PC_1^{\mathrm{T}}$ versus $\operatorname{tr} P$ solves the problem. Such minimization can be easily written similar to Theorems 1 and 2.

Other extensions relate to various performance criteria replacing sum of squared halfaxes of $\mathcal{E}$. For instance one can desire to minimize L_∞ norm of $e(t)$ (as is done in Abedor et al. (1996)). This problem can be written as $r \to \min$ subject to above LMIs and additional constraint $P \leqslant rI$. Other performance criteria can be treated in similar way.

Finally, robust version of filtering problem can be considered. That is we suppose that system matrices A, D contain uncertainty $\Delta A, \Delta D$. The goal is to obtain guaranteed error bounds for filtering which hold for all admissible uncertainties. The solution of the problem will be presented in a separate paper.

5. EXAMPLE

Double pendulum in viscous media can serve as an example for the proposed technique. Its state vector is $x = \big(x_1^{\mathrm{T}}\ x_2^{\mathrm{T}}\ v_1^{\mathrm{T}}\ v_2^{\mathrm{T}}\big)^{\mathrm{T}}$, where x_1, x_2 are coordinates of upper and lower loads and v_1, v_2 are their velocities. The measured output is $y = \big(x_1^{\mathrm{T}}\ x_2^{\mathrm{T}}\big)^{\mathrm{T}}$, while estimated output is $y_1 = \big(v_1^{\mathrm{T}}\ v_2^{\mathrm{T}}\big)^{\mathrm{T}}$. Exterior perturbation w is applied to the lower load. For unit values of system parameters and viscosity coefficient 0.2 we obtain the equation

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.2 & 0 \\ 2 & -2 & 0 & -0.2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} w.$$

A priory information on initial value of state is $P_0 = 0.01I$. Applying the proposed approach we get the optimal filter

$$\widehat{F} = \begin{pmatrix} 3.7623 & 0.5417 \\ -0.3920 & 6.0278 \\ 3.9006 & 1.8213 \\ 0.3855 & 43.7484 \end{pmatrix}$$

and guaranteed error bound for the output y_1: its error e_1 lies in the ellipse with matrix $\begin{pmatrix} 0.0102 & 0 \\ 0 & 0.0805 \end{pmatrix}$. Fig. 1 presents this ellipse and two trajectories of $e_1(t)$ (for small and large deviation). Noise was chosen as the locally worst one — which maximizes $\dot{V}(e)$ for given e. It reads

$$w^* = \frac{D^{\mathrm{T}}\widehat{P}^{-1}e}{\|D^{\mathrm{T}}\widehat{P}^{-1}e\|}.$$

Fig. 2 provides trajectory of $v_2(t)$ (bold line) and its estimate $\widehat{v}_2(t)$ (dotted line). The accuracy of filtering is good enough (for $v_1(t)$ it is even higher).

6. CONCLUSIONS

We present simple and effective method of filtering for linear systems with nonrandom bounded noises. The approach is based on invariant ellipsoids technique; its use makes possible to reduce the problem to LMIs, while finding the optimal filter can be performed by using SDP and one-dimensional optimization. Estimation of velocities of double pendulum is taken as an example. The authors acknowledge helpful discussions with P.Scherbakov.

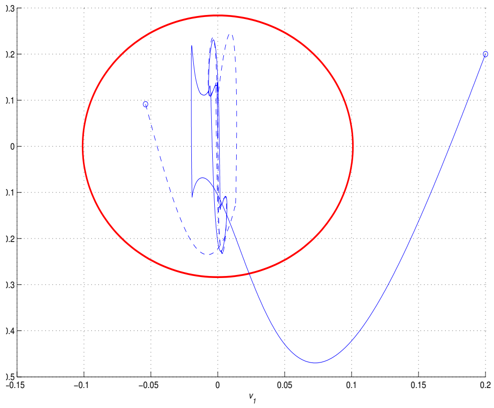

Fig. 1. Guaranteed bound (ellipse) and error trajectories

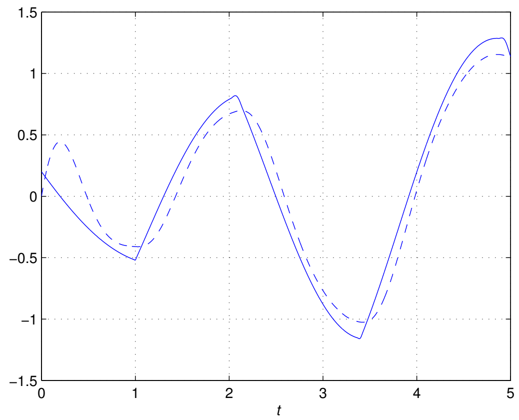

Fig. 2. Filtering of v_2

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