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Extensions of Petersen's Lemma on Matrix Uncertainty ⋆

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Abstract: Proposed are generalizations and refinements of a well-known result on robust matrix sign-definiteness, which is extensively exploited in quadratic stability, design of robust quadratically stabilizing controllers, robust LQR-problem, etc. The main emphasis is put on formulating the results in terms of linear matrix inequalities.

1. INTRODUCTION

The subject of this work is the following model of matrix uncertainty studied in Petersen (1987). Considered is a real symmetric matrix $G \in \mathbb{S}^{n\times n}$ ($\mathbb{S}^{n\times n}$ denotes the space of such $n \times n$–matrices) and its perturbation of the form

$$G + M\Delta N + N^\top \Delta^\top M^\top, \tag{1}$$

where $\Delta \in \mathbb{R}^{p\times q}$ is the perturbation matrix and $M \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{q\times n}$ are fixed "frame" matrices of appropriate dimensions, which define the uncertainty structure. Note that the perturbation for a symmetric matrix G is specified by a matrix Δ, which is not necessarily symmetric and even square.

Such an uncertainty model for symmetric matrices arises naturally when constructing quadratic Lyapunov functions for a dynamic system whose state matrix contains uncertainty Δ. It is this generality that explains a wide range of applications, where model (1) has been found useful. For this model, a necessary and sufficient condition was obtained in Petersen (1987) for the inequality $G + M\Delta N + N^\top\Delta^\top M^\top \le 0$ to hold for all norm-bounded perturbations Δ; in this note, these conditions will be referred to as *Petersen's lemma*. Throughout the exposition, the notion $A \le B$ for matrices stands for the negative semidefiniteness of the matrix $A-B$, i.e., $x^\top(A-B)x \le 0$ for all $x \in \mathbb{R}^n$.

In the seminal paper Petersen (1987), the result discussed here was exploited in the solution of the robust LQR-problem, and in Xie (1996), Khargonekar et al. (1990), it was used in the robust H_∞ control design. Based on this result, common quadratic Lyapunov function for interval matrix families was constructed in Mao and Chu (2003, 2006); a similar uncertainty model was considered in Alamo et al. (2007) to derive a reduced vertex result on the quadratic stability of interval systems.

The present paper is devoted to the analysis and generalizations of Petersen's lemma, with the main emphasis put on the connection of the results obtained with linear matrix inequalities (LMI) and semidefinite programming (SDP). The authors faced the need in such generalizations when working on the problem of robust rejection of exogenous bounded disturbances in systems containing norm-bounded matrix uncertainty, see Polyak et al. (2007). The results obtained are collected in the present paper.

2. PETERSEN'S LEMMA

One of the problems analyzed in Petersen (1987) was the issue of robust sign-definiteness of the matrix G in setup (1). The following result was obtained.

Petersen's lemma (Petersen (1987)). Let $G = G^\top$, $M \neq 0$, $N \neq 0$, be matrices of compatible dimensions. Then for all $\Delta \in \mathbb{R}^{p\times q}$, $\|\Delta\| \le 1$, the inequality

$$G + M\Delta N + N^\top\Delta^\top M^\top \le 0 \tag{2}$$

holds [1] if and only if there exists $\varepsilon > 0$ such that

$$G + \varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N \le 0. \tag{3}$$

We note that in the lemma, the spectral matrix norm is used: $\|\Delta\| = \max_i \lambda_i^{1/2}(\Delta\Delta^\top)$, where $\lambda_i(A)$ are the eigenvalues of A.

Prior to presenting our main results, we propose a simple proof of Petersen's lemma.

Proof. Let

$$G + M\Delta N + N^\top\Delta^\top M^\top \le 0$$

for all $\|\Delta\| \le 1$. This is equivalent to

$$x^\top Gx + 2x^\top M\Delta Nx \le 0$$

for all $x \in \mathbb{R}^n$ and $\|\Delta\| \le 1$. Denoting $x^\top M\Delta \doteq y^\top$, we write the inequality above in the form

$$x^\top Gx + 2y^\top Nx \le 0$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^q$ such that

$$y^\top y \;=\; x^\top M\Delta\Delta^\top M^\top x \;\le\; x^\top MM^\top x.$$

Introducing

$$z \doteq \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+q},\; A_0 \doteq \begin{pmatrix} G & N^\top \\ N & \mathbf{0} \end{pmatrix},\; A_1 \doteq \begin{pmatrix} -MM^\top & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$$

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[1] In the original statement in Petersen (1987), the condition $M \neq 0$, $N \neq 0$ was not imposed since strict inequality was considered in (2).

(where I and $\mathbf{0}$ denote the identity and zero matrices of appropriate dimensions), we re-write it in the following form: $z^\top A_0 z \le 0$ for all z such that $z^\top A_1 z \le 0$.

By applying the S-procedure with one constraint (e.g., see Boyd et al. (1994)), we conclude that the fulfillment of this condition is equivalent to the existence of $\varepsilon \ge 0$ such that $A_0 \le \varepsilon A_1$, i.e.,

$$\begin{pmatrix} G + \varepsilon M M^\top & N^\top \\ N & -\varepsilon I \end{pmatrix} \le 0. \tag{4}$$

Finally, by restricting our considerations to $\varepsilon > 0$ and applying the Schur lemma (see Boyd et al. (1994)), we arrive at the desired result:

$$G + \varepsilon M M^\top + \frac{1}{\varepsilon} N^\top N \le 0.$$

Several comments are due at this point.

First, the proof above is substantionally based on the use of S-procedure. This simplifies considerably the derivation of the result as compared to the one proposed in the original paper Petersen (1987), where the authors had to formulate several auxiliary propositions on the properties of quadratic forms.

Second, Petersen's lemma reduces the robustness test for family (1) to condition (3), i.e., to a simple one-dimensional search. Note that in the equivalent form, condition (3) can be written as the LMI (4) with respect to one scalar uncertainty ε.

3. RADIUS OF SIGN–DEFINITENESS AND THE WORST–CASE PERTURBATION

Petersen's lemma is an analysis result; it provides a necessary and sufficient condition of robust sign-definiteness of family (1) for a fixed level of uncertainty. A natural extension of this result would be finding the maximal admissible level of perturbation Δ in (2) that retains sign-definiteness; this will be referred to as the *radius of sign-definiteness* of the family

$$G(\Delta, \gamma) = G + M\Delta N + N^\top \Delta^\top M^\top, \ \ \|\Delta\| \le \gamma, \tag{5}$$

defined by

$$\gamma_{\max} \doteq \sup\{\gamma : \ G + M\Delta N + N^\top \Delta^\top M^\top < 0 \ \ \forall \ \|\Delta\| \le \gamma\}.$$

Throughout this section, it is assumed that $G < 0$.

Prior to formulating the main result of this section, we note that the quantity $\gamma_{\max}$ can equally be defined as the maximal value of γ retaining the validity of the condition

$$G + \gamma(M\Delta N + N^\top \Delta^\top M^\top) \le 0 \ \ \forall \, \|\Delta\| \le 1. \tag{6}$$

On the other hand, by Petersen's lemma, the fulfillment of (6) for a fixed $\gamma > 0$ is equivalent to the existence of $\varepsilon > 0$ such that

$$G + \gamma\big(\varepsilon M M^\top + \frac{1}{\varepsilon} N^\top N\big) \le 0. \tag{7}$$

These expressions will be used in the sequel.

Proposition 1. Let λ^* be a solution of the following semidefinite program:

$$\min \lambda \ \ \text{s.t.} \ \ \begin{pmatrix} \lambda G + \varepsilon M M^\top & N^\top & \mathbf{0} \\ N & -\varepsilon I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\lambda \end{pmatrix} \le 0 \tag{8}$$

with respect to $\varepsilon, \lambda \in \mathbb{R}$. Then $\gamma_{\max} = 1/\lambda^*$.

Proof. We re-write condition (7) as

$$\frac{1}{\gamma} G + \varepsilon M M^\top + \frac{1}{\varepsilon} N^\top N \le 0,$$

or in the block form by using the Schur lemma:

$$\begin{pmatrix} \frac{1}{\gamma} G + \varepsilon M M^\top & N^\top \\ N & -\varepsilon I \end{pmatrix} \le 0.$$

Introducing $\lambda = 1/\gamma$ and noting that $\gamma > 0$, we incorporate this condition in the constraints to obtain the augmented matrix:

$$\begin{pmatrix} \lambda G + \varepsilon M M^\top & N^\top & \mathbf{0} \\ N & -\varepsilon I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\lambda \end{pmatrix} \le 0. \tag{9}$$

Negative semidefiniteness of the original matrix (7) for some $\varepsilon > 0, \gamma > 0$ is equivalent to feasibility of the LMI (9) with respect to ε, λ. By solving the semidefinite program

$$\min \lambda \quad \text{subject to constraints} \ \ (9), \tag{10}$$

we obtain the desired maximal value $\gamma_{\max} = 1/\lambda^*$, where λ^* is the solution of problem (10).

With this result, the computation of the robustness radius reduces to a standard semidefinite program which can be easily solved numerically, e.g., by using well-known toolboxes SeDuMi and Yalmip in MATLAB.

Proposition 1 presents a convenient tool for computing the robustness radius; however, it gives no insight to the "physical meaning" of the quantities involved. Also, an important issue of worst-case perturbation cannot be addressed.

Namely, the perturbation $\Delta = \Delta_{cr}$, $\|\Delta_{cr}\| = \gamma_{\max}$, is said to be *worst-case* or *critical* if it violates the strict inequality $G(\Delta, \gamma_{\max}) < 0$, i.e., $G(\Delta, \gamma_{\max})$ becomes singular, $\lambda_{\max}\big(G(\Delta_{cr}, \gamma_{\max})\big) = 0$.

The theorem below clarifies these issues.

Theorem 1. Let λ^*, ε^* be solutions of problem (8); then

$$\lambda^* = \lambda_{\max}\big(\varepsilon^* \tilde{M}\tilde{M}^\top + \frac{1}{\varepsilon^*} \tilde{N}^\top \tilde{N}\big),$$

where $\tilde{M} = (-G)^{-1/2} M$, $\tilde{N} = N(-G)^{-1/2}$. Let e be the eigenvector of the matrix $\varepsilon^* \tilde{M}\tilde{M}^\top + \frac{1}{\varepsilon^*} \tilde{N}^\top \tilde{N}$ associated with the eigenvalue λ^*. Then the worst-case perturbation has the form

$$\Delta_{cr} = \frac{1}{\lambda^*} \frac{\tilde{M}^\top e e^\top \tilde{N}^\top}{\|\tilde{M}^\top e\| \, \|\tilde{N} e\|}, \tag{11}$$

and the following relation holds:

$$\varepsilon^* = \|\tilde{N} e\| / \|\tilde{M}^\top e\|. \tag{12}$$

Proof. Let us pre-multiply and post-multiply inequality (7) by $(-G)^{-1/2} > 0$ and re-write it in the equivalent form

$$\gamma\big(\varepsilon \tilde{M}\tilde{M}^\top + \frac{1}{\varepsilon} \tilde{N}^\top \tilde{N}\big) \le I$$

(i.e., without loss of generality, $G = -I$ can be considered in problem (5)). It is seen that the maximal value of γ retaining the last inequality for a fixed ε is equal to

$$\gamma_{\max}(\varepsilon) = \frac{1}{\lambda_{\max}\big(\varepsilon\tilde M\tilde M^\top + \frac{1}{\varepsilon}\tilde N^\top\tilde N\big)}\,, \tag{13}$$

so that the subsequent maximization in $\varepsilon > 0$ yields

$$\gamma_{\max} = \frac{1}{\min\limits_{\varepsilon>0}\lambda_{\max}\big(\varepsilon\tilde M\tilde M^\top + \frac{1}{\varepsilon}\tilde N^\top\tilde N\big)}\,, \tag{14}$$

which is, therefore, the maximal value of γ such that inequality (7) is satisfied for some $\varepsilon > 0$. By equivalence of (7) and (6), this coincides with the radius of robustness. Hence, according to (14), for the solution λ^* of problem (8) we have

$$\lambda^* = \min_{\varepsilon>0}\lambda_{\max}\big(\varepsilon\tilde M\tilde M^\top + \frac{1}{\varepsilon}\tilde N^\top\tilde N\big). \tag{15}$$

Next, to find the worst-case uncertainty, we consider inequality (6) and pre-multiply and post-multiply it by $(-G)^{-1/2} > 0$ to obtain the equivalent condition

$$\gamma\big(\tilde M\Delta\tilde N + \tilde N^\top\Delta^\top\tilde M^\top\big) \;\le\; I \quad \text{for all} \quad \|\Delta\| \le 1,$$

or

$$\gamma\, x^\top\big(\tilde M\Delta\tilde N + \tilde N^\top\Delta^\top\tilde M^\top\big)x \le 1 \quad \forall\ \|\Delta\| \le 1,\, \|x\| = 1,$$

i.e.,

$$2\gamma\, x^\top\tilde M\Delta\tilde N x \le 1 \quad \forall\ \|\Delta\| \le 1,\, \|x\| = 1. \tag{16}$$

Denote $a = \tilde M^\top x$, $b = \tilde N x$. It is easily shown that for any vectors a, b and matrix Δ of compatible dimensions, the following relation holds:

$$\max_{\|\Delta\|\le 1} a^\top\Delta b = \|a\|\,\|b\|$$

(in the spectral or Frobenius matrix norm), and the maximum is attained with

$$\Delta^* = \frac{ab^\top}{\|a\|\,\|b\|}.$$

Therefore, for a fixed x, the maximum of the left-hand side of (16) with respect to $\|\Delta\| \le 1$ is attained with the rank-one matrix

$$\Delta^*(x) = \frac{\tilde M^\top x x^\top\tilde N^\top}{\|\tilde M^\top x\|\,\|\tilde N x\|}\,, \tag{17}$$

and the maximal value is equal to $2\gamma\|\tilde M^\top x\|\,\|\tilde N x\|$. It now remains to maximize this quantity with respect to $\|x\| = 1$ to obtain

$$\gamma_{\max} = \frac{1}{2\max\limits_{\|x\|=1}\|\tilde M^\top x\|\,\|\tilde N x\|}\,. \tag{18}$$

This relation will be used in the sequel.

Finally, let ε^* denote the value of ε which attains the minimum in (15); then λ^* is the corresponding maximal eigenvalue, and let e be the normalized eigenvector of the matrix $\varepsilon^*\tilde M\tilde M^\top + \frac{1}{\varepsilon^*}\tilde N^\top\tilde N$ associated with this eigenvalue λ^*. We have

$$\lambda^* = e^\top\big(\varepsilon^*\tilde M\tilde M^\top + \frac{1}{\varepsilon^*}\tilde N^\top\tilde N\big)e = \varepsilon^*\|\tilde M^\top e\|^2 + \frac{1}{\varepsilon^*}\|\tilde N e\|^2 \ge 2\|\tilde M^\top e\|\,\|\tilde N e\|;$$

moreover, the equality is attained with $\varepsilon^* = \|\tilde N e\|/\|\tilde M^\top e\|$. In accordance with (17), consider the admissible perturbation

$$\Delta = \frac{\tilde M^\top e e^\top\tilde N^\top}{\|\tilde M^\top e\|\,\|\tilde N e\|}\,, \quad \|\Delta\| = 1,$$

so that $-I + \frac{1}{\lambda^*}\big(\tilde M\Delta\tilde N + \tilde N^\top\Delta^\top\tilde M^\top\big) \le 0$ holds. We have

$$\begin{aligned}
\big(\tilde M\Delta\tilde N &+ \tilde N^\top\Delta^\top\tilde M^\top\big)e = \\
&= \Big(\tilde M\frac{\tilde M^\top e e^\top\tilde N^\top}{\|\tilde M^\top e\|\,\|\tilde N e\|}\tilde N + \tilde N^\top\frac{\tilde N e e^\top\tilde M}{\|\tilde M^\top e\|\,\|\tilde N e\|}\tilde M^\top\Big)e \\
&= \Big(\frac{\|\tilde N e\|}{\|\tilde M^\top e\|}\tilde M\tilde M^\top + \frac{\|\tilde M^\top e\|}{\|\tilde N e\|}\tilde N^\top\tilde N\Big)e \\
&= \big(\varepsilon^*\tilde M\tilde M^\top + \frac{1}{\varepsilon^*}\tilde N^\top\tilde N\big)e \\
&= \lambda^* e
\end{aligned}$$

by the definition of the quantities ε^*, λ^*, e. In other words, this means $\lambda_{\max}\big(-I + \frac{1}{\lambda^*}(\tilde M\Delta\tilde N + \tilde N^\top\Delta^\top\tilde M^\top)\big) = 0$; i.e., the considered perturbation is a worst-case one. Proof of the theorem is complete.

The quantity $\gamma_{\max}$ can be determined by other means, namely, using the notion of boundary oracle, see Polyak and Shcherbakov (2006). Indeed, inequality (7) specifies the domain of negative semidefiniteness of the matrix family $W(\gamma,\varepsilon) = G + \gamma\big(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N\big)$ in the space of parameters γ, ε under the additional constraint $\gamma, \varepsilon > 0$. For any fixed $\varepsilon > 0$ we have $W(0,\varepsilon) = G < 0$, and the minimal value of $\gamma > 0$ that violates sign-definiteness of $W(\gamma,\varepsilon)$ is equal to the minimal generalized eigenvalue (all of them are positive), see Lemma 1 in Polyak and Shcherbakov (2006):

$$\gamma(\varepsilon) \;=\; \min_i \lambda_i\Big(G,\; -(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N)\Big). \tag{19}$$

Recall that a number λ and a vector e are said to be a generalized eigenvalue and the associated generalized eigenvector of the pair of matrices A and B, if $Ae = \lambda Be$.

By optimizing (19) with respect to ε we obtain

$$\gamma_{\max} \;=\; \max_{\varepsilon>0}\lambda_{\min}\Big(G,\; -(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N)\Big), \tag{20}$$

which is the largest γ for which there exists $\varepsilon > 0$ such that the matrix (7) is negative semidefinite.

The results of Theorem 1 can accordingly be formulated in terms of generalized eigenvalues. Namely,

$$\gamma_{\max} = \lambda_{\min}\Big(G,\; -(\varepsilon^* MM^\top + \frac{1}{\varepsilon^*}N^\top N)\Big) \;\doteq\; \lambda^*_{\min}.$$

Respectively, the worst-case perturbation has the form $\Delta_{cr} = \lambda^*_{\min}M^\top e e^\top N^\top/\|M^\top e\|\,\|Ne\|$, where e is the generalized eigenvector associated with the indicated generalized eigenvalue, and the relation $\varepsilon^* = \|Ne\|/\|M^\top e\|$ holds.

We also note that, since $G < 0$ (i.e., nonsingular), usual (not generalized) eigenvalues $\lambda_i(\varepsilon)$ of the matrix $G^{-1}(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N)$ can be taken and the smallest among them (they are all non-positive) is then to be chosen. Then $\gamma(\varepsilon) = -1/\lambda_{\min}(\varepsilon)$ and $\gamma_{\max} = \max\limits_{\varepsilon>0}\gamma(\varepsilon)$.

Example 1. We consider $G = -\mathrm{diag}\,(1\ \ 2\ \ 3)$ and the following randomly generated square matrices M, N:

$$M = \begin{pmatrix} 0.1822 & 0.0450 & 0.3062 \\ -0.2952 & 0.3110 & 0.1072 \\ -0.3844 & -0.1084 & -0.2866 \end{pmatrix};$$

$$N = \begin{pmatrix} -0.3328 & 0.4952 & -0.6984 \\ 0.3360 & 0.2248 & -0.4464 \\ 0.7896 & 0.6108 & -0.2980 \end{pmatrix}.$$

Figure 1 depicts the curve $\gamma(\varepsilon)$ (19); together with the line $\gamma = 0$ it defines the domain of negative-definiteness of the family $G + \gamma\big(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N\big)$, $\gamma, \varepsilon > 0$. Clearly, the plot of the function $\gamma(\varepsilon)$ coincides with $\gamma_{\max}(\varepsilon)$ (13); the maximum over $\varepsilon > 0$ is equal to 1.2044. Solving problem (8) leads to the same value $\gamma_{\max} = 1.2044$.

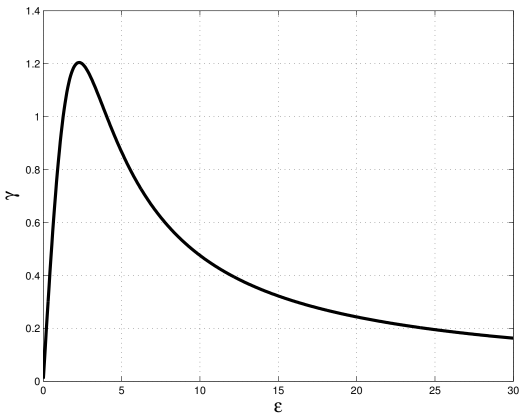

Fig. 1. Computation of the robustness radius.

4. RADIUS OF NONSINGULARITY

The next extension of Petersen's lemma relates to finding the *radius of nonsingularity* in problem (6) under the condition that the matrix G is symmetric but sign-indefinite. Reasonings similar to those used in the previous section lead to the following result, which we formulate in terms of generalized eigenvalues.

Theorem 2. Let $G \in \mathbb{S}^{n\times n}$ be nonsingular; then the *radius of nonsingularity*

$$\rho(G, M, N) \doteq \sup\{\|\Delta\|\colon\ G + M\Delta N + N^\top\Delta^\top M^\top \text{ is nonsingular}\}$$

is given by

$$\rho(G, M, N) = \max_{\varepsilon>0}\min_i |\lambda_i(\varepsilon)|, \tag{21}$$

where

$$\lambda_i(\varepsilon) = \lambda_i\Big(G,\ -(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N)\Big)$$

are the generalized eigenvalues of the pair of matrices G and $-(\varepsilon MM^\top + \frac{1}{\varepsilon}N^\top N)$. The worst-case perturbation is given by

$$\Delta_{cr} = \overline{\lambda}\frac{M^\top e e^\top N^\top}{\|M^\top e\|\|Ne\|},$$

where $\overline{\lambda}$ is the generalized eigenvalue that attains the optimum in (21), and e is the associated generalized eigenvector.

It is noted that the only difference between formula (21) and expression (20) for the radius of sign-definiteness is the presence of the absolute value sign for the eigenvalues. This result is expected, since nonsingularity is lost at an eigenvalue which is "closest to zero."

Theorem 2 generalizes Petersen's lemma to the case of symmetric sign-indefinite matrices, and for $G < 0$ we arrive at Proposition 1. On the other hand, for a specific case where the frame matrices in (2) are identities, $M = N = I$, and the perturbation matrix Δ is assumed to be symmetric, the following result on the *symmetric* radius of nonsingularity is known.

Lemma 1 (Polyak and Shcherbakov (2006)). For a nonsingular matrix $G \in \mathbb{S}^{n\times n}$, its symmetric radius of nonsingularity defined as

$$r(G) \doteq \sup\{\|\Delta\|\colon\ \Delta \in \mathbb{S}^{n\times n},\ \ G + \Delta\ \text{ is nonsingular}\}$$

is given by

$$r(G) = 1/\|G^{-1}\| = \min_i |\lambda_i(G)|,$$

and the critical value of Δ is equal to $\Delta_{cr} = -\lambda ee^\top$, where λ is the minimal (by absolute value) eigenvalue of G and e is the associated normalized eigenvector.

Hence, Theorem 2 extends this result to the more general uncertainty structure (2). For $M = N = I$ we have $G + (\Delta + \Delta^\top)/2$, and denoting $\Delta_1 = (\Delta + \Delta^\top)/2 \in \mathbb{S}^{n\times n}$, we arrive at the conditions of Lemma 1 and obtain the relation $r(G) = 2\rho(G, I, I)$.

5. MULTIPLE UNCERTAINTIES

In this section, we analyze Petersen's lemma in the situation where several uncertainties are brought into the picture. We consider $\ell > 1$ independent perturbations $\Delta_i \in \mathbb{R}^{p_i\times q_i}$ in (6), where the matrices M_i, N_i have dimensions $n \times p_i$ and $q_i \times n$ respectively. The goal is to check the condition

$$G + \sum_{i=1}^{\ell}(M_i\Delta_i N_i + N_i^\top\Delta_i^\top M_i^\top) \le 0 \qquad \text{for all } \|\Delta_i\| \le \gamma,\quad i = 1,\ldots,\ell, \tag{22}$$

for a given level γ and to find the robustness radius $\gamma_{\max}$, which is the maximal value of γ retaining the validity of (22). Clearly, the span γ can be considered common for all Δ_i by introducing appropriate scalar multipliers in the matrices M_i (or N_i).

In that case, Petersen's lemma provides only sufficient conditions, which is formulated next.

Theorem 3. For a given $\gamma > 0$, condition (22) is satisfied if there exist positive scalars $\varepsilon_1, \ldots, \varepsilon_\ell$ such that

$$G + \gamma\sum_{i=1}^{\ell}\big(\varepsilon_i M_i M_i^\top + \frac{1}{\varepsilon_i}N_i^\top N_i\big) \le 0. \tag{23}$$

The maximal value of γ retaining the validity of (23) for some $\varepsilon_1, \ldots, \varepsilon_\ell > 0$ is equal to $\gamma_{\text{est}} = 1/\lambda^*$, where λ^* is the solution of the semidefinite program

$$\min \lambda \ \text{ s.t. } \begin{pmatrix} \lambda G + \sum_{i=1}^{\ell} \varepsilon_i M_i M_i^\top & N_1^\top & \dots & N_\ell^\top & \mathbf{0} \\ N_1 & -\varepsilon_1 I & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & & \\ N_\ell & & & -\varepsilon_\ell I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} & -\lambda \end{pmatrix} \le 0 \quad (24)$$

with respect to $\lambda, \varepsilon_1, \dots, \varepsilon_\ell \in \mathbb{R}$.

The proof is rather technical and straightforward and is therefore ommited.

To the best of our knowledge, this result has never been formulated in the present LMI form. Theorem 3 provides a simple way for computing the lower bound γ_{est} for the robustness radius $\gamma_{\max}$ in the situation with multiple uncertainties, — by reducing to the appropriate semidefinite program. The accuracy of this estimate will be discussed in the subsections to follow.

5.1 Special cases

Relation (23) provides only a sufficient condition for (22) to hold. This is explained by the fact that the S-procedure with $\ell \ge 2$ constraints is lossy; e.g., see Petersen et al. (2000). However, in certain situations, this restriction can be removed.

First, assume that all the quantities involved are defined over the field of complex numbers. In that case, the S-procedure is lossless for two constraints (see Brickman (1961)), and Petersen's lemma provides *necessary and sufficient* conditions of robust sign-definiteness of a matrix subjected to $\ell = 2$ independent perturbations. We do not analyze this case in detail since it is out of the scope of this paper.

The second case relates to more stringent conditions on the uncertainty structure in (22). Specifically, assume that the uncertainties in (22) are subjected to the joint constraint $\|\Delta\| \le 1$, where

$$\Delta = (\Delta_1 \ \dots \ \Delta_\ell)$$

is the compound $p \times \hat{q}$-matrix and $\hat{q} = \sum_{i=1}^{\ell} q_i$. Moreover, let $M_i \equiv M$. We have the following result.

Lemma 2. Consider $M_1 = \dots = M_\ell = M$ in (22). Then the inequality

$$G + \sum_{i=1}^{\ell} \big(M_i \Delta_i N_i + N_i^\top \Delta_i^\top M_i^\top \big) \le 0$$

holds for all Δ_i such that $\big\|\Delta_1 \cdots \Delta_\ell\big\| \le 1$ if and only if there exists $\varepsilon > 0$ such that

$$G + \varepsilon M M^\top + \frac{1}{\varepsilon} \sum_{i=1}^{\ell} N_i^\top N_i \ \le \ 0.$$

Proof. We have

$$G + \sum_{i=1}^{\ell} \big(M_i \Delta_i N_i + N_i^\top \Delta_i^\top M_i^\top \big) =$$

$$= G + M \big(\Delta_1 \ \dots \ \Delta_\ell\big) \begin{pmatrix} N_1 \\ \vdots \\ N_\ell \end{pmatrix} + \begin{pmatrix} N_1 \\ \vdots \\ N_\ell \end{pmatrix}^\top \big(\Delta_1 \ \dots \ \Delta_\ell\big)^\top M^\top.$$

Denoting $\widehat{N} = \begin{pmatrix} N_1 \\ \vdots \\ N_\ell \end{pmatrix} \in \mathbb{R}^{\hat{q} \times n}$, we arrive at the standard Petersen's setup $G + M \Delta \widehat{N} + \widehat{N}^\top \Delta^\top M^\top$.

Obviously, a similar result holds for the case $N_i \equiv N$, $i = 1, \dots, \ell$.

5.2 On the accuracy of γ_{est}

In Mao and Chu (2003), the fulfilment of (22) was claimed to be *equivalent* to the existence of positive scalars $\varepsilon_1, \dots, \varepsilon_\ell$ such that (23) holds. Later, the proof was found to be erroneous, and in Mao and Chu (2006), an anecdotal counterexample has been proposed showing that condition (23) is only sufficient for family (22) to be robustly sign-definite. The respective values in Mao and Chu (2006) were reported to be $\gamma_{\max} \approx 2.696$ and $\gamma_{\text{est}} \approx 2.663$; below, a meaningful counterexample is constructed, where the difference is much bigger.

Since the perturbations Δ_i in (22) are independent, similarly to the derivation of formulae (18) and (14), we obtain

$$\gamma_{\max} = \frac{1}{2 \max\limits_{\|x\|=1} \sum\limits_{i=1}^{\ell} \|\tilde{M}_i^\top x\| \, \|\tilde{N}_i x\|}, \quad (25)$$

$$\gamma_{\text{est}} = \frac{1}{\min\limits_{\varepsilon_i > 0} \lambda_{\max} \Big(\sum\limits_{i=1}^{\ell} \big(\varepsilon_i \tilde{M}_i \tilde{M}_i^\top + \frac{1}{\varepsilon_i} \tilde{N}_i^\top \tilde{N}_i \big) \Big)}, \quad (26)$$

where $\tilde{M} = (-G)^{-1/2} M$, $\tilde{N} = N(-G)^{-1/2}$. We now indicate specific matrices G, M_i, N_i leading to $\gamma_{\text{est}} < \gamma_{\max}$.

Example 2. Likewise Mao and Chu (2003, 2006), we consider the case of $\ell = 2$ *scalar* uncertainties $\Delta_1, \Delta_2 \in \mathbb{R}$; then the matrices M_1, M_2 are column vectors, and the matrices N_1, N_2 are row vectors. We take $G = -I$ and

$$M_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad N_1 = (\, 1 \ \ 0 \,),$$

$$M_2 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \quad N_2 = \big(\sqrt{2}/2 \ \ -\sqrt{2}/2 \big).$$

Note that the vectors $M_1, N_1^\top$ and $M_2, N_2^\top$ are normalized, pairwise orthogonal, and the angle between the pairs is equal to $\pi/2\ell$.

Straightforward algebraic manipulations performed with expressions (25) and (26) lead to

$$\gamma_{\max} = \sqrt{2}/2, \qquad \gamma_{\text{est}} = \frac{1}{2},$$

so that $\gamma_{\max}/\gamma_{\text{est}} = \sqrt{2}$.

Analogous examples where $\gamma_{\text{est}} < \gamma_{\max}$ are easy to construct for $\ell > 2$ scalar uncertainties; e.g., by choosing the two-dimensional vectors in the form

$$M_i = \begin{pmatrix} \cos(\omega + (i-1)\delta) \\ \sin(\omega + (i-1)\delta) \end{pmatrix},$$

$$N_i = \big(\sin(\omega + (i-1)\delta) \quad -\cos(\omega + (i-1)\delta) \big),$$

for $i = 1, \dots, \ell$, where $\delta = \pi/2\ell$ and ω is arbitrary.

At present, for the general case of matrix-valued uncertainties, the question on how conservative the estimate γ_{est} may be remains open.

However, the accuracy of $\gamma_{\rm est}$ can be tested in the following way. Since the perturbations Δ_i vary independently of each other, it is readily seen that the relations similar to (12) are valid for the solutions ε_i^* of problem (24) for ε_i. Namely, we have $\varepsilon_i^* = \|\tilde{N}_i e\|/\|\tilde{M}_i^\top e\|$, where e is the eigenvector associated with the maximal eigenvalue λ^* of the matrix $\sum_{i=1}^{\ell}(\varepsilon_i^*\tilde{M}_i\tilde{M}_i^\top + \frac{1}{\varepsilon_i^*}\tilde{N}_i^\top\tilde{N}_i)$. Hence, by analogy with (11), we consider

$$\Delta_{cr,i} \;=\; \gamma_{\rm est}\frac{\tilde{M}_i^\top e e^\top \tilde{N}_i^\top}{\|\tilde{M}_i^\top e\|\|\tilde{N}_i e\|}, \qquad i = 1,\ldots,\ell,$$

as candidate worst-case perturbations. It then follows that $\gamma_{\max} \approx \gamma_{\rm est}$ if

$$\lambda_{\max}\Big(G + \sum_{i=1}^{\ell}(M_i\Delta_{cr,i}N_i + N_i^\top\Delta_{cr,i}^\top M_i^\top)\Big) \;\approx\; 0.$$

5.3 More on scalar uncertainties

This situation closely relates to the problem of robust stability of the $n\times n$ interval symmetric matrix family $A+\gamma\Delta$, where $A = A^\top < 0$, $\Delta = \Delta^\top$, $|\Delta_{ij}| \le r_{ij}$. Theorem 2 in Rohn (1994) reduces this problem to checking the negative definiteness of 2^{n-1} vertex matrices $G+\gamma S_m R S_m$, where $R = (r_{ij})$ is the symmetric matrix composed of the positive numbers r_{ij}, and $S_m = \mathrm{diag}\,(1, s_1, \ldots, s_{n-1})$, where $s_k = \pm 1$. For very high dimensions n, solution becomes computationally intractable, since the constraint matrix in the respective SDP problem is of dimension $n2^{n-1}$. Instead, Theorem 3 yields an estimate which is easily computable by solving a semidefinite program with $n(n+1)/2+1$ scalar variables and the LMI constraints of size $n+n(n+1)/2+1$.

As mentioned above, the quality of the estimate $\gamma_{\rm est}$ is hard to evaluate satisfactorily. Even for the case of scalar uncertainties, we could only obtain an obvious and very rough bound $\gamma_{\rm est}/\gamma_{\max} \le \ell$. However, simulations show that "on average," the quantity $\gamma_{\rm est}$ just slightly differs from $\gamma_{\max}$. Thus, for 100,000 randomly generated 4×4 interval matrices ($\ell = 10$ independent scalar uncertainties in Petersen's setup), the maximal value of the ratio $\gamma_{\max}/\gamma_{\rm est} \approx 1.15$ was observed, and for approximately 96% of the samples it did not exceed 1.01.

6. CONCLUSION

Several new results are obtained, which seem useful for both a deeper understanding of the criterion considered, and for computing the robustness radius or its lower bound. Among these, we distinguish a transparent proof of Petersen's lemma, the result on the radius of nonsingularity, and the SDP formulation for computing the robustness radius or its estimate $\gamma_{\rm est}$ in the case of multiple uncertainties. Numerical experiments testify to high accuracy of this estimate in the case of scalar uncertainties; however, it seems very important to accurately evaluate the quality of this estimate in closed form.

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