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Frequency content in an axially impacted bar subject to boundary conditions ⋆

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Abstract: We analyze the frequency contents of strain waves in an axially excited bar, where one end of the bar is subject to a free end boundary condition. The analysis is treated in the context of identification of viscoelastic materials. The free end boundary condition leads to very particular constraints on the wave propagation in the bar and it is shown how this influences the identification. Wave propagation in both elastic and viscoelastic materials is treated, and the validity of the analysis confirmed through simulated and (in the case of viscoelastic materials) experimental data. The analysis is then used in order to interpret the large frequency variations in previous studies concerning the accuracy of the estimate and optimal input signal, respectively.

Keywords: Wave propagation, frequency content, viscoelastic materials, nonparametric identification

1. INTRODUCTION

Wave propagation experiments on an axially excited bar specimen is commonly used to identify material functions of viscoelastic materials, see for example Blanc (1993); Hillström et al. (2000); Pintelon et al. (2004); Sogabe and Tsuzuki (1986). In many cases, the experiment is designed so that one end of the bar is left free. This free end condition leads to very particular constraints on the wave propagation in the bar, which in turn influences the identification. In previous studies it was for example found that the quality of the estimated material function varies significantly with frequency, a behavior that can be explained through the work in this paper. The aim of the work is to examine how the boundary conditions affect the frequency content of the waves propagating in the bar, and to use this study in order to further understand what influence this design aspect may have on the identification.

We here focus on the identification problem described in Hillström et al. (2000), where nonparametric identification of the frequency dependent complex modulus $E(\omega)$ of a viscoelastic material is concerned. In Hillström et al. (2000), the complex modulus of the material is identified based on the frequency domain wave equation

$$\frac{\partial^2 \varepsilon(x,\omega)}{\partial x^2} = -\frac{\rho\omega^2}{E(\omega)}\,\varepsilon(x,\omega), \qquad (1)$$

which describes the propagation of strain waves in an axially excited bar, see Fig. 1. In (1), $\varepsilon(x,\omega)$ denotes the Fourier transform at frequency ω of strain measured at location x along the bar and ρ denotes the density of the material. This identification problem has been further studied in for example Mossberg et al. (2001) where the variance of the achieved estimates was treated, and in

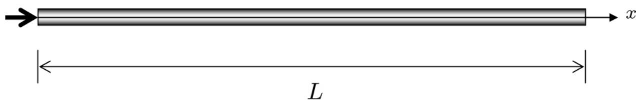

Fig. 1. An axially excited bar

Rensfelt and Söderström (2006) where optimal excitation of the bar was considered. In both these cases, a study of the frequency contents of the measured signal may further elucidate the results.

This paper treats bars made of linearly elastic as well as viscoelastic materials. The study on elastic bars is closely related to the eigenvalue problem for rods in axial vibration, as described in Meirovitch (1997), and serves as foundation for the more complex viscoelastic case. A description of elastic and viscoelastic materials, and the difference between them, can be found in Section 2. In Section 3 the analysis of the frequency contents in the measured signal is carried out. The case of elastic materials is first addressed followed by a similar analysis in the viscoelastic case. In Section 4, the identification problem described in Hillström et al. (2000) is treated and it is shown how the frequency contents of the measured signal may influence the results of the identification. Finally, conclusions are given in Section 5.

2. ELASTIC AND VISCOELASTIC MATERIALS

The relationship between stress (force per unit area) and strain (elongation per unit length) in an linearly elastic material is described by the simple Hooke's law

$$\sigma(t) = E\varepsilon(t) \qquad (2)$$

where E is the Young's modulus of linear elasticity. Young's modulus is a material specific *constant*, which means that there is a instantaneous relationship between

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the stress and the strain in an elastic material. In other words, as soon as the force put on to the material is released, the material will immediately go back to its original shape. Viscoelastic materials on the other hand exhibit viscous, as well as elastic, effects. The viscous effect in the material acts as a form of damping, which means that when the stress put onto the material is released, the material will only *slowly* return to its original shape. For a viscoelastic material we hence have a time dependence between the stress and the strain, *i.e.* the material response is determined not only by the current state of stress but also by all past states. Similarly, if the deformation is specified, the current stress depends on the entire deformation history. For a linearly viscoelastic material, this can be expressed as

$$\sigma(t) = \int_{-\infty}^{t} Y(t-\tau)\dot{\varepsilon}(t)d\tau, \tag{3}$$

where $Y(t)$ is called the relaxation modulus and $\dot{\varepsilon}(t)$ is the time derivative of the strain $\varepsilon(t)$. The convolution relationship in (3) is in frequency domain translated into a multiplication

$$\sigma(\omega) = E(\omega)\varepsilon(\omega), \tag{4}$$

where $\sigma(\omega)$ and $\varepsilon(\omega)$ denote the Fourier transformed stress and strain, respectively. Note that $E(\omega) = \mathrm{i}\omega Y(\omega)$, where $Y(\omega)$ is the Fourier transform of the relaxation modulus $Y(t)$. Knowledge about the complex modulus is very important in order to understand how a viscoelastic material behaves when used in an environment where it is subject to stress of various kind.

The complex modulus can be identified through strain waves propagating in an axially excited bar. This is a convenient technique for estimating $E(\omega)$ in the frequency band $10^2 - 10^4$ Hz. The wave propagation is then be described by the frequency domain wave equation (1). The general solution to this equation is given by

$$\varepsilon(x,\omega) = P(\omega)\mathrm{e}^{-\gamma(\omega)x} + N(\omega)\mathrm{e}^{\gamma(\omega)x} \tag{5}$$

where $P(\omega)$ and $N(\omega)$ are complex valued functions of ω, and can be interpreted as amplitudes of waves traveling in positive and negative x direction respectively. The complex valued, frequency dependent function $\gamma(\omega)$ in (5) is called the wave propagation function and is in the viscoelastic case given by

$$\gamma(\omega) = \sqrt{-\frac{\rho\omega^2}{E(\omega)}}. \tag{6}$$

If $\gamma(\omega)$ is expressed in terms of its real and imaginary parts, *i.e.*

$$\gamma(\omega) = \alpha(\omega) + \mathrm{i}k(\omega), \tag{7}$$

it is clearly seen from (5) that the real part of the wave propagation function, $\alpha(\omega)$, acts as a damping factor for waves at frequency ω, while the imaginary part, $k(\omega)$, is the corresponding wave number. The wave number $k(\omega)$ is an odd function, positive for $\omega > 0$, and the damping factor $\alpha(\omega)$ a positive even function, Hunter (1960). By definition, the wavelength for waves at frequency ω can be obtained through the wave number as

$$\lambda(\omega) = 2\pi/|k(\omega)|. \tag{8}$$

The frequency domain wave equation (1) also describes wave propagation in an elastic bar. The complex modulus $E(\omega)$ is then replaced by the constant Young's modulus E in (2) and the wave propagation function $\gamma(\omega)$ in (5) is given by

$$\gamma(\omega) = \sqrt{-\frac{\rho\omega^2}{E}}, \tag{9}$$

which is a purely imaginary function of ω. Hence, in the elastic case

$$\gamma(\omega) = \mathrm{i}k(\omega), \quad k(\omega) = \sqrt{\rho/E(\omega)}\omega \tag{10}$$

and the damping factor $\alpha(\omega) = 0$. This implies that if the material is excited initially and then allowed to vibrate freely, the vibrations will continue eternally without ever damping out, *i.e.* the system is conservative. This, however, is a mathematical idealization, and ideally elastic materials do not exist in practice. Elastic theory is nevertheless useful when the damping is very low, and many materials such as most metals follow Hooke's law with high accuracy. Examples of materials where the damping is not negligible, and has to be considered in a viscoelastic framework, include many types of plastics.

3. ANALYSIS OF FREQUENCY CONTENTS

When an experiment is designed so that an end of the bar is left free, the strain at that end will be identically equal to zero. If we for example have a free end at $x = 0$, then $\varepsilon(0,t) = 0$ for all t. Consequently, the Fourier transformed strain at that end will also equal zero, *i.e.* $\varepsilon(0,\omega) = 0$. In the following two subsections we will examine the frequency contents of the measured strain signal in bars made of elastic and viscoelastic material, respectively, where one end of the bar is free. The bar is excited (strained) at the end opposite to the free end, and it is here assumed that the only influence at this end is that of the excitation. The frequency domain strain at the excited end is therefore the same as the Fourier transform of the strain excitation. This assumption can, however, be relaxed and the analysis modified to cover all cases where the power of the strain at the excited end is limited for all frequencies.

3.1 Elastic materials

Consider a bar as in Fig. 1 made of an elastic material with ends in $x = 0$ and $x = L$. The bar is suspended in such a way that there is a free end at $x = 0$ and then excited at $x = L$. This gives the following boundary conditions

$$\varepsilon(0,\omega) = 0 \tag{11}$$

$$\varepsilon(L,\omega) = u(\omega), \tag{12}$$

where $u(\omega)$ denotes the Fourier transform of the strain excitation. As the wave propagation function of an elastic material is purely imaginary, see (10), the solution to the frequency domain wave equation (5) can be written as

$$\varepsilon(x,\omega) = A(\omega)\cos\big(k(\omega)x\big) + B(\omega)\sin\big(k(\omega)x\big). \tag{13}$$

The complex valued amplitudes $A(\omega)$ and $B(\omega)$ are given by

$$A(\omega) = N(\omega) + P(\omega) \tag{14}$$

and

$$B(\omega) = \mathrm{i}(N(\omega) - P(\omega)). \tag{15}$$

By applying boundary condition (11) to (13) we may conclude that $A(\omega) = 0$, and equation (13) hence simplifies to

$$\varepsilon(x,\omega) = B(\omega)\sin\big(k(\omega)x\big). \tag{16}$$

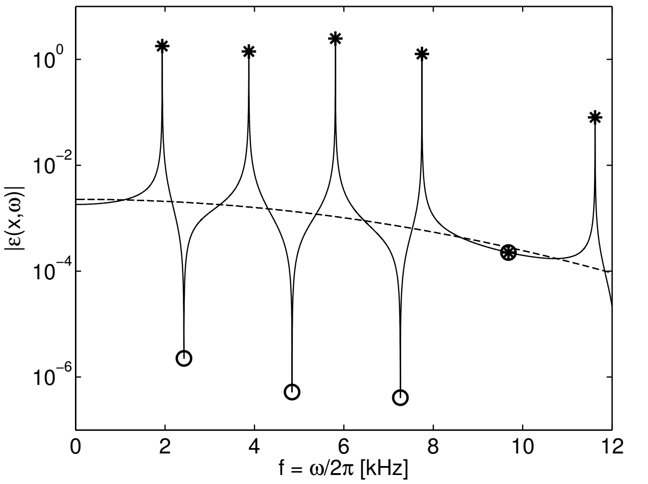

Fig. 2. Frequency domain strains $|\varepsilon(x,\omega)|$ for simulated aluminium bar (solid line), and frequency response of the simulated input signal in Fig. 3 (dashed). Frequencies fulfilling (18) marked with $(*)$, and frequencies fulfilling (19) marked with (o).

If the second boundary condition, (12), is applied to this equation, we get the following expression for the amplitude $B(\omega)$

$$B(\omega) = \frac{u(\omega)}{\sin\big(k(\omega)L\big)}. \tag{17}$$

Under the assumption that the power of the input signal is finite for each frequency, a condition that is fulfilled in most practical applications, we see from (17) that $|B(\omega)|$ approaches infinity for those frequencies where $|\sin\big(k(\omega)L\big)| = 0$. This is fulfilled when $k(\omega)L = n\pi$ or equivalently when

$$k(\omega) = \frac{n\pi}{L}. \tag{18}$$

The frequency domain strains will hence be amplified for frequencies where the wave number fulfills (18). This corresponds to the eigenmodes of a bar in axial vibration where the free end condition is fulfilled at both ends of the bar, see Meirovitch (1997). On the other hand, from (16) we can expect the frequency domain strain measured at position x to vanish when $\sin\big(k(\omega)x\big) = 0$, *i.e.* when $k(\omega)x = n\pi$ or equivalently when

$$k(\omega) = \frac{n\pi}{x}. \tag{19}$$

Note that from (8), this corresponds to measuring the signal at integral multiples of half a wavelength from the free end at $x = 0$, *i.e.* at the nodes of the sine wave in (16).

In Fig. 2, simulated frequency domain strains from a wave propagation experiment on an elastic bar is shown. The bar is axially excited at $x = L$ by the pulse shown in Fig. 3, and then left to vibrate freely. The length of the bar is $L = 2$ m and the strains are simulated through (5) and (9). The amplitudes $P(\omega)$ and $N(\omega)$ are given by (11) and (12), where $u(\omega)$ is the Fourier transform of the exciting pulse in Fig. 3. In the simulations, Young's modulus was set to $E = 71$ GPa which corresponds well to the elastic behavior of aluminum (Al). The simulated strain is measured at position $x = 1.6$ m. As can be seen from Fig. 2, the frequencies that fulfills (18) corresponds well with the peaks of the measured frequency domain strain signal. Furthermore, the frequencies for which the wave number fulfills (19) corresponds well with the dips in the strain signal. At $f = 9.7$ kHz, the two conditions coincide and the peak is counteracted by the dip.

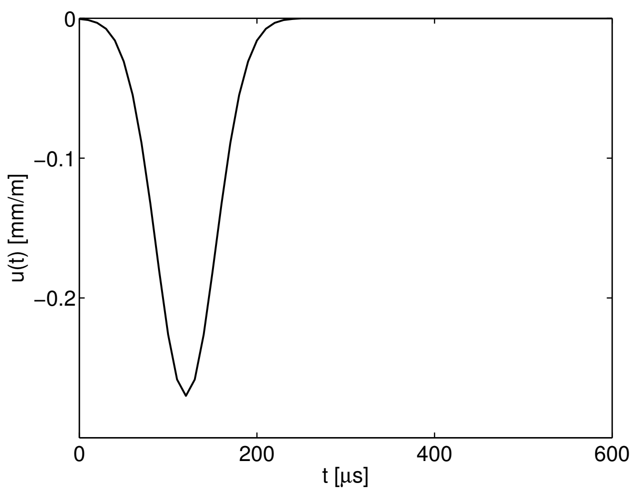

Fig. 3. The excitation pulse.

3.2 Viscoelastic materials

The same type of analysis as for the elastic case can also be performed for a viscoelastic material. Now instead consider a viscoelastic bar with a free end in $x = 0$. The bar is excited at $x = L$, which gives the same set of boundary conditions as in the elastic case, see (11) and (12). If boundary condition (11) is applied to (5) this expression simplifies to

$$\varepsilon(x,\omega) = C(\omega)\sinh\big(\gamma(\omega)x\big), \tag{20}$$

where $C(\omega)$ is given by

$$C(\omega) = 2N(\omega). \tag{21}$$

By applying the second boundary condition, (12), to (20) we get the following expression for the amplitude $C(\omega)$

$$C(\omega) = \frac{u(\omega)}{\sinh\big(\gamma(\omega)L\big)}. \tag{22}$$

As in the elastic case, we may conclude that $|C(\omega)|$ is big when $|\sinh\big(\gamma(\omega)L\big)|$ is small, and hence that we should expect the frequency domain strains to be amplified for frequencies where this is fulfilled. On closer inspection, $|\sinh\big(\gamma(\omega)L\big)|$ can be written as

$$|\sinh\big(\gamma(\omega)L\big)| = \Big(F\big(\alpha(\omega), k(\omega)\big)\Big)^{1/2}, \tag{23}$$

where

$$\begin{aligned} F\big(\alpha(\omega), k(\omega)\big) &= \cosh^2\big(\alpha(\omega)L\big)\sin^2\big(k(\omega)L\big) \\ &\quad + \sinh^2\big(\alpha(\omega)L\big)\cos^2\big(k(\omega)L\big). \end{aligned} \tag{24}$$

From (24) we see that in order for $|\sinh\big(\gamma(\omega)L\big)|$ to be small, then $\sin^2\big(k(\omega)L\big)$ must be small since $\cosh(y) \geq 1$ for all y. This gives the approximate condition

$$k(\omega) \approx \frac{n\pi}{L}, \tag{25}$$

which corresponds to (18) in the elastic case. Secondly, if (25) is fulfilled we also need $\sinh^2\big(\alpha(\omega)L\big)$ to be small. This corresponds to $\alpha(\omega)$ being sufficiently close to zero. We can hence expect the frequency domain strains to

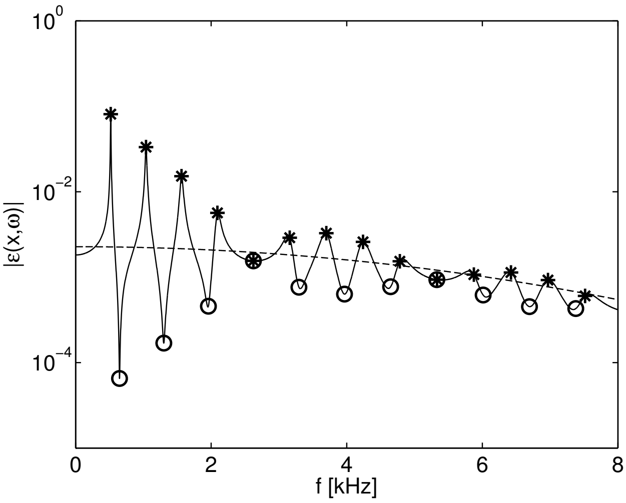

Fig. 4. Frequency domain strains $|\varepsilon(x, \omega)|$ for simulated PMMA bar (solid line), and frequency response of the simulated input signal in Fig 3 (dashed). Frequencies fulfilling (25) marked with (∗), and frequencies fulfilling (26) marked with (o).

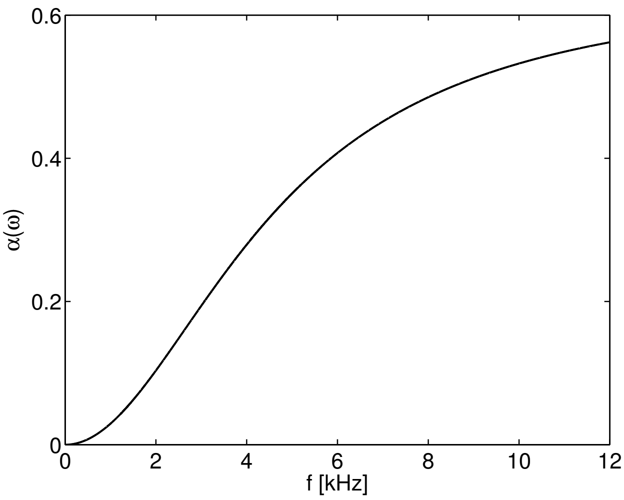

Fig. 5. Damping factor $\alpha(\omega)$ of PMMA (simulated).

be amplified for frequencies for which the wave number fulfills (25), and for which the damping factor $\alpha(\omega)$ is low enough. To determine what value of $\alpha(\omega)$ can be considered as sufficiently low, or to determine the exact minima of $|\sinh\big(\gamma(\omega)L\big)|$, further analysis is needed, which is beyond the scope of this paper.

We may also expect a decrease in signal power when $|\sinh\big(\gamma(\omega)x\big)|$ is small, see (20). Following the analysis above we get the approximate condition

$$k(\omega) \approx \frac{n\pi}{x}, \tag{26}$$

and that the damping factor $\alpha(\omega)$ is sufficiently low. Similarly to the elastic case, the condition (26) corresponds to the signal being measured at integral multiples of half a wavelength from the free end at $x = 0$, see (8).

In Fig. 4, simulated frequency domain strains from a wave propagation experiment on an viscoelastic bar is shown. The bar is axially excited at $x = L$ by the pulse shown in Fig. 3, and then left to vibrate freely. The length of the bar is $L = 2$ m and the strains are simulated through (5) and (6). The amplitudes $P(\omega)$ and $N(\omega)$ are given by (11) and (12), where $u(\omega)$ is the Fourier transform of the exciting pulse in Fig. 3. Data for the complex modulus of the specimen was generated from the standard linear solid model Zener (1948),

$$E(\omega) = \frac{E_2(E_1 + \mathrm{i}\omega\eta)}{E_1 + E_2 + \mathrm{i}\omega\eta)}, \tag{27}$$

with model parameters $E_1 = 56$ GPa, $E_2 = 5.6$ GPa and $\eta = 2$ kPa·s. This choice of parameters applies of the dynamic behavior of the weakly damped PMMA (plexiglass). The corresponding damping factor $\alpha(\omega)$ is shown in Fig. 5. The simulated strain is measured at position $x = 1.6$ m. As can be seen from Fig. 4, the frequencies for which the wave number fulfills (25) corresponds well with the peaks in the measured frequency domain strain signal. As the damping $\alpha(\omega)$ grows higher with increasing ω, the peaks become less prominent and starts to deviate slightly from the frequencies given by (25). Furthermore, the frequencies for which the wave number fulfills (26) corresponds well with the dips in the measured frequency domain strain signal. These two conditions coincide at $f = 2.6$ kHz and $f = 5.3$ kHz, where the peak is accordingly counteracted by the dip.

3.3 Experiment on a viscoelastic bar

In Hillström et al. (2000), experiments were made on a viscoelastic PMMA bar in order to identify the complex modulus of the material. The ends of the bar were located at $x = 0$ and $x = L$, where $L = 2$ m is the length of the bar. The bar was suspended in such a way that the end at $x = 0$ was free, and then axially excited at $x = L$ through a strain pulse generated by the use of a steel hammer. The resulting strain data was collected at sensor locations $\mathbf{x} = \{0,\ 0.290,\ 0.646,\ 1.078,\ 1.600\}$ m for $N = 4096$ discrete time instances with a sampling interval of $T = 20$ μs, and then transformed into the frequency domain using the discrete Fourier transform (DFT). In Mossberg et al. (2001), the time domain measurement noise was found to be very close to white with variance $\sigma_t^2 \approx 3.6 \cdot 10^{-14}$ for the frequency band of interest. The Fourier transformed noise will then also be approximately white with variance $\sigma_f^2 = N \cdot \sigma_t^2$, see the analysis in Mahata et al. (2003).

In Fig. 6, the frequency domain strain signal measured at sensor location $x = 1.6$ m is shown. As in the simulated data case we see that the frequencies for which the wave number fulfills (25) corresponds well with the peaks in the strain signal and that the frequencies for which the wave number fulfills (26) corresponds well with the dips. At $f = 2.7$ kHz and $f = 5.4$ kH, these two conditions coincide, and accordingly no prominent peak or dip can be detected.

4. IDENTIFICATION OF VISCOELASTIC MATERIALS

In this section, we investigate how the frequency contents in the measured signal influences the identification problem described in Hillström et al. (2000). In Hillström

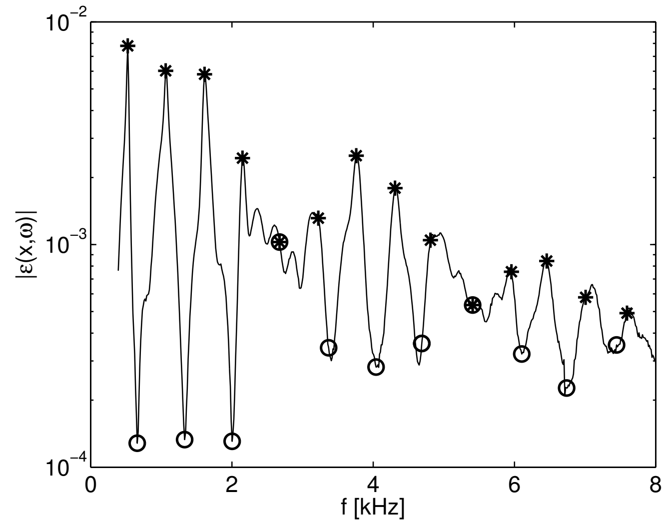

Fig. 6. Frequency domain strains $|\varepsilon(x, \omega)|$ from experiment on a PMMA bar (solid line). Frequencies fulfilling (25) marked with ($*$), and frequencies fulfilling (26) marked with (o).

et al. (2000), the experimental data in the previous subsection were used to identify the complex modulus of the viscoelastic material through the frequency domain wave equation (1). Using (1), the strain at each of the sections $\mathbf{x} = \{x_i\}_{i=1}^n$ form the system of equations

$$\boldsymbol{\varepsilon}(\omega) = \mathbf{A}(\omega)\mathbf{z}(\omega), \tag{28}$$

where

$$\boldsymbol{\varepsilon}(\omega) = \left[\varepsilon(x_1, \omega) \;\cdots\; \varepsilon(x_n, \omega)\right]^T, \tag{29}$$

$$\mathbf{A}(\omega) = \begin{bmatrix} e^{-\gamma(\omega)x_1} & e^{\gamma(\omega)x_1} \\ \vdots & \vdots \\ e^{-\gamma(\omega)x_n} & e^{\gamma(\omega)x_n} \end{bmatrix}, \tag{30}$$

$$\mathbf{z}(\omega) = \left[P(\omega) \;\; N(\omega)\right]^T. \tag{31}$$

At each section x_i a pair of strain gauges are mounted and connected to a bridge amplifier to extract the contribution from the extensional waves, Hillström et al. (2000). An estimate of the wave propagation function $\gamma(\omega)$, and through (6) the complex modulus $E(\omega)$, can be found by minimizing the loss function

$$U\big(\gamma(\omega), \mathbf{z}(\omega)\big) = \|\boldsymbol{\varepsilon}_{\mathcal{M}}(\omega) - \mathbf{A}(\omega)\mathbf{z}(\omega)\|^2 \tag{32}$$

with respect to $\gamma(\omega)$ and $\mathbf{z}(\omega)$. In (32), the vector $\boldsymbol{\varepsilon}_{\mathcal{M}}(\omega)$ contains the DFTs of the actual strain measurements which are assumed to be corrupted by measurement noise. The unknown amplitudes $P(\omega)$ and $N(\omega)$ act as nuisance parameters, giving a total of three unknowns, and the number of sensors needed for identification must thus be $n \geq 3$.

4.1 Accuracy of estimate

In order to get accurate estimates of the complex modulus, it is important that the measured signal is of good quality and that the signal-to-noise ratio (SNR) is sufficiently large for all frequencies under consideration. The quality of the achieved estimates for the identification problem described in this section is treated in Mossberg et al. (2001) and Mahata et al. (2003), where it was found that the variance of the estimate at any particular frequency is inversely proportional to the SNR at that frequency.

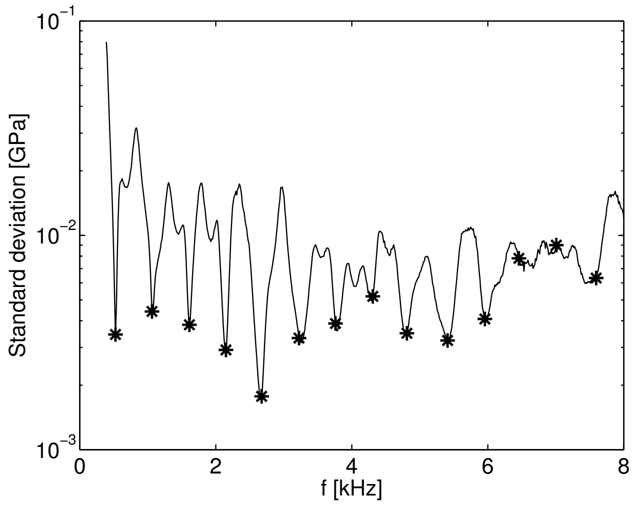

Fig. 7. Standard deviation of estimate (solid). Frequencies fulfilling (25) marked with ($*$).

In Fig. 7, the standard deviation of the estimates from the experimental data described in Section 3.3 is shown. As can be seen, the dips in standard deviation corresponds well with the peaks in the measured strain signal, *i.e.* the standard deviation is low for those frequencies for which the wave number fulfills (25). The reason for this behavior is that since there is more signal power at these frequencies, the SNR will become larger with a lower variance as a result. The dips in the strain signal does not have an as obvious effect on the resulting standard deviations. This is because these dips are dependent on where on the bar the signal is measured, see (26). In the experiment described in Section 3.3, the strain waves are measured at five distinct points along the bar. The spacing between the sensors are chosen so as to minimize the possibility of condition (26) being fulfilled at all five sensors for any particular frequency. For each frequency, we hence have at least a few measurements with reasonable signal power and large SNR, which in turn counteracts ensuing peaks in the standard deviation. If instead all sensors were spaced uniformly on the bar, so that there was a equal distance h between any two adjacent sensors, then (26) would be fulfilled at all sensors for some frequencies. The signal power would then be low for all measurements, with a low SNR as a result, and peaks in the standard deviation could be expected, see the discussion in Hillström et al. (2000).

4.2 Optimal input signal

In Rensfelt and Söderström (2006), optimal excitation for the given identification problem was investigated [1]. The basis for the investigation was that there is a linear dependency $g(x, \omega)$ between the input signal and the strain measured at position x, *i.e.* $\varepsilon(x, \omega) = g(x, \omega)u(\omega)$.

[1] In this study the input signal was chosen as normal force rather than strain. The relationship between the strain $\varepsilon(\omega)$ and normal force $N(\omega)$ is $\varepsilon(\omega) = N(\omega)/(AE(\omega))$, where A is the cross sectional area of the bar and $E(\omega)$ the complex modulus. The optimal input is to be implemented using a shaker.

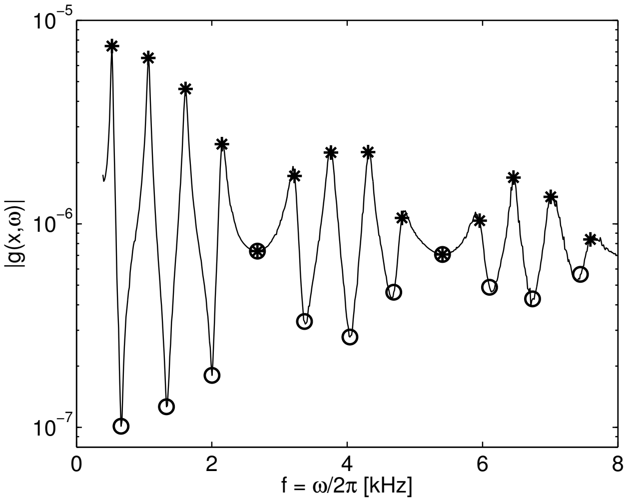

Fig. 8. Amplitude of transfer function from input to measured strain signal at $x = 1.6$ (solid). Frequencies fulfilling (25) marked with (∗), and frequencies fulfilling (26) marked with (o).

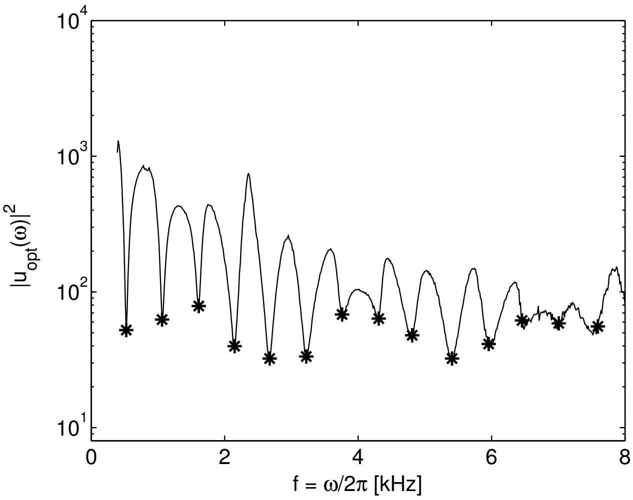

Fig. 9. Optimal input signal (solid). Frequencies fulfilling (25) marked with (∗).

The amplitude of the transfer function $g(x, \omega)$ for $x = 1.6$ m is shown in Fig. 8. As may be expected, there is good agreement between the peaks in $|g(x, \omega)|$ and the frequencies where the strain signal is amplified, *i.e.* the frequencies for which the wave propagation function satisfies (25). The frequencies where the strain signal is instead damped, *i.e.* the frequencies for which the wave propagation function satisfies (26), also corresponds well with the dips of $|g(x, \omega)|$. The poles of this transfer function can be used to estimate a parametric model for the complex modulus, as in Pintelon et al. (2004).

In Fig. 9, the D-optimal input spectrum from Rensfelt and Söderström (2006) is shown. As can be seen, the input power is significantly lower for those frequencies for which the wave propagation function satisfies (25). Since these frequencies are amplified in the bar, it is instead preferable to concentrate energy of the input signal to frequencies that are not amplified or even damped out.

5. CONCLUSION

In this paper, we investigated how the frequency content of strain waves propagating in an axially excited bar influence the identification of material functions of viscoelastic materials. One end of the bar was assumed to be free and it was shown how this boundary condition causes the measured strain signal to be amplified for some frequencies, while others are damped out. How this may influence the identification was illustrated through the identification problem described in Hillström et al. (2000), where nonparametric identification of the complex modulus of a viscoelastic material is concerned.

Wave propagation in bars made of elastic and viscoelastic materials were considered, respectively. In the elastic case, the analysis yield exact results and is closely related to the eigenvalue problem for bars in axial vibration described in Meirovitch (1997). For viscoelastic materials the results are approximate and apply to frequencies where the damping is sufficiently low. The validity of the approximate results are confirmed through experiments on bars made of the weakly damped material PMMA (plexiglass). In particular the analysis explains why the variances of the estimates as well as the spectrum of the optimal excitation vary strongly with frequency.

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