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Stochastic Reachability and Measurement Feedback under Control-Dependent Noise ⋆

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Abstract: This paper deals with reachability under unknown disturbances and incomplete information on the state space variables. The unknown disturbances are described by a special type of vector-valued stochastic Brownian input noise which depends on the values of vector-valued control. The control may be either unbounded or bounded by hard bounds. The reachabilty sets introduced here are deterministic. They consist of all points whose mean-square deviations from a tube of given controlled trajectories are small. The "reach" sets are presented in terms of level sets to solutions of appropriate types of Hamilton-Jacobi-Bellman equation, which depend on the presence or absence of additional hard bounds on the controls. These allow explicit representation of the reach sets when the controls are unbounded and are presented in terms of solutions to some dual optimization problems when the controls are bounded. Accordingly, the reach sets are either ellipsoids or convex compact sets of more complicated structure. The last fact introduces significant changes with transition from scalar to vector-valued control-dependent noise. Finally the notions of reachability and control under incomplete feedback are introduced.

Keywords: Stochastic optimal control problems, Nonlinear observer and filter design, Output feedback control.

1. INTRODUCTION

This paper deals with the problem of reachability — one of the central topics of modern control theory. Its motivation comes from problems in control design, verification of algorithms and other problems in automation, navigation and related areas. Of recent interest are problems of reachability under disturbances and resets. In the case of unknown but bounded disturbances the problem was treated by Kurzhanski and Varaiya [2000, 2002], while reachability for hybrid systems was dealt with by Kurzhanski and Varaiya [2002], Lygeros et al. [1999].

The present report deals with the problem of reachability under stochastic disturbances. In contrast with previous investigations, considered is a continuous-time linear system subjected to perturbations generated by vector-valued Brownian noise with parameters dependent on the values of the vector-valued control (see Digailova and Kurzhanski [2005]). Two cases are considered, namely, those, when the controls are unbounded and those when they are bounded by hard bounds.

The reachability sets introduced here are deterministic. They are presented in terms of level sets to solutions of certain types of the Hamilton-Jacobi-Bellman (HJB) equation. The respective value functions are given by explicit expressions in the domains where the controls are unbounded and are given in terms of solutions to some dual optimization problems where the controls are bounded. Finally the notion of *reachability under incomplete feedback* and stochastic noise is introduced.

Other settings of the problem of stochastic reachability are considered by Lygeros and Watkins [2003], Blom and Lygeros [2006].

Throughout this paper, A' denotes the matrix transpose of A, $\mathrm{tr}(A)$ — the trace of the matrix A, I^s_q is an $s \times s$ matrix with all zero elements except one $i_{q,q} = 1$, I is an identity matrix of appropriate dimension. The notation $A > 0$ means that the matrix A is positive definite. We denote by $E\{x|\cdot\}$ the conditional expectation of random vector x. We denote by (x, y) the scalar product and $\|x\|^2_S = (x, Sx)$. For vector b and set $\mathcal{A}$ consider the distance $d^2_L(b, \mathcal{A}) = \min_a\{\|b - a\|^2_L | a \in \mathcal{A}\}$. We denote $d^2(b, \mathcal{A}) = d^2_I(b, \mathcal{A})$. Also consider distance between sets $\mathcal{A}$ and $\mathcal{B}$ as $d_L(\mathcal{B}, \mathcal{A}) = \min_b\{d_L(b, \mathcal{A}) \mid b \in \mathcal{B}\}$. Finally denote by $\mathcal{E}(a, A)$ an ellipsoid with center a and shape matrix A: $\mathcal{E}(a, A) = \{a : (a, Aa) \le 1\}$ and by $\mathcal{B}_1 = \mathcal{E}(0, I)$ the closed unit ball.

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2. THE SYSTEM AND THE REACHABILITY PROBLEM

Given is a continuous-time stochastic control system (see Astrom [1970], Fleming and Rishel [1975], Fleming and Soner [1993], Liptser and Shiryayev [1977], Kurzhanski [1965])

$$dx = A(t)xdt + B(t)udt + R(t)J(u)d\omega, \qquad (1)$$

with state vector $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^m$, $d\omega \in \mathbb{R}^m$ stands for the normalized vector-valued Brownian motion (see Doob [1953]), with pairwise independent components. Continuous matrix $A(t)$, $B(t)$ and $R(t)$ of appropriate dimensions are given. Also suppose that $R'(t)R(t) > 0$ $\forall\, t \in [\alpha, \beta]$, $[\tau, \vartheta] \subset [\alpha, \beta]$. Matrix $J(u) = \sum_{k=1}^{m} I_k^m u$.

Thus the noise in the system (1) is control dependent. Such types of noise appear in communication systems, reliability problems and financial models. They may also turn out equivalent to output-dependent noise.

An important problem is to describe the *reachability set* (*reach set*) for system (1) (see Lygeros and Watkins [2003] for other approaches).

Definition 1. The γ-reach set $\mathcal{Z}_\gamma[\vartheta] = \mathcal{Z}_\gamma(\vartheta, \tau, x_\tau)$ of system (1) at time ϑ, from position $\{\tau, x_\tau\}$, is the set of all points $z \in \mathbb{R}^n$ for which the inequality

$$V(\tau, x_\tau, z) =$$
$$\min_u \Big\{ E\{\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\} \Big| u \in \mathcal{U}[\tau, \vartheta] \Big\} \le \gamma^2$$

is true.

Here $x[\cdot] = x(\cdot, \tau, x_\tau)$, is the trajectory of system (1), emanating from position $\{\tau, x_\tau\}$ and $\mathcal{U}[\tau, \vartheta]$ is the class of admissible feedback controls $u = u(t, x, z)$, for the reachability problem (we shall consider two types of such classes).

For $\mathcal{Z}_\gamma[\vartheta] \neq \emptyset$ it is necessary that $\gamma^2 \ge \gamma_0^2$, where γ_0^2 will be indicated below.

The problem to be studied is to calculate the set $\mathcal{Z}_\gamma[\vartheta]$ for the following types of controlled systems:

(a) $u \in \mathbb{R}^m$ — control is unbounded,
(b) $\big(u, Q(t)u\big) \le 1$, where $Q(t) = Q'(t) > 0$ is continuous $\forall\, t \in [\alpha, \beta]$, $[\tau, \vartheta] \subset [\alpha, \beta]$.

The main problem is thus to find the value function

$$V(\tau, x_\tau, z) = \min_u E\{\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\}$$

under control constrains (a) or (b).

We shall reduce this problem to an optimization procedure, starting with case (a).

3. REACHABILITY UNDER UNBOUNDED CONTROLS

Consider case (a) (vector $u \in \mathbb{R}^m$).

Problem (a): Find value function

$$V^{(a)}(\tau, x_\tau, z) =$$
$$= \min_u \left\{ E\{\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\} \Big| u \in \mathcal{U}^{(a)}[\tau, \vartheta] \right\}.$$

Class $\mathcal{U}^{(a)}[\tau, \vartheta]$ in its feedback representation comprises all continuous functions of $\{t, x\}$, $t \in [\tau, \vartheta]$, $x \in \mathbb{R}^n$, Lipschitz in x and such that allow extension of solutions $x[t]$ throughout the finite interval $[\tau, \vartheta]$ under consideration.

Function $V^{(a)}(\tau, x_\tau, z)$ satisfies the "principle of optimality" (see Bertsekas [1995], Krasovskii [1960, 1963])

$$V^{(a)}(\tau, x_\tau, z) =$$
$$= \min_u \Big\{ E\Big\{ \min_u \{ E(\|x(\vartheta, t, x[t]) - z\|)^2 | t, x[t] = x\} \Big|$$
$$\Big| u \in \mathcal{U}^{(a)}[t, \vartheta]\} \Big| \tau, x_\tau \Big\} \Big| u \in \mathcal{U}^{(a)}[\tau, t] \Big\} =$$
$$= \min_u \left\{ E\{\|x(\vartheta, \tau, x[\tau]) - z\|^2 | \tau, x_\tau\} \Big| u \in \mathcal{U}^{(a)}[\tau, \vartheta] \right\}.$$

Taking $t = \tau + \sigma$, and applying this principle along the standard lines of Dynamic Programming Theory (see Fleming and Soner [1993]) we obtain (assuming differentiability of $V^{(a)}(\tau, x_\tau, z)$, which is later proved to be true) the following sufficient condition for optimality.

Theorem 1. Function $V^{(a)}(t, x, z)$ satisfies the relation

$$\min_u \left\{ dV^{(a)}(t, x, z)/dt \Big| u \in \mathbb{R}^m \right\} = 0, \qquad (2)$$

under boundary condition

$$V^{(a)}(\vartheta, x, z) = (x - z, x - z). \qquad (3)$$

Here

$$dV^{(a)}(t, x, z)/dt =$$
$$= \lim_{\sigma \to +0} \sigma^{-1} \big(E\{\|x(\vartheta, t + \sigma, x[t + \sigma]) - z\|^2 | t, x\} -$$
$$- V^{(a)}(t, x, z) \big)$$

and equation (2) is actually a backward HJB equation

$$V_t^{(a)} + \min_u \Big\{ (V_x^{(a)}, A(t)x + B(t)u) +$$
$$+ \tfrac{1}{2}(u, K^{(a)}(t)u) \Big| u \in \mathbb{R}^m \Big\} = 0. \qquad (4)$$

Here $V_t^{(a)}$ is the partial derivative of $V^{(a)}(t, x, z)$ in t, $V_x^{(a)}$ — its n-dimensional gradient vector and $V_{xx}^{(a)}$ — its matrix of second partials (the Hessian). Matrix $K^{(a)}(t) = \sum_{q=1}^{n} I_q^n R'(t) V_{xx}^{(a)} R(t) I_q^n$.

Minimizing over u, we have

$$u_*^{(a)}(t, x, z) = -(K^{(a)}(t))^{-1} B'(t) V_x^{(a)}, \qquad (5)$$

where u_* is the minimizer in (4).

Function $V^{(a)}(t, x, z)$ may be found in explicit form, namely, as

$$V^{(a)}(t, x, z) = (x - z, P^{(a)}(t)(x - z)) +$$
$$+ 2(q^{(a)}(t), x - z) + \kappa^{(a)}(t), \qquad (6)$$

where matrix $P^{(a)}(t) = (P^{(a)}(t))' > 0$ and vector $q^{(a)}(t)$ with scalar function $\kappa^{(a)}(t)$ are continuous.

Then

$$u_*^{(a)}(t, x, z) =$$
$$= -(K^{(a)}(t))^{-1} B'(t) \left(P^{(a)}(t)(x - z) + q^{(a)}(t) \right), \qquad (7)$$

where now $K^{(a)}(t) = \sum_{q=1}^{n} I_q^n R'(t) P^{(a)}(t) R(t) I_q^n$.

Substituting (6) and (7) in (4) and equating terms with multipliers $(x_i - z_i)(x_j - z_j)$, $(x_i - z_i)$, $i, j = 1, \ldots, n$ and also the free terms, we come to the next system of equations:

$$\dot{P}^{(a)}(t) + A'(t)P^{(a)}(t) + P^{(a)}(t)A(t) -$$

$$-P^{(a)}(t)B(t)(K^{(a)}(t))^{-1}B'(t)P^{(a)}(t) = 0 \quad (8)$$

$$\dot{q}^{(a)}(t) + (A'(t) - P^{(a)}(t)B(t)(K^{(a)}(t))^{-1}B'(t))q^{(a)}(t) + \\ +P^{(a)}(t)A(t)z = 0, \quad (9)$$

$$\dot{\kappa}^{(a)}(t) + 2(q^{(a)}(t))'A(t)z - \\ -(q^{(a)}(t))'B(t)(K^{(a)}(t))^{-1}B'(t)q^{(a)}(t) = 0 \quad (10)$$

with boundary conditions which follow from (3):

$$P^{(a)}(\vartheta) = I, \quad q^{(a)}(\vartheta) = 0, \quad \kappa^{(a)}(\vartheta) = 0. \quad (11)$$

Note that since $K^{(a)}(t)$ depends on $P^{(a)}(t)$, equation (8) *is not a Riccati equation.*

Nevertheless, system (8)–(11) is solvable, consisting of a linear equation in vector $q^{(a)}(t)$, an integral equality for $\kappa^{(a)}(t)$ and a well-posed matrix equation (8) in $P^{(a)}(t)$. The proof follows the lines of (Kurzhanski [1965]) wherein it is shown that equation (8) with initial condition (11) has a unique solution, extendable throughout the whole interval. This fact indicates that $V^{(a)}(t, x, z)$ is differentiable.

As follows from Definition 1, the desired reach set $\mathcal{Z}_\gamma[\vartheta] = \mathcal{Z}_\gamma^{(a)}[\vartheta]$ may now be described within the next statement.

Theorem 2. The following representation is true

$$\mathcal{Z}_\gamma^{(a)}[\vartheta] = \{z : V^{(a)}(\tau, x_\tau, z) \leq \gamma^2\}.$$

Note that $q^{(a)}(t)$ and $\kappa^{(a)}(t)$ may be represented as

$$q^{(a)}(t) = F^{(a)}(t, \vartheta)z, \quad \kappa^{(a)}(t) = (z, H^{(a)}(t, \vartheta)z),$$

where

$$F^{(a)}(t, \vartheta) = \int_t^\vartheta X_{M^{(a)}}(t, s)P^{(a)}(s)A(s)ds,$$

$$M^{(a)}(t) = A'(t) - P^{(a)}(t)B(t)(K^{(a)}(t))^{-1}B'(t),$$

$$\dot{X}_{M^{(a)}}(t, s) = -M^{(a)}(t)X_{M^{(a)}}(t, s), \quad X_{M^{(a)}}(s, s) = I,$$

$$H^{(a)}(t, \vartheta) = \int_t^\vartheta \Big(2A'(s)F^{(a)}(s, \vartheta) - \\ -(F^{(a)}(s, \vartheta))'B(s)(K^{(a)}(s))^{-1}B'(s)F^{(a)}(s, \vartheta)\Big)ds.$$

Therefore

$$V^{(a)}(t, x, z) = (x - z, P^{(a)}(t)(x - z)) + \\ +2(z, (F^{(a)}(t, \vartheta))'(x - z)) + (z, H^{(a)}(t, \vartheta)z) = \\ = (z - x, \mathcal{P}^{(a)}(t, \vartheta)(z - x)) - 2(x, \mathcal{N}^{(a)}(t, \vartheta)(z - x) - \\ -(x, H^{(a)}(t, \vartheta)x).$$

Here

$$\mathcal{P}^{(a)}(t, \vartheta) = P^{(a)}(t) - 2(F^{(a)}(t, \vartheta))' + H^{(a)}(t, \vartheta),$$

$$\mathcal{N}^{(a)}(t, \vartheta) = (F^{(a)}(t, \vartheta))' - H^{(a)}(t, \vartheta).$$

One may further observe, after some calculations, that function $V^{(a)}(\tau, x_\tau, z)$ may be represented in following form

$$V^{(a)}(\tau, x_\tau, z) = \\ = \|z - \big(I - (\mathcal{P}^{(a)}(\tau, \vartheta))^{-1}(\mathcal{N}^{(a)}(\tau, \vartheta))'\big)x_\tau\|^2_{\mathcal{P}^{(a)}(\tau,\vartheta)} + \\ +(k^{(a)}[\vartheta])^2,$$

where

$$(k^{(a)}[\vartheta])^2 = \Big(x_\tau, \mathcal{R}^{(a)}(\tau, \vartheta)x_\tau\Big),$$

$$\mathcal{R}^{(a)}(\tau, \vartheta) = \\ = \mathcal{N}^{(a)}(\tau, \vartheta)(\mathcal{P}^{(a)}(\tau, \vartheta))^{-1}(\mathcal{N}^{(a)}(\tau, \vartheta))' + H^{(a)}(t, \vartheta).$$

Therefore, the following theorem is true.

Theorem 3. The γ-reach set of system (1) $\mathcal{Z}_\gamma^{(a)}[\vartheta]$ may be represented as an ellipsoid

$$\mathcal{Z}_\gamma^{(a)}[\vartheta] = \mathcal{E}\Big(z^{(a)}[\vartheta], \big(\gamma^2 - (k^{(a)}[\vartheta])^2\big)\big(Z^{(a)}[\vartheta]\big)^{-1}\Big) = \\ = \Big\{z : \big(z - z^{(a)}[\vartheta], Z^{(a)}[\vartheta](z - z^{(a)}[\vartheta])\big) \leq \gamma^2 - (k^{(a)}[\vartheta])^2\Big\}$$

with center

$$z^{(a)}[\vartheta] = \Big(I + (\mathcal{P}^{(a)}(\tau, \vartheta))^{-1}(\mathcal{N}^{(a)}(\tau, \vartheta))'\Big)x_\tau$$

and shape matrix $\big(Z^{(a)}[\vartheta]\big)^{-1} = \mathcal{P}^{(a)}(\tau, \vartheta)$. This ellipsoid is nonempty iff $\gamma^2 \geq \gamma_0^2 = (k^{(a)}[\vartheta])^2$.

4. REACHABILITY UNDER HARD BOUNDS ON THE CONTROLS

Let $G(t, \vartheta)$ be the fundamental transition matrix of the homogeneous part of system (1). Applying the transformation $z = G^{-1}(t, \vartheta)x$ to equation (1) and retaining the original notations we come to

$$dx = B(t)udt + R(t)J(u)d\omega. \quad (12)$$

Consider first the system (12) with $J(u) = I$. Then the term $R(t)d\omega$ will be *independent* of u, but with control $u \in \mathbb{R}^m$ restricted by additional *hard bound*:

$$(u, Q(t)u) \leq 1, \quad (13)$$

where $Q(t) = Q'(t) > 0$ is continuous $\forall\ t \in [\alpha, \beta]$, $[\tau, \vartheta] \subset [\alpha, \beta]$. In other words $u \subset \mathcal{E}(0, Q^{-1}(t))$.

Problem (b): Find value function

$$V^{(b)}(\tau, x_\tau, z) = \\ = \min_u \Big\{E\{\|x[\vartheta] - z\|^2|\tau, x(\tau) = x_\tau\}\Big|\mathbf{u} \in \mathcal{U}^{(b)}[\tau, \vartheta]\Big\},$$

where

$$\mathcal{U}^{(b)}[\tau, \vartheta] = \Big\{\mathbf{u}(t, x, z) : \mathbf{u} \subset \mathcal{E}(0, Q^{-1}(t))\Big\}$$

are set-valued controls with convex compact values, upper semicontinuous in $\{t, x\}$.

With noise independent of u this problem is immediately reduced to the next one, which is deterministic:

Problem (b-1): Find

$$V_1^{(b)}(\tau, x_\tau, z) = \\ = \min_u \Big\{\|x[\vartheta] - z\|^2\Big|\tau, x(\tau) = x_\tau, \mathbf{u} \in \mathcal{U}^{(b)}[\tau, \vartheta]\Big\}.$$

Direct calculation through methods of convex analysis (see Rockafellar [1970]) gives

$$V_1^{(b)}(\tau, x_\tau, z) = d^2(z, \mathcal{X}[\vartheta]),$$

where $\mathcal{X}[\vartheta]$ is the reachability set of system (12)–(13) at time ϑ, from position $\{\tau, x_\tau\}$, with $\omega \equiv 0$.

Let

$$h^2(\vartheta) = \int_\tau^\vartheta \mathrm{tr}(R'(t)R(t))dt. \quad (14)$$

Lemma 1. The reachability set

$$\mathcal{Z}_\gamma[\vartheta] = \{z : d^2(z, \mathcal{X}[\vartheta]) + h^2(\vartheta) \leq \gamma^2\}.$$

Hence $\mathcal{Z}_\gamma[\vartheta] \neq \emptyset$ iff $\gamma^2 \geq h^2(\vartheta)$.

Now consider system (12) with term $R(t)J(u)d\omega$ taken as in previous section, but with control $u \in \mathbb{R}^m$ restricted by (13). A standard transformation allows us to assume $Q(t) \equiv I$.

Here the solution follows the lines of Section 3, and the problem is solved within the class of functions $\mathcal{U}^{(a)}[\tau,\vartheta]$, iff control $u \in \operatorname{int}\mathcal{B}_1$. This property holds within the domain

$$\mathcal{D}^{(a)} = \{x : (u_*^{(a)}(t,x,z), u_*^{(a)}(t,x,z)) \le 1\},$$

where $u_*^{(b)}(t,x,z) = u_*^{(a)}(t,x,z)$ is as defined above, in (7). Beyond the domain $\mathcal{D}^{(a)}$ lies domain $\mathcal{D}^{(b)} = \mathbb{R}^n \setminus \mathcal{D}^{(a)}$. Here we will now solve Problem (b) for equation (12)–(13), with $\omega \neq 0$.

Similarly to the previous section, we come to the HJB equation with hard bound on u:

$$V_t^{(b)} + \min_{u\in\mathcal{B}_1}\big\{(V_x^{(b)}, B(t)u) + \tfrac{1}{2}(u, K^{(b)}(t)u)\big\} = 0. \quad (15)$$

under boundary condition

$$V^{(b)}(\vartheta,x,z) = (x-z, x-z). \quad (16)$$

Here $K^{(b)} = \sum_{q=1}^{n} I_q^n R'(t) V_{xx}^{(b)} R(t) I_q^n$.

After the minimization, we come to the equation

$$V_t^{(b)} + \mathcal{H}^{(b)}(t,x,V_x^{(b)},V_{xx}^{(b)}) = 0, \quad (17)$$

Thus, having in mind equation (12), in the domain

$$\mathcal{D}^{(a)} = \{x : (K^{(a)}(t))^{-1}B'(t)V_x^{(a)} \in \operatorname{int}\mathcal{B}_1\}, \quad (18)$$

we have

$$\mathcal{H}^{(b)}(t,x,V_x^{(b)},V_{xx}^{(b)}) \equiv \mathcal{H}^{(a)}(t,x,V_x^{(a)},V_{xx}^{(a)}) =$$
$$= -\tfrac{1}{2}\|B'(t)(P^{(a)}(t)(x-z) + q^{(a)}(t)\|^2_{(K^{(a)}(t))^{-1}},$$

with related control $u_*^{(b)}(t,x,z) = u^{(a)}(t,x,z)$ being as indicated in (7). (Recall that variable z is present in the boundary condition for $V^{(b)}(\vartheta,x,z)$ as it was for $V^{(a)}(t,x,z)$).

The domain $\mathcal{D}^{(b)} = \mathbb{R}^m \setminus \mathcal{D}^{(a)}$ consists of all points where the minimum over u in (15) is achieved through a problem of constrained optimization, which may be solved through modified Lagrangian techniques (Kuhn-Tucker theorem)

$$\mathcal{H}^{(b)}(t,x,V_x^{(b)},V_{xx}^{(b)}) =$$
$$= \min_{u\in\mathcal{B}_1}\Big\{(V_x^{(b)}, B(t)u) + \tfrac{1}{2}(u, K^{(b)}(t)u)\Big\}.$$

This gives

$$\mathcal{H}^{(b)}(t,x,V_x^{(b)},V_{xx}^{(b)}) = -\tfrac{1}{2}(V_x^{(b)})'B(t)(K_\lambda^{(b)}(t))^{-1}B'(t)V_x^{(b)}$$

with minimum achieved at

$$u_*^{(b)}(t,x,z) = -(K_\lambda^{(b)}(t))^{-1}B'(t)V_x^{(b)}.$$

Here

$$K_\lambda^{(b)}(t) = K^{(b)}(t) + 2\lambda I, \quad \lambda \ge 0.$$

Multiplier $\lambda = 0$ in $\mathcal{D}^{(a)}$ and $\lambda > 0$ in $\operatorname{int}\mathcal{D}^{(b)}$ ensuring in this domain that value $(u_*^{(b)}(t,x,z), u_*^{(b)}(t,x,z)) \le 1$.

Since $K^{(b)}(t)$ is diagonal, the matrix $(K_\lambda^{(b)}(t))^{-1}$ is also diagonal, with elements $(k_{ii}^{(b)} + 2\lambda)^{-1}$ and λ may be found as the positive root of equation

$$\Big((K_\lambda^{(b)}(t))^{-1}B'(t)V_x^{(b)}, (K_\lambda^{(b)}(t))^{-1}B'(t)V_x^{(b)}\Big) = 1. \quad (19)$$

Note that at the boundary of domain $\mathcal{D}^{(a)}$ we have $\lambda = 0$ and

$$(u_*^{(a)}(t,x,z), u_*^{(a)}(t,x,z)) = (u_*^{(b)}(t,x,z), u_*^{(b)}(t,x,z)) = 1$$

with

$$\mathcal{H}^{(a)}(t,x,V_x^{(a)},V_{xx}^{(a)}) = \mathcal{H}^{(b)}(t,x,V_x^{(b)},V_{xx}^{(b)}).$$

Multiplier λ is a continuous function of the distance $d(u_*^{(b)}(t,x,z), \mathcal{D}^{(a)})$ and $K_\lambda^{(b)}(t) \to K^{(b)}(t)$ with $\lambda \to 0$.

Function $V^{(b)}(t,x)$ coincides with $V^{(a)}(t,x)$ in $\mathcal{D}^{(a)}$ with continuous transition to $\mathcal{D}^{(b)}$ over the boundary of $\mathcal{D}^{(a)}$.

As indicated in Section 3, in $\mathcal{D}^{(a)}$ the control $u_*^{(a)}(t,x,z)$ is affine. Namely,

$$V^{(a)}(t,x,z) =$$
$$= (x-z, P^{(a)}(t)(x-z)) + 2(q^{(a)}(t), x-z) + \kappa^{(a)}(t),$$

where matrix $P^{(a)}(t) = (P^{(a)}(t))' > 0$ and vector $q^{(a)}(t)$ with scalar function $\kappa^{(a)}(t)$ are described in (8)–(11).

Taking

$$V^{(b)}(t,x,z) =$$
$$= (x-z, P^{(b)}(t)(x-z)) + 2(q^{(b)}(t), x-z) + k^{(b)}(t).$$

and considering the equation for $V^{(b)}(t,x,z)$ in $\mathcal{D}^{(b)}$, substitute $V^{(b)}(t,x,z)$ into (15), (16) and equalize the terms with multipliers $(x_i - z_i)(x_j - z_j)$, $(x_i - z_i)$, $i,j = 1,\ldots,n$ and also the free terms. Then we come to equations which are formally similar to (8)–(10), namely:

$$\dot{P}^{(b)}(t) - P^{(b)}(t)B(t)(K_\lambda^{(b)}(t))^{-1}B'(t)P^{(b)}(t) = 0, \quad (20)$$

$$\dot{q}^{(b)}(t) - P^{(b)}(t)B(t)(K_\lambda^{(b)}(t))^{-1}B'(t))q^{(b)}(t) = 0, \quad (21)$$

$$\dot{\kappa}^{(b)}(t) - (q^{(b)}(t))'B(t)(K_\lambda^{(b)}(t))^{-1}B'(t)q^{(b)}(t) = 0 \quad (22)$$

with boundary conditions

$$P^{(b)}(\vartheta) = I, \quad q^{(b)}(\vartheta) = 0, \quad \kappa^{(b)}(\vartheta) = 0. \quad (23)$$

But $K^{(b)}(t) = \sum_{q=1}^{n} I_q^n R'(t)P^{(b)}(t)R(t)I_q^n$, so that the multiplier λ depends on $\{t,x,z,P^{(b)}\}$ and with fixed $P^{(b)} = P^{(b)}(t \mid x,z)$. However, for any realization of $\lambda(t)$ that may appear throughout the interval $[\tau,\vartheta]$, whatever be the realizations of $x(t)$ emanating from x_τ under control $u_*^{(b)}(t,x,z) \in \mathcal{B}_1$, equations (20)–(23) allow the existence of solutions, extendable throughout $[\tau,\vartheta]$. This justifies the made assumptions, though calculating the exact solution $V^{(b)}(t,x,z)$, which is proved to exist, may require rather subtle technique. In the case of scalar controls the procedure is substantially simpler.

From another perspective, under the same assumptions, the value function given by equation (15) with boundary condition (16) may be also generated by the deterministic control problem of finding

$$\overline{V}^{(b)}(t,x,z) = \min_u\Big\{\int_t^\vartheta \frac{1}{2}(u, K^{(b)}(s)u)ds +$$
$$+ (x(\vartheta) - z, x(\vartheta) - z) \,\Big|\, x(t) = x, u \in \mathcal{B}_1\Big\},$$

so that $\overline{V}^{(b)}(t,x,z) = V^{(b)}(t,x,z)$.

Applying the techniques of convex analysis along the lines of (Kurzhanski and Varaiya [2002], Digailova and Kurzhanski [2005]) we find $\overline{V}^{(b)}(t,x,z)$ as follows:

$$\overline{V}^{(b)}(t,x,z) =$$
$$= \max_l \min_u\Big\{(l, x(\vartheta) - z) + \tfrac{1}{2}\int_t^\vartheta (u, K^{(b)}(s)u)ds -$$

$$-\tfrac{1}{4}(l,l)\Big| l = l(t,x,z) \in \mathbb{R}^n, u \in \mathcal{B}_1\Big\} =$$
$$= \max_l \Big\{ (l, x-z) - \tfrac{1}{2}(l, W(t,\lambda)l) - \tfrac{1}{4}(l,l) \Big| l \in \mathbb{R}^n \Big\} =$$
$$= \tfrac{1}{2}\left(x - z, (W(t,\lambda_*) + \tfrac{1}{2}I)^{-1}(x-z)\right) \qquad (24)$$

Here

$$W(t,\lambda) = \int_t^{\vartheta} (B(s)(K_\lambda^{(b)}(s))^{-1}B'(s))ds,$$

where $K_\lambda^{(b)} = K^{(b)} + \lambda I$. With $\lambda_* = \lambda_*(t,x,z) > 0$ is uniquely obtained from equation $f(\lambda) - 1 = 0$, where $f(\lambda) = (u_*^{(b)}, u_*^{(b)})$, then for $\lambda = \lambda_*$

$$u_*^{(b)}(t,x,z) = -(K_{\lambda_*}^{(b)}(t))^{-1}B'(t)l_*(t,x,z),$$
$$l_*(t,x,z) = (W(t,\lambda_*) + \tfrac{1}{2}I)^{-1}(x-z).$$

Note that here

$$\overline{P}^{(b)}(t) = \tfrac{1}{2}\overline{V}_{xx}^{(b)}(t,x,z) = \tfrac{1}{2}(W(t,\lambda_*) + \tfrac{1}{2}I)^{-1}$$

and

$$\dot{\overline{P}}^{(b)} - \overline{P}^{(b)}(B(s)(K_{\lambda_*}^{(b)}(t))^{-1}B'(s))\overline{P}^{(b)} = 0,$$
$$\overline{P}^{(b)}(\vartheta) = I.$$

Here $K_{\lambda_*}^{(b)}(s) = (K_{\lambda_*}^{(b)}(s))' > 0$ is uniformly positive and the realization $\lambda_* = \lambda_*(t,x,z) \geq 0$ is a continuous uniformly bounded function, (whatever be the realizations of $x(t)$ emanating from x_τ under control $u(t,x,z) \in \mathcal{B}_1$), so that $\overline{P}^{(b)}(t) = \big(\overline{P}^{(b)}(t)\big)' > 0$ is extendable throughout $[\alpha, \vartheta]$ due to same considerations as for the Riccati equation.

Lemma 2. The value function $V^{(b)}(t,x,z) \equiv \overline{V}^{(b)}(t,x,z)$ and $\overline{V}^{(b)}(\vartheta,x,z) = (x-z, x-z)$.

Substituting $\overline{V}^{(b)}(t,x,z)$ of (24) into (15) we observe that $V^{(b)}(t,x,z) \equiv \overline{V}^{(b)}(t,x,z)$.

Theorem 4. The synthesized control in domain $\mathcal{D}^{(b)}$ is a nonlinear function

$$u_*^{(b)}(t,x,z) = -(K_{\lambda_*}^{(b)}(t))^{-1}B'(t)l_*(t,x,z)$$

which is correctly represented in set-valued form as

$$\mathbf{u}_*^{(b)}(t,x,z) = \begin{cases} u_*^{(b)}(t,x,z), & B'(t)l_*(t,x,z) \neq 0, \\ \mathcal{B}_1, & B'(t)l_*(t,x,z) = 0. \end{cases}$$

The overall system may be then presented in $\mathcal{D}^{(b)}$ as a stochastic differential inclusion (see Bensoussan et al. [2007])

$$dx = B(t)udt + R(t)J(u)d\omega(t), \;\; u \in \mathbf{u}_*^{(b)}(t,x,z).$$

The indicated controls solve Problem b for given z. Summarizing the above we come to the following conclusion.

Theorem 5. (i) The γ-reach set $\mathcal{Z}_\gamma[\vartheta]$ under control-dependent noise with hard bound on the control is of two forms. Namely, in the domain $\mathcal{D}^{(a)}$ it is an ellipsoid $\mathcal{Z}_\gamma^{(a)}[\vartheta]$ (see Theorem 3), whose boundary may be reached by controls $u_*^{(a)}(t,x,z)$ which are affine in x.

In the complementary domain $\mathcal{D}^{(b)}$ it is a convex compact set

$$\mathcal{Z}_\gamma^{(b)}[\vartheta] = \{x : V^{(b)}(t,x,z) \leq \gamma^2\},$$

whose boundary may be reached by set-valued controls $\mathbf{u}_*^{(b)}(t,x,z)$ which in the case of scalar controls turn out to be of the bang-bang type.

5. REACHABILITY UNDER INCOMPLETE MEASUREMENTS

Among the problems which may be approached through the techniques discussed here is the one of *finding reachability sets* for system (1) *under* additional information given by *on-line measurements*. Here as in the previous section, without lost of generality, we consider system (12) instead of system (1).

$$dy = G(t)xdt + d\xi, \; y \in \mathbb{R}^p, \qquad (25)$$

where measurement noise $d\xi$ stands for the Brownian motion with mean value 0 and covariance matrix $T(t) > 0$, *independent* of $d\omega$ and control u and the system input noise is normalized Brownian, independent of control u. This problem may be solved in three steps.

The first step is to solve the stochastic Problem (b) with control-independent noise but under the additional constraint (25). A standard procedure, similar to the derivation of the Kalman filter, gives the following system of differential equations

$$d\bar{x} = B(t)udt + (P^{(c)}(t))^{-1}G'(t)T^{-1}(t)(dy(t) - G(t)\bar{x}dt), \qquad (26)$$
$$\dot{P}^{(c)} = S(t) - P^{(c)}G'(t)T^{-1}(t)G(t)P^{(c)}, \qquad (27)$$

where $S(t) = \sum_{q=1}^n I_q^n R'(t)R(t)I_q^n$. With initial conditions

$$\bar{x}(\tau) = x_\tau, \quad P^{(c)}(\tau) = 0.$$

Since the conditional expectation under given measurement $y(\cdot)$ produces

$$E||x(\vartheta) - z||^2 = E||x(\vartheta) - \bar{x}(\vartheta)||^2 + ||\bar{x}(\vartheta) - z||^2,$$

this brings us to the following deterministic problem.

Problem c. Find

$$V^{(c)}(\tau, x_\tau, z) =$$
$$= \min_{\mathbf{u}} \Big\{ \|\bar{x}[\vartheta] - z\|^2_{P^{(c)}(\vartheta)} | \tau, \bar{x}(\tau) = x_\tau, \mathbf{u} \in \mathcal{U}^{(c)}[\tau,\vartheta] \Big\}$$

due to systems (26), (27). The class $\mathcal{U}^{(c)}[\tau,\vartheta]$ is similar to $\mathcal{U}^{(b)}[\tau,\vartheta]$ of the previous section, but due to the certainty principle (see Astrom [1970], Davis and Varaiya [1972]) $\mathbf{u}$ will now depend on $\{t, \bar{x}\}$.

The second step is the solution of Problem c. This may be described in terms of the dynamic programming equation for value function $V^{(c)}(\tau, x_\tau, z)$ (see Fleming and Soner [1993] for more details).

Similarly to Problem (b-1) it can be shown, by applying methods of convex analysis, that $V^{(c)}(\tau, x_\tau, z)$ may be represented as

$$V^{(c)}(\tau, x_\tau, z) = d^2_{P^{(c)}(\vartheta)}(z, \bar{\mathcal{X}}[\vartheta]),$$

where $\bar{\mathcal{X}}[\vartheta]$ is the reachability set of system (26) with hard bounds (13), at time ϑ, from position $\{\tau, x_\tau\}$. The last step is to find the reachability set $\mathcal{Z}_\gamma^{(c)}[\vartheta]$.

Theorem 6. The γ-reach set $\mathcal{Z}_\gamma^{(c)}[\vartheta]$ under incomplete observation is $\mathcal{Z}_\gamma^{(c)}[\vartheta] = \left\{ z : V^{(c)}(\tau, \bar{x}_\tau, z) + h^2(\vartheta) \leq \gamma^2 \right\}$.

where $h^2(\vartheta)$ is like in (14). The boundary of set $\mathcal{Z}^{(c)}_\gamma[\vartheta]$ may be reached by controls

$$u^{(c)}_*(t, \bar{x}, z) =$$
$$= -\left((V^{(c)}_{\bar{x}})' B(t)Q(t)B'(t)V^{(c)}_{\bar{x}}\right)^{-1/2} Q(t)B'(t)V^{(c)}_{\bar{x}},$$

which, being dependent on $\bar{x}[\vartheta]$, turn out to be functionals of the measurement $y(t),\ t \in [\tau, \vartheta]$. Note that the correct representation of the control $u^{(c)}_*(t, \bar{x}, z)$ in set-valued form is

$$\mathbf{u}^{(c)}_*(t, \bar{x}, z) = \begin{cases} u^{(c)}_*(t, \bar{x}, z), & B'(t)V^{(c)}_{\bar{x}}(t, \bar{x}, z) \neq 0, \\ \mathcal{B}_1, & B'(t)V^{(c)}_{\bar{x}}(t, \bar{x}, z) = 0. \end{cases}$$

Using the last scheme one may define a *measurement feedback control* which steers the system to a convex compact *target set* $\mathcal{M} \subset \mathbb{R}^n$. Since $d_L(\mathcal{B}, \mathcal{A})$ is a minimal distence between sets $\mathcal{A}$ and $\mathcal{B}$, we just have solve the problem

$$\mathcal{V}(t, \bar{x}) = d^2_{P^{(c)}(\vartheta)}(\mathcal{M}, \bar{\mathcal{X}}[\vartheta]) =$$
$$= \min_z \{V^{(c)}_{\bar{x}}(t, \bar{x}, z) \mid z \in \mathcal{M}\} = V^{(c)}_{\bar{x}}(t, \bar{x}, z^0)$$

and check that along the optimal trajectory $x^0[s] = x^0(s, \bar{x}, t)\ \ s \in [t, \vartheta]$, the vector parameter z^0 does not change. The optimal control solution is then defined as $\mathbf{u}^0(t, \bar{x}) = \mathbf{u}^{(c)}_*(t, \bar{x}, z^0)$. With $V^{(c)}_{\bar{x}}(t, \bar{x}, z^0) = 0$ we have $x^0[\vartheta] \in \mathcal{M}$.

6. CONCLUSION

This paper introduces the notion of reachability for controlled systems subjected to stochastic Brownian noise which depends on the control parameters which may be either unbounded or bounded by hard bounds. The emphasis is on systems with vector-valued noise which also depends on vector-valued controls. This presents a substantially more different situation than the scalar case (Digailova and Kurzhanski [2005]) especially when the controls are bounded. The reach set of each type is described through level sets of solutions to appropriate stochastic HJB equations. This paper clarifies the structure of solutions and indicates the routes to calculate reach sets in both cases. Finally a scheme for calculation reachability sets *under measurement feedback* is indicated for systems with bounded controls and control-independent noise. A forthcoming group of problems which now follow from here is to investigate reachability under incomplete feedback and control-dependent stochastic noise as well as the related filtering equations and other problems of measurement feedback control.

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