Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Impulse Control Inputs and the Theory of Fast Feedback Control ⋆

A. N. Daryin and A. B. Kurzhanski

Moscow State (Lomonosov) University
Faculty of Computational Mathematics and Cybernetics

Abstract: This paper deals with problems of impulse control which allow control inputs consisting not only of delta functions but also of their higher derivatives (impulses of higher order). The controls are sought for in the form of feedback strategies which leads to the application of respective generalized dynamic programming techniques, where the role of traditional Hamilton–Jacobi–Bellman equations is taken by respective variational inequalities of similar structure. Further proposed are physically realizable approximations which converge to these ideal solutions. Since the ideal solutions allow to transfer a controllable system from one given position to another in zero time, their approximations lead us to physically realizable "fast" controls with piecewise constant realizations. Such feedback control inputs are then compared with traditional bang-bang type strategies and turn out to be more robust. Computational schemes for related problems of reachability and control synthesis are further described with examples of damping oscillating systems of high order in minimal time being demonstrated.

Keywords: Switching stability and control, control design, optimization based controller synthesis, impulse feedback control, generalized dynamic programming, variational inequalities, reachability

1. INTRODUCTION

The present paper deals with problems of impulse control. It differs from other papers in this area due to the application of input controls which allow not only delta-functions, but also their higher derivatives (impulses of higher order). Besides that the controls are sought for in the form of closed-loop feedback strategies. We thus combine the properties of generalized functions with the dynamic nature of feedback control. This leads to the application of generalized dynamic programming techniques based on reducing the original problem to another one which is posed in the class of solutions which allow only delta-impulses. In such schemes the traditional Hamilton–Jacobi–Bellman equation is substituted by variational inequalities of similar structure. However these "ideal" solutions may not allow a physical realization. In order to make the solutions applicable we further introduce and array of physically realizable approximations with piecewise-constant control realizations. The approximations converge to the exact ideal solutions. Thus, since the ideal controls allow transition of the original controllable system from one given position to another in zero time, their physically realizable approximations — the so-called "fast" controls — allow to solve the same problem in finite time which is arbitrary small. The suggested impulse control synthesis is then compared with traditional bang-bang-type feedback strategies, turning out to be robust in situations where the latter are not. Computational schemes for related problems of reachability and closed-loop control are thus conceived and examples of damping oscillations in high-order systems within small time are demonstrated.

⋆ This work is supported by Russian Foundation for Basic Research (grant 06-01-00332). It has been realized within the programs "State Support of the Leading Scientific Schools" (NS-5344.2006.1) and "Development of Scientific Potential of the Higher School" (RNP 2.1.1.1714).

2. IMPULSE CONTROL PROBLEMS

2.1 The Ordinary Impulse Controls

We start by recalling the classical impulse control problem (see Krasovski [1957], Neustadt [1964]). Consider a linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t_0 \le t \le t_1. \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, and $u(t)$ is the control input of form

$$u(t) = \frac{dU(t)}{dt},$$

where $U(\cdot)$ is a function from the space $BV([t_0, t_1]; \mathbb{R}^m)$ of m-vector functions of bounded variation, Riesz and Sz-.Nagy [1972]. Given matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are continuous.

The problem is to minimize the variation of control

$$\operatorname*{Var}_{[t_0,t_1]} U(\cdot) \to \min \tag{2}$$

subject to conditions $x(t_0) = x_0$, $x(t_1 + 0) = x_1$. The initial time t_0 and terminal time t_1 are fixed. Here and below we assume that control $U(t)$ and trajectories $x(t)$ are left-continuous functions.

The problem (2) has been extensively studied in the class of open-loop controls. In particular, the minimum

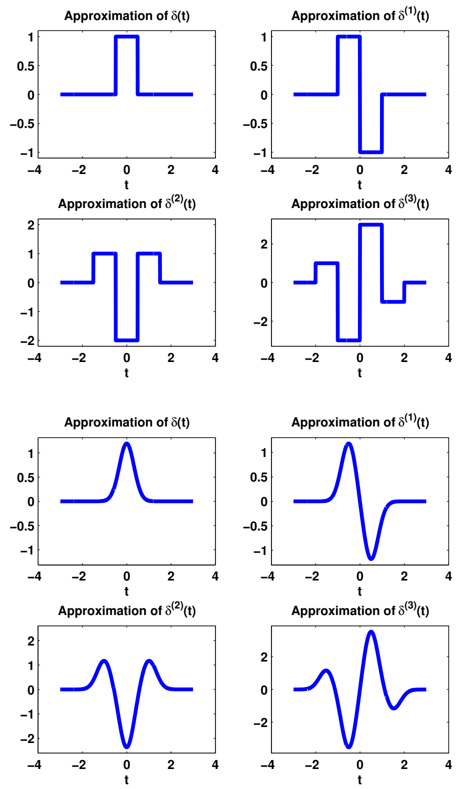

Fig. 1. Approximations of delta-function and its derivatives

variation is [1]

$$\min \operatorname*{Var}_{[t_0,t_1]} U(\cdot) = \sup_{p\neq 0} \frac{\langle p, x_1 - X(t_1,t_0)x_0\rangle}{\max_{t\in[t_0,t_1]} \|B^T(t)X^T(t_1,t)p\|}.$$

Furthermore, among the controls of minimum variation there is at least one of the form [2]

$$U(t) = \sum_{i=1}^{\ell} h_i \chi(t-\tau_i), \qquad \operatorname*{Var}_{[t_0,t_1]} U(\cdot) = \sum_{i=1}^{\ell} \|h_i\|,$$

where $\ell \le n$, $\tau_i \in [t_0,t_1]$, and $h_i \in \mathbb{R}^m$. The corresponding control input is $u(t) = \sum_{i=1}^{\ell} h_i \delta(t-\tau_i)$. In other words, the minimum in (2) is attained on a control which is the sum of a finite number of impulses. This number does not exceed the dimension of the system. Each of the impulses is represented by a delta-function.

2.2 The Higher-Order Generalized Impulse Controls

The problem (2) may be generalized such that the control may have not only terms with delta-functions, but also with derivatives of delta-functions up to order k (see Kurzhanski and Osipov [1969]). The delta-function and its derivatives are limits of sequences of functions presented in Fig. 1.

A rigorous mathematical formulation of the generalized problem is given in Appendix A. Here we note that the optimal open-loop controls of such problem have the form

$$u(t) = \sum_{i=1}^{\ell} \sum_{j=0}^{k} h_{i,j} \delta^{(j)}(t-\tau_i), \qquad \ell \le n,$$

and its norm (which replaces the variation in case of ordinary impulse controls) is defined as

$$\rho^*[u] = \sum_{i=1}^{\ell} \gamma^*[\gamma_0^*[h_{i,0}], \ldots, \gamma_k^*[h_{i,k}]],$$

where γ, γ_0, …, γ_k are norms in the corresponding finite-dimension spaces, $*$ denotes the adjoint norm.

The generalized problem may be reduced to a problem of type (2) as follows.

Assume that matrix functions $A(t)$ and $B(t)$ are k times continuously differentiable on the interval $(\alpha,\beta) \supset [t_0,t_1]$. Define functions $L_j(t)$ by relations

$$L_0(t) = B(t), \quad L_j(t) = A(t)L_{j-1}(t) - \frac{dL_{j-1}}{dt}, \quad j = \overline{1,k}$$

and form the matrix $\mathscr{B}(t) = (L_0(t)\ L_1(t)\ \cdots\ L_k(t))$. For a fixed $h \in \mathbb{R}^m$ the vector $L_j(\tau)h$ is equal to a jump of the trajectory of (1) at time τ under control input $u(t) = (-1)^j h\delta^{(j)}(t-\tau)$.

Control inputs $U(t) = \left(U_0^T(t)\ U_1^T(t)\ \cdots\ U_k^T(t)\right)^T$ are chosen in the class $BV([t_0,t_1];\mathbb{R}^{m(k+1)})$ of functions of bounded variations on the interval $[t_0,t_1]$ with values in $\mathbb{R}^{m(k+1)}$ (each function $U_j(t)$ is with values in $\mathbb{R}^m$). This space is endowed with the following norm

$$\operatorname*{Var}_{[t_0,t_1],G^*[\cdot]} U(\cdot) = \sup \sum_i G^*[U(t_{i+1}) - U(t_i)],$$

$$G^*(U) = \gamma^*[\gamma_0^*[U_0], \ldots, \gamma_k^*[U_k]].$$

The corresponding generalized control is

$$u(t) = \sum_{j=0}^{k} (-1)^j \frac{d^{j+1}U_j}{dt^{j+1}}, \qquad \rho^*[u] = \operatorname*{Var}_{[t_0,t_1],G^*[\cdot]} U(\cdot).$$

The generalized control problem is now equivalent to the problem of type (2) for system $\dot{x}(t) = A(t)x(t) + \mathscr{B}(t)u(t)$.

3. THE FAST CONTROLS

3.1 The Ideal Zero-Time Controls

By applying the generalized control inputs system (1) may be controllable in zero time (i.e. $t_0 = t_1$) even if the dimension of control m is less than the state space dimension n. Indeed, this is the case when $\operatorname{im}\mathscr{B}(t_1) = \mathbb{R}^n$. Then the minimal norm of the control is

$$\min \rho^*[u] = \sup\{\langle x, p\rangle \mid G(\mathscr{B}^T(t_\beta)p) \le 1\}.$$

Here the supremum is taken over a bounded set since $\ker \mathscr{B}(t_1) = \{0\}$.

However, such zero-time controls are idealistic and in real applications their physically realizable bounded approximations should be used. Such controls will allow a solution of the two-point boundary control problem in arbitrary small time.

3.2 The Realistic Fast Controls

Here we construct the so-called "fast" controls — bounded approximations of the control which solves the problem

---

[1] $X(t,\tau)$ is the fundamental matrix of the homogeneous system, i.e. the solution to the matrix differential equation $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$

[2] $\chi(t)$ is the Heaviside step function, $\delta(t) = \chi'(t)$ is Dirac's delta-function.

in zero time. For sufficiently small $h > 0$ (such that the entire interval $[t_1, t_1 + kh]$ lies within (α, β)) define scalar functions [3]

$$\Delta_h^0(t) = \frac{1}{h}\mathbf{1}_{[0,h]}(t),\ \Delta_h^j(t) = \frac{1}{h}(\Delta_h^{j-1}(t) - \Delta_h^{j-1}(t-h)),$$

$j = 1, \dots, k$. These functions approximate the generalized functions $\delta^{(j)}(t)$. Consider controls of form $u(t) = \sum_{j=0}^k u_j \Delta_h^j(t - t_1)$, where $u_j \in \mathbb{R}^m$. Denote $U = (u_0^T\ u_1^T \cdots u_k^T)$,

$$M_h^{(j)}(t) = \int_t^{t+kh} X(t+kh, \tau)B(\tau)\Delta_h^j(\tau - t)d\tau,$$

$$\mathscr{M}_h(t) = \left(M_h^{(0)}(t)\ M_h^{(1)}(t)\ \cdots\ M_h^{(h)}(t)\right),$$

then $x(t_1+kh) = X(t_1+kh, t_1)x(t_1) + \mathscr{M}_h U$ and a natural analogue of the generalized control problem with $t_0 = t_1$ is the following finite-dimension optimization problem:

$$\begin{cases} G^*(U) \to \inf, \\ \mathscr{M}_h(t_1)U = x_1 - X(t+kh, t_1)x_0 = c. \end{cases} \tag{3}$$

Theorem 1. Suppose that $\operatorname{rank} \mathscr{B}(t_1) = n$, then the problem (3) is solvable.

Proof. As $h \to 0+$, the functions $\Delta_h^j(t)$ weakly converge to $\delta^{(j)}(t)$ in the space of distributions $D_{k,1}^*[\alpha, \beta]$. Thus

$$M_h^{(j)}(t) \to (-1)^j \left.\frac{\partial X(t,\tau)B(\tau)}{\partial \tau}\right|_{\tau=t} = L_j(t),$$

and $\mathscr{M}_h(t) \to \mathscr{B}(t)$. Since by our assumption $\operatorname{rank} \mathscr{B}(t_1) = n$, then for sufficiently small $h > 0$ we also have $\operatorname{rank} \mathscr{M}_h(t_1) = n$ and the admissible controls do exist. Let $\bar{U}$ be an arbitrary admissible control. The set $\{U | G^*(U) \le G^*(\bar{U}),\ \mathscr{M}_h U = c\}$ is a compact, and the finite-dimensional norm $G^*(U)$ is continuous. Therefore, the problem (3) has a solution.

Next we calculate the minimum value in problem (3):

$$V_h = \min_{\mathscr{M}_h(t_1)U=c} \max_{G(q)\le 1} \langle q, U\rangle = \max_{G(q)\le 1} \min_{\mathscr{M}_h(t_1)U=c} \langle q, U\rangle =$$
$$= \max\{\langle (\mathscr{M}_h^T(t_1))^{\oplus} q, c\rangle \mid G(q) \le 1, q \in \ker \mathscr{M}_h(t_1)\}.$$

Here $\oplus$ denotes a pseudo-inverse matrix (see Lancaster [1969]). We set $p = (\mathscr{M}_h^T(t_1))^{\oplus} q$, then

$$V_h = \max\{\langle p, c\rangle \mid G(\mathscr{M}_h^T(t_1)p) \le 1\}.$$

When $h \to 0$, we have $c \to x_1 - x_0$, $\mathscr{M}_h^T(t_1) \to \mathscr{B}^T(t_1)$. Therefore, $V_h \to \min \rho^*[u]$.

3.3 The Norm of Fast Controls

Here we assume that $A(t) \equiv A$, $B(t) \equiv B$. According to Seidman and Yong [1997], the minimum variation of the impulse control is asymptotically

$$N(\Delta) \sim \Delta t^{-k}, \quad k = \min\{j | x_0 \in R_j\}, \tag{4}$$

where $R_j = \operatorname{im}\left(B\ AB \cdots A^j B\right)$.

It follows from (4) that for generalized controls (distributions) of order r

$$N(\Delta) \sim \Delta t^{-(k-r)},\ k = \min\{j | x_0 \in R_j\}. \tag{5}$$

(since the problem in this class of controls is equivalent to the impulse control problem with matrix B replaced by $(B\ -AB\ \cdots\ (-1)^r A^r B)$).

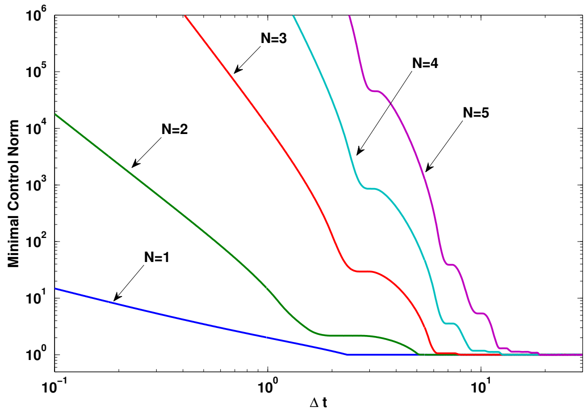

Fig. 2. Dependence of minimum control norm on N — the number of links in the oscillating system

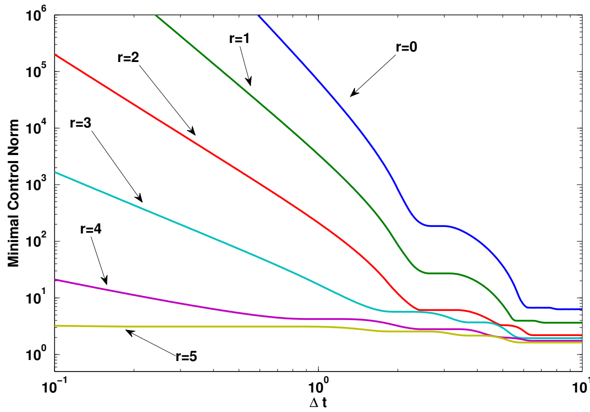

Fig. 3. Dependence of minimum control norm on r — the order of distributions taken as control inputs

Example 1. Consider an oscillating system

$$\begin{cases} \dot{x}_i = x_{N+i}, \quad i = \overline{1, N}; \\ \dot{x}_{N+1} = x_2 - 2x_1; \\ \dot{x}_{N+i} = x_{i+1} + x_{i-1} - 2x_i, \quad i = \overline{2, N-1}; \\ \dot{x}_{2N} = x_{N-1} - x_N + u \end{cases} \tag{6}$$

with norm $G^*(u) = \|u\|$. Fig. 2 shows in logarithmic scale how the minimum norm of impulse controls depends on the dimension of the state space (which is $2N$ here).

In Fig. 3 we show how the minimum control norm depends on the order of distributions used as control inputs. Here $N = 3$.

Interpretation in terms of generalized controls. Suppose that $x_0 \in R_k \setminus R_{k-1}$. This means that the system is controllable in zero time by generalized controls of order (not less than) k of the form $u(t) = \sum_{j=0}^k u_j \delta^{(j)}(t - t_1)$, where vectors u_j solve the linear system $\sum_{j=0}^k L_j u_j = x_0$ with matrices $L_j = (-1)^j A^j B$. Note that $u_k \ne 0$ (otherwise controls of order $k - 1$ would suffice, which contradicts our assumption).

[3] $\mathbf{1}_A(t)$ denotes the membership function of the set A, equal to 1 in A and 0 in other points.

Now if the control is a distribution of order $r < k$, then the terms with derivatives of order j greater than r are replaced with finite-difference approximations of these derivatives. The sum of coefficients of such approximations are asymptotically of order $\Delta t^{-(j-r)}$, which agrees with (5).

A proof of (4). In (Seidman and Yong [1997]) the estimate (4) follows from a general result for controls in the class L_p. Here we give a straightforward (and more simple) proof of this estimate directly for scalar impulse controls ($B = b \in \mathbb{R}^n$).

According to Neustadt [1964] the minimum norm of the impulse control is attained on a control with at most n impulses: $u(t) = \sum_{j=1}^{n} u_j \delta(t - \tau_j)$, $t_1 - \Delta t \le \tau_j \le t_1$. We assume that $x_0 \in R_k \setminus R_{k-1}$, and without any loss of generality, that $t_1 = 0$. Then the numbers u_j solve the linear system $\sum_{j=1}^{n} \mathrm{e}^{-\tau_j A} b u_j = -x_0$. Using the Taylor-Maclaurin series for the matrix exponential, we have

$$\sum_{j=1}^{n}\sum_{s=0}^{k} \frac{(-\tau_j)^s}{s!} A^s b u_j + O(\Delta t^{k+1}) u = -x_0.$$

Vectors $A^s b$, $s = 0, \ldots, k$ are linearly independent, so x_0 may be expressed through them (with some coefficients ξ_i):

$$\sum_{j=1}^{n}\sum_{s=0}^{k} \frac{(-\tau_j)^s}{s!} A^s b u_j + O(\Delta t^{k+1}) u = \sum_{s=0}^{k} \xi_s A^s b.$$

Taking coefficients of $A^s b$ on both sides, we come to the linear system for u_j:

$$\sum_{j=1}^{n} \frac{(-\tau_j)^s}{\Delta t^s} u_j + O(\Delta t^{k+1-s}) u = \frac{s!\xi}{\Delta t^s}, \qquad s = 0, \ldots, k.$$

The norm of the operator in the left-hand side is bounded for small Δt uniformly over all choices of τ_j, the right-hand side is of order Δt^{-k}. Therefore, the control should have a norm which asymptotically is not less than $C \Delta t^{-k}$.

To prove the upper bound, we consider particular instants of time $\tau_j = -\frac{j}{n} \Delta t$:

$$\sum_{j=1}^{n} \left(\frac{j}{n}\right)^s u_j + O(\Delta t^{k+1-s}) u = \frac{s!\xi}{\Delta t^s}, \qquad s = 0, \ldots, k.$$

The operator M in the left-hand side is injective and thus satisfies $\|Mu\| \ge C\|u\|$. From here it follows that $N(\Delta t) \le C \Delta t^{-k}$.

The Realistic controls. Suppose that a physical realization of the impulse $u(t) = u_j \delta(t - t_j)$ is a "column" $u(t) = u_j h^{-1} \mathbf{1}_{[t_j, t_j + h]}(t)$. Then the respective "jump" of the trajectory (actually not a jump, but a fast change in the trajectory) is

$$\Delta x(t_j) = e^{-hA} x(t_j + h) - x(t_j) = \\ = h^{-1} \int_{t_j}^{t_j + h} e^{(t_j - t)A} B u_j dt = F_h B u_j,$$

where $F_h = h^{-1} \int_0^h e^{-tA} dt$.

In the case of invertible matrix A, $F_h = h^{-1} A^{-1} (I - \mathrm{e}^{-hA})$.

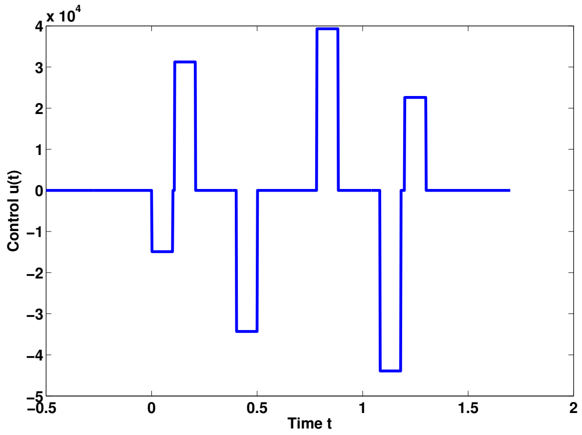

Fig. 4. The high-order realistic impulse controls

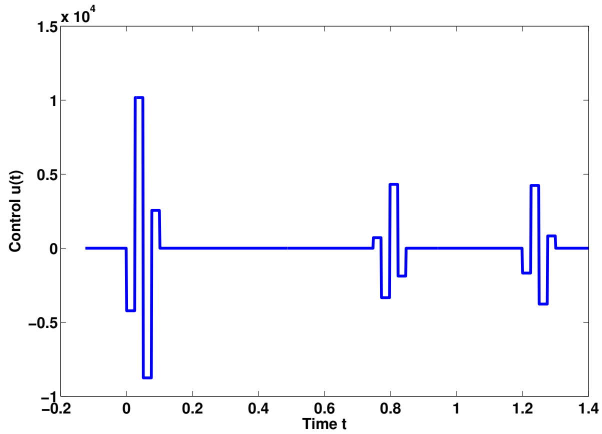

Fig. 5. A realistic approximation of a generalized impulse control of order 3.

We see that the problem with physical realization of controls is equivalent to the problem of impulse control with matrix B replaced by $F_h B$.

In a similar way we replace a generalized impulse $u_j \delta^{(s)}(t - t_j)$ with its physical realization, namely, with a finite-difference approximation of s-th derivative of delta-function, $\Delta_h^{(s)}(t - t_j)$. The corresponding jump of x is $\Delta x(t_j) = F_h^{(s)} B u_j$, where $F_h^{(s)} = h^{-1}(1 - e^{-hA}) F_h^{(s-1)}$ with $F_h^{(0)} = F_h$. We have $F_h^{(s)} = h^{-s}(1 - e^{-hA})^s F_h$, and for invertible matrix A, $F_h^{(s)} = h^{-(s+1)}(1 - e^{-hA})^{s+1} A^{-1}$.

It follows that the problem with realistic controls — finite-difference approximations of generalized functions of order k — is equivalent to the problem of impulse control with matrix B replaced with $\left(F_h^{(0)} B \; F_h^{(1)} B \cdots F_h^{(k)} B\right)$. In particular, the estimate 4 holds for this problem.

Note that $F_h^{(s)} \to A^s$ as $h \to 0$.

Example 2. Fig. 4 shows a realization of the control for Example 1 in the class of realistic impulses ($h = 0.1$). One may observe that this control corresponds to a finite-difference approximation of the fifth derivative $\delta^{(5)}(t - \tau)$.

Fig. 5 shows another realization of the control for Example 1, now in the class of realistic controls of order $r = 3$.

This control has only three (generalized) impulses, but is also an approximation of the fifth derivative $\delta^{(5)}(t-\tau)$.

The selection of step h here and in other calculations must be correlated with the sampling rate. It may also depend on requirements given in advance for practical reasons.

4. THE CLOSED-LOOP CONTROL

4.1 The Dynamic Programming Approach

In order to formulate the principle of optimality, we consider the problem of minimizing a functional of Mayer–Bolza type:

$$\rho^*[u] + \Phi(x(t_1+0)) \to \inf . \tag{7}$$

The previously considered problems are particular cases of (7) if the function Φ is chosen as[4] $\Phi(x) = \mathscr{I}(x|\{x_1\})$.

Denote the minimum value in problem (7) as $V(t_0, x_0) = V(t_0, x_0; t_1, \Phi(\cdot))$. The principle of optimality holds in the form of a semi-group property

$$V(t_0, x_0; t_1, \Phi(\cdot)) = V(t_0, x_0; t_\epsilon, V(t_\epsilon, \cdot; t_1, \Phi(\cdot))),$$

where $t_0 \le t_\epsilon \le t_1$.

The value function $V(t,x)$ satisfies a quasi-variational inequality (Bensoussan and Lions [1982]; Daryin, Kurzhanski and Seleznev [2005]) (in the points of non-smoothness of the value function the inner products stand for the respective directional derivatives)

$$\min\{H_1(t,x,V_t,V_x), H_2(t,x,V_t,V_x)\} = 0, \tag{8}$$

$$V(t_\beta, x) = V(t_\beta, x; t_\beta, \Phi(\cdot)).$$

$$H_1(t,x,\xi_t,\xi_x) = \xi_t + \langle \xi_x, A(t)x \rangle,$$

$$H_2(t,x,\xi_t,\xi_x) = \min_{u \in \mathbb{R}^{m(k+1)}} \{\langle \xi_x, \mathscr{B}(t)u \rangle + 1 \mid G^*(u) \le 1\} =$$

$$1 - G(\mathscr{B}^T(t)V_x) = 1 - \gamma[\gamma_0[L_0^T(t)V_x], \ldots, \gamma_k[L_k^T(t)V_x]].$$

Here the Hamiltonian H_1 corresponds to motion without control ($dU = 0$), and H_2 accounts for the jumps generated by control impulses. Therefore, equation (8) may be interpreted as follows: either $H_1 = 0$ and in the respective on-line position one may choose zero control (no jump), or $H_2 = 0$ and the control will have an impulse.

4.2 A Formalization of the Closed-Loop System

Although a feedback control strategy may be formally derived from the Hamilton–Jacobi–Bellman equation, it is not clear what a closed-loop system would be under such control. A possible formalization for ordinary impulse controls is based on the double constraint approach from (Daryin, Kurzhanski and Seleznev [2005]).

Definition 1. The pair of functions $\mathscr{U} = \{u(t,x;\mu), \theta(t,x;\mu)\}$ ("magnitude" and "duration"), such that

$$u(t,x;\mu) \in S_1 \cup \{0\}, \qquad u(t,x;\mu) \overset{\mu\to\infty}{\to} u_\infty(t,x),$$

$$\theta(t,x;\mu) \ge 0, \qquad \mu\theta(t,x;\mu) \overset{\mu\to\infty}{\to} m_\infty(t,x),$$

is called the impulse feedback control strategy.

The component $u(t,x)$ is the direction of the control impulse which is issued on interval $[t, t+\theta(t,x)]$. Note that as $\mu \to \infty$, we have $\theta \to 0$ and in the limit one has a delta-function as control.

[4] By $\mathscr{I}(x|A)$ we denote the indicator function of a set A, which is zero on this set and equal to infinity elsewhere.

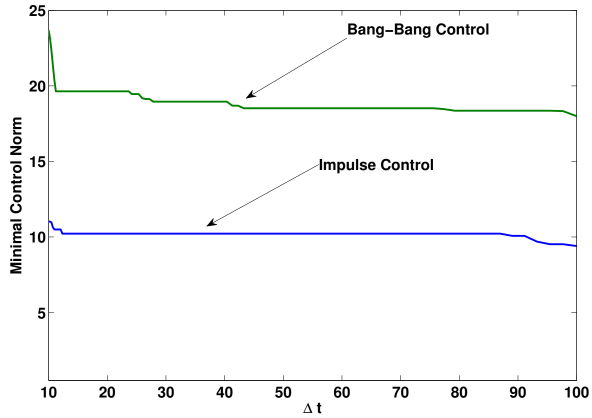

Fig. 6. Comparison of bang-bang controls vs impulse controls in L_1 norm

Definition 2. Fix a control strategy $\mathscr{U}$, number $\mu > 0$ and a partition $t_0 = \tau_0 < \tau_1 < \ldots < \tau_s = t_1$ of interval $[t_0, t_1]$. An approximating motion of the system (1) is the solution to the differential equation

$$\tau_i^* = \tau_i \wedge \theta(\tau_{i-1}, x_\Delta(\tau_{i-1}); \mu),$$

$$\dot{x}_\Delta(\tau) = \mu B(t) u(\tau_{i-1}, x_\Delta(\tau_{i-1}); \mu), \quad \tau_{i-1} < \tau < \tau_i^*,$$

$$x_\Delta(\tau_i) = x_\Delta(\tau_i^*).$$

Number $\sigma = \max\{\tau_i - \tau_{i-1}\}$ is the diameter of the partition.

Definition 3. A constructive motion of system (1) under feedback control $\mathscr{U}$ is a piecewise continuous function $x(t)$, which is the pointwise (weak) limit of approximating motions $x_\Delta(t)$ as $\mu \to \infty$ and $\sigma \to 0$.

4.3 Impulses vs Bang-Bang Controls

In Fig. 6 we compare bang-bang controls from Pontryagin's Maximum Principle with impulse controls in terms of the L_1 norm. We observe that the norm of impulse controls is twice lower than that of bang-bang controls.

As the control time, $t_1 - t_0$, decreases, the norm of the impulse control grows, and so does the minimum possible amplitude of bang-bang controls. Consider such a bang-bang feedback control with minimum amplitude μ. Suppose that the switching of bang-bang control occurs after each θ time units ($\theta \ll t_1 - t_0$). Then the error $x(t_1) - x_1$ will be of order $\theta\mu$, which may be large unless $\theta \ll \mu^{-1}$. In other words, to get such feedback control functioning properly, one has to ensure very high sampling rate, which may be infeasible. The proposed formalization of impulse control is free of the indicated issue (although too low sampling rate may result in increase of the variation of control).

5. ON NUMERICAL SCHEMES

To avoid calculation of the exact value function, its upper and lower bounds may be described through quadratic approximations similar to those developed in ellipsoidal calculus (Kurzhanski and Vályi [1997], Kurzhanski and Varaiya [2000]). Such results may be then applied to the calculation of forward and backward reach sets under generalized impulse controls of the present paper. An ellipsoidal bound for reach sets under impulse controls is presented in detail in (Daryin and Kurzhanski [2007]).

6. CONCLUSION

In this paper we presented an approach to problems of generalized impulse feedback control in the class of inputs which allows not only delta-functions, but also their higher derivatives as controls. We also introduce the notion of realizable fast controls which solve the problems of this paper in arbitrary small "nano"-time. The suggested generalized inputs taken as virtual controls may be also relevant for systems with jumps and for on-line resets of the system structure and/or position, Kurzhanski [2006]. Further work is expected to include calculation and formalization of impulse feedback controls for systems under uncertainty, as well as the treatment of complex state constraints.

Appendix A. THE GENERALIZED IMPULSE CONTROL PROBLEM

Here the control $u(t)$ is chosen from the class $D^*_{k,m}[\alpha,\beta]$ of continuous linear functionals over the linear normed space $D_{k,m}[\alpha,\beta]$ (see Gelfand and Shilov [1964], Schwartz [1950]). The latter consists of k times differentiable functions $\phi(t) : [\alpha,\beta] \to \mathbb{R}^m$ with support contained in $[\alpha,\beta]$. The norm in $D_{k,m}$ is defined as

$$\rho[\phi] = \max_{t\in[\alpha,\beta]} \gamma[\gamma_0(\phi(t)), \gamma_1(\phi'(t)), \ldots, \gamma_k(\phi^{(k)}(t))],$$

where γ_k, γ are finite-dimensional norms in spaces $\mathbb{R}^m$ and $\mathbb{R}^{k+1}$. The norm $\rho[\phi]$ determines its adjoint norm $\rho^*[u]$ in the space $D^*_{k,m}[\alpha,\beta]$. Hence the control is a distribution of order $k_u \le k$. The trajectories of the system (1) are distributions from $D^*_{k-1,n}[\alpha,\beta]$.

The admissible controls $u(t)$ are distributions from $D^*_{k,m}[\alpha,\beta]$ for which there exists a distribution $x(t) \in D^*_{k-1,n}[\alpha,\beta]$ satisfying $\dot{x}(t) = A(t)x + B(t)u + f^{(\alpha)} - f^{(\beta)}$ with support of $x(t)$ enclosed in $[t_0,t_1]$, where $\alpha < t_0 \le t_1 < \beta$. Here $f^{(\alpha)}$ and $f^{(\beta)}$ are distributions from $D^*_{k,n}[\alpha,\beta]$, concentrated at points t_0 and t_1 respectively. These distributions may be interpreted as initial and final conditions for the trajectory $x(t)$ and may be represented as $f^{(\alpha)} = \sum_{j=0}^k \alpha_j \delta^{(j)}(t-t_0)$, $f^{(\beta)} = \sum_{j=0}^k \beta_j \delta^{(j)}(t-t_1)$. Recall that any distribution $u \in D^*_{k,m}[\alpha,\beta]$ may be written as (see Gelfand and Shilov [1964])

$$\langle u, \phi\rangle = \sum_{j=0}^{k} \int_\alpha^\beta \frac{d^j\phi}{dt^j} dU_j(t),$$

where U_j are functions of bounded variation on $[\alpha,\beta]$, taking values in $\mathbb{R}^m$ and constant when $\alpha \le t \le t_0$, $t_1 < t < \beta$.

A rigorous formulation of problem 7 sounds as follows: for a given distribution $f^{(\alpha)}$ and a time interval $[t_0,t_1]$ find a distribution $f^{(\beta)}$ and an admissible control $u(t) \in D^*_{k,m}[\alpha,\beta]$ minimizing the functional $J(u, f^{(\beta)}) = \rho^*[u] + \phi(f^{(\beta)})$. Here $\phi(f)$ is a convex, bounded from below terminal function.

REFERENCES

Bensoussan, A. and J.-L. Lions. *Contrôle impulsionnel et inéquations quasi-variationnelles.* Dunod, Paris, 1982.

Daryin, A. N. and A. B. Kurzhanski. Generalized functions of high order as feedback controls. *Differenc. Uravn.*, 43 (11), 2007. In Russian.

Daryin, A. N., A. B. Kurzhanski, and A. V. Seleznev. A dynamic programming approach to the impulse control synthesis problem. In *Proc. Joint 44th IEEE CDC-ECC 2005*, Seville, 2005. IEEE.

Dykhta, V. A. and O. N. Samsonuk. *Optimal impulsive control with applications.* Fizmatlit, Moscow, 2003. In Russian.

Gelfand, I. M. and G. E. Shilov. *Generalized Functions.* Academic Press, N.Y., 1964.

Krasovski, N. N. On a problem of optimal regulation. *Prikl. Math. & Mech.*, 21(5):670–677, 1957. In Russian.

Krasovski, N. N. *The Theory of Control of Motion.* Nauka, Moscow, 1968. In Russian.

Kurzhanski, A. B. On synthesis of systems with impulse controls. *Mechatronics, Automatization, Control*, (4): 2–12, 2006. In Russian.

Kurzhanski, A. B. and Yu. S. Osipov. On controlling linear systems through generalized controls. *Differenc. Uravn.*, 5(8):1360–1370, 1969. In Russian.

Kurzhanski, A. B. and I. Vályi. *Ellipsoidal Calculus for Estimation and Control.* SCFA. Birkhäuser, Boston, 1997.

Kurzhanski, A. B. and P. Varaiya. Ellipsoidal techniques for reachability analysis. Internal approximation. *Systems and Control Letters*, 41:201–211, 2000.

Lancaster, P. *Theory of Matrices.* Academic Press, N.Y., 1969.

Miller, B. M. and E. Ya. Rubinovich. *Impulsive Control in Continuous and Discrete-Continuous Systems.* Kluwer, N.Y., 2003.

Neustadt, L. W. Optimization, a moment problem and nonlinear programming. *SIAM Journal on Control*, 2 (1):33–53, 1964.

Riesz, F. and B. Sz-.Nagy. *Leçons d'analyse fonctionnelle.* Akadémiai Kiadó, Budapest, 1972.

Schwartz, L. *Théorie des distributions.* Hermann, Paris, 1950.

Seidman, Th. I. and J. Yong. How violent are fast controls? II. *Math. of Control, Signals, Syst.*, (9):327–340, 1997.