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Fast Nonlinear Model Predictive Control Using Set Membership Approximation ⋆

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Abstract: Set Membership function estimation methodologies under the Nearest Point approach are employed to compute an approximating function for a nonlinear model predictive controller (NMPC). The method is based on the off-line computation of a finite number ν of exact NMPC control solutions. The obtained approximating functions fulfill input constraints, have computational time which is independent on the control horizon and guarantee a level of accuracy which tends to zero by increasing ν. A nonlinear oscillator example is used to demonstrate the effectiveness of the presented results.

1. INTRODUCTION

Nonlinear Model Predictive Control (NMPC) (see e.g. Allgöwer and Zheng (2000)) is a model based control technique where the control action is computed by solving at each sampling time a constrained optimization problem using the predicted behaviour of the controlled nonlinear system. The control move u_t at time t, for time invariant systems, results to be a nonlinear static function of the system state x_t, i.e. $u_t = \kappa^0(x_t)$. A serious limitation in using NMPC is the presence of fast plant dynamics which require small sampling periods that do not allow to perform the optimization problem online. This motivates the research efforts devoted to develop computationally tractable NMPC solutions, or suitable approximations of NMPC control laws. In particular, the use of explicit functions which approximate the control law κ^0 appears to be a viable solution. A first contribution along this line of research was given by Parisini and Zoppoli (1995), who considered the use of a neural approximation of κ^0. However, besides the computational problems related, from one side, to the "curse of dimensionality" causing an exponential dimension increase in the neural network parameter space and, from the other side, to possible deteriorations in approximation due to trapping in local minima, the neural network approach does not appear to be well suited to obtain conditions that guarantee the stabilizing properties of the approximated controller and the satisfaction of the constraints required by the control problem. Moreover, no information on the guaranteed approximation error with the neural network approach can be derived. Another methodology to approximate a nonlinear MPC controller is proposed by Johansen (2004), who gives an off-line algorithm for the construction of a piecewise linear (PWL) explicit approximation of the nominal predictive control law. Then, the PWL approximation is implemented via a binary search tree. Stability properties are also obtained. However, with this approach the computational times depend on the number of the state space partitions, which increases as the required error tolerance decreases. Moreover, the stability and constraint satisfaction properties rely on the assumption of the convexity of the optimal cost function. If such assumption is not met, ad-hoc solutions have to be used.
To overcome these problems and to obtain a systematic method to derive approximations of NMPC laws with guaranteed accuracy, an alternative approach based on Set Membership (SM) approximation techniques has been introduced in Canale and Milanese (2005) and Canale et al. (2007b). In such approach an optimal (i.e. with minimum guaranteed approximation error) approximating function is derived on the basis of a number ν of exact off-line solutions of the NMPC optimization problem involved in the design. The obtained approximation error is bounded and converges to zero as ν increases, thus with the proper value of ν it is possible to achieve any desired level of approximation accuracy. However, on-line computation times of such an approximating function increase with ν and may still result too high for applications requiring very fast computation times (e.g. less than 1 ms). In this paper, another SM approach is introduced, denoted as "Nearest Point" (NP) approximation, whose guaranteed approximation error is not minimal, but whose computation is simpler and independent on ν. Using the proposed approach, input constraints are always satisfied and it is possible to introduce conditions under which closed loop system stability and state constraint satisfaction are guaranteed. Moreover, it is possible to guarantee a computable bound on the maximum distance between the state trajectories obtained with the nominal and the approximated predictive control laws. Besides, the computational time is independent from the MPC control horizon. The only required assumption is the continuity of the stabilizing nominal MPC control law over the set considered for the approximation. The effectiveness of the proposed methodology is tested on a nonlinear oscillator example.

2. MODEL PREDICTIVE CONTROL

Consider the following nonlinear state space model:

$$x_{t+1} = f(x_t, u_t) \tag{1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and control input respectively. In this paper, it is assumed that function f in (1) is continuous over $\mathbb{R}^n \times \mathbb{R}^m$. Assume that the control objective is to regulate the system state to the origin under some input and state constraints. The latter are represented by a convex set $\mathbb{X} \subseteq \mathbb{R}^n$ and a compact set $\mathbb{U} \subseteq \mathbb{R}^m$, both containing the origin in their interiors, in which the state trajectories x_t and input values u_t should be kept respectively.
Denoting by N_p and $N_c \leq N_p$ the prediction horizon and the control horizon respectively, the following objective function J can be defined:

$$J(U, x_{t|t}, N_p) = \Phi(x_{t+N_p|t}) + \sum_{k=0}^{N_p-1} L(x_{t+k|t}, u_{t+k|t}) \tag{2}$$

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where $x_{t+k|t}$ denotes k step ahead state prediction using the model (1), given the input sequence $u_{t|t}, \ldots, u_{t+k-1|t}$ and the "initial" state $x_{t|t} = x_t$. $U = \left[u_{t|t}^T, \ldots, u_{t+N_c-1|t}^T\right]^T$ is the vector of the control moves to be optimized. The remaining predicted control moves $[u_{t+N_c|t}, \ldots, u_{t+N_p-1|t}]$ can be computed with different strategies, e.g. by setting $u_{t+k|t} = u_{N_c-1|t}$ or $u_{t+k|t} = K\, x_{t+k|t}$, $\forall k \in [N_c, N_p-1]$, where K is a suitable matrix.
The MPC control law is then obtained applying the following receding horizon strategy:

(1) At time instant t, get x_t.
(2) Solve the optimization problem:

$$\min_U \quad J(U, x_{t|t}, N_p) \tag{3a}$$

s. t.

$$x_{t+1} = f(x_t, u_t) \tag{3b}$$

$$x_{t+k|t} \in \mathbb{X},\ k = 1, \ldots, N_p \tag{3c}$$

$$u_{t+k|t} \in \mathbb{U},\ k = 0, \ldots, N_p \tag{3d}$$

$$\tag{3e}$$

(3) Apply the first element of the solution sequence U to the optimization problem as the actual control action $u_t = u_{t|t}$.
(4) Repeat from step (1) at time $t+1$.

It is assumed that the optimization problem (3) is feasible over a set $\mathcal{F} \subseteq \mathbb{R}^n$ which will be referred to as the "feasibility set". The application of the receding horizon controller gives rise to the following nonlinear state feedback configuration:

$$x_{t+1} = f(x_t, \kappa^0(x_t)) = F^0(x_t) \tag{4}$$

where the control law κ^0 results to be a time invariant static function of the state x_t at time t, i.e.:

$$u_t = [u_{t,1} \ldots u_{t,m}]^T = [\kappa_1^0(x_t) \ldots \kappa_m^0(x_t)]^T = \kappa^0(x_t)$$
$$\kappa^0 : \mathcal{F} \to \mathbb{R}^m$$

Moreover, it is assumed that the horizons N_p and N_c and the cost functions L and Φ have been suitably chosen (see e.g. Mayne et al. (2000) and Goodwin et al. (2005) for details) such that the nonlinear autonomous system (4) is uniformly asymptotically stable at the origin for any initial state value $x_0 \in \mathcal{F}$, i.e. it is stable and

$$\begin{gathered}\forall \epsilon > 0,\ \forall \delta > 0\ \exists T \in \mathbb{N} \text{ s.t.}\\ \|\phi^0(t+T, x_0)\|_2 < \epsilon, \forall t \geq 0,\ \forall x_0 \in \mathcal{F} : \|x_0\|_2 \leq \delta\end{gathered} \tag{5}$$

where $\phi^0(t, x_0) = \underbrace{F^0(F^0(\ldots F^0}_{t \text{ times}}(x_0)\ldots))$ is the solution of (4) at time instant t with initial condition x_0. Note that, according to (3c), for any $x_0 \in \mathcal{F}$ the state constraints are always satisfied after the first time step, i.e.:

$$\phi^0(t, x_0) \in \mathbb{X}, \forall x_0 \in \mathcal{F}, \forall t \geq 1 \tag{6}$$

Thus, the set $\mathbb{X} \cap \mathcal{F}$ is positively invariant with respect to system (4):

$$\phi^0(t, x_0) \in \mathbb{X}, \forall x_0 \in \mathbb{X} \cap \mathcal{F}, \forall t \geq 0 \tag{7}$$

Moreover, due to (3d) the input constraints are satisfied for any $x \in \mathcal{F}$:

$$\kappa^0(x) \in \mathbb{U}, \forall x \in \mathcal{F} \tag{8}$$

Finally, it is assumed that the nominal control law κ^0 is continuous over the feasibility set $\mathcal{F}$. Such property has been investigated e.g. by Ohno (1978), where the continuity of the control law is studied in the context of general finite-horizon nonlinear optimal control. Note that stronger regularity assumptions (e.g. differentiability) cannot be made, since even in the simple case of linear dynamics, linear constraints and quadratic objective function, κ^0 is a piece-wise linear continuous function (see for example the papers of Bemporad et al. (2002) and Seron et al. (2003)).

3. SM "NEAREST POINT" APPROXIMATION OF MPC

To reduce the relatively high computational time needed to solve the optimization problem (3), in this paper a Set Membership approximation technique, denoted as "Nearest Point" (NP), is proposed for the computation of an approximating function $\kappa^{\text{NP}} \approx \kappa^0$. Such an approximating function, defined on a compact set $\mathcal{X} \subseteq \mathcal{F}$, is derived by the off-line computation of $\kappa^0(x)$ for a suitable number ν of states $\tilde{x}^k \in \mathcal{X}, k = 1, \ldots, \nu$. These state values define the set $\mathcal{X}_\nu = \{\tilde{x}^k, k = 1, \ldots, \nu\} \subset \mathcal{X}$. Function κ^{NP} will be also referred to as "FMPC" (Fast Model Predictive Control) control law.

3.1 Prior information

The approximation of function κ^0 is performed on a compact set $\mathcal{X} \subseteq \mathcal{F}$. Since $\mathcal{X}$ and the image set $\mathbb{U}$ of κ^0 are compact, continuity of κ^0 over $\mathcal{F}$ implies that its components κ_i^0, $i = 1, \ldots, m$ are Lipschitz continuous functions over $\mathcal{X}$, i.e. there exist finite constants γ_i, $i = 1, \ldots, m$ such that:

$$\begin{gathered}\forall x^1, x^2 \in \mathcal{X},\ \forall i \in [1, m],\\ |\kappa_i^0(x^1) - \kappa_i^0(x^2)| \leq \gamma_i \|x^1 - x^2\|_2\end{gathered} \tag{9}$$

$$\begin{gathered}\forall x^1, x^2 \in \mathcal{X},\\ \|\kappa^0(x^1) - \kappa^0(x^2)\|_2 \leq \|\gamma\|_2\, \|x^1 - x^2\|_2\end{gathered} \tag{10}$$

where $\gamma = [\gamma_1, \ldots, \gamma_m]^T$.
As already pointed out, it is assumed that a set $\mathcal{X}_\nu \subset \mathcal{X}$ of ν initial conditions $\tilde{x}^k$ is chosen and that the corresponding exact solutions $\tilde{u}^k = \kappa^0(\tilde{x}^k)$, $k = 1, \ldots, \nu$ are computed off-line and stored. The set $\mathcal{X}_\nu$ is supposed to be chosen such that the following property holds:

$$\lim_{\nu \to \infty} d_H(\mathcal{X}, \mathcal{X}_\nu) = 0 \tag{11}$$

where $d_H(\mathcal{X}, \mathcal{X}_\nu)$ is the Hausdorff distance between $\mathcal{X}$ and $\mathcal{X}_\nu$, defined as (see e.g. Blagovest (1990)):

$$d_H(\mathcal{X}, \mathcal{X}_\nu) = \max\left(\sup_{x \in \mathcal{X}} \inf_{\tilde{x} \in \mathcal{X}_\nu} (\|x - \tilde{x}\|_2),\ \sup_{\tilde{x} \in \mathcal{X}_\nu} \inf_{x \in \mathcal{X}} (\|x - \tilde{x}\|_2)\right)$$

Note that uniform gridding over $\mathcal{X}$ satisfies condition (11).
All this prior information can be summarized by concluding that

$$\kappa^0 \in FFS,$$

where the set FFS (Feasible Functions Set) is defined as:

$$FFS = \{\kappa \in \mathcal{A}_\gamma : \kappa(\tilde{x}) = \tilde{u},\ \forall \tilde{x} \in \mathcal{X}_\nu\}$$

where $\mathcal{A}_\gamma$ is the set of all continuous functions $\kappa : \mathcal{X} \to \mathbb{U}$, such that (9) holds.

3.2 "Nearest Point" approximation

NP approximation technique is now presented. The aim is to derive, from the ν off-line computed values of $\tilde{u}^k$ and $\tilde{x}^k$ and from the known properties of κ^0, an approximation κ^{NP} of κ^0 with the following properties:

i) the input constraints are always satisfied:

$$\kappa^{\text{NP}}(x) \in \mathbb{U},\ \forall x \in \mathcal{X} \tag{12}$$

ii) for a given ν, a bound $\zeta(\nu)$ on the pointwise approximation error can be computed:

$$\|\kappa^0(x) - \kappa^{\text{NP}}(x)\|_2 \leq \zeta \in \mathbb{R}^+,\ \forall x \in \mathcal{X} \tag{13}$$

iii) $\zeta(\nu)$ is convergent to zero:

$$\lim_{\nu\to\infty} \zeta = 0 \tag{14}$$

Such properties are needed to prove the stability results reported in Section 4.

For any $x \in \mathcal{X}$, denote with $\tilde{x}^{\mathrm{NP}}$ a state value such that:

$$\tilde{x}^{\mathrm{NP}} \in \mathcal{X}_\nu : \|\tilde{x}^{\mathrm{NP}} - x\|_2 = \min_{\tilde{x}\in\mathcal{X}_\nu} \|\tilde{x} - x\|_2 \tag{15}$$

Then, the NP approximation $\kappa^{\mathrm{NP}}(x)$ is computed as:

$$\kappa^{\mathrm{NP}}(x) = \kappa^0(\tilde{x}^{\mathrm{NP}}) \tag{16}$$

Such approximation trivially satisfies condition (12). The next Theorem 1 shows that NP approximation (16) satisfies also properties (13) and (14).

Theorem 1.

i) The pointwise NP approximation error $\|\kappa^0(x) - \kappa^{\mathrm{NP}}(x)\|_2$ is bounded:

$$\|\kappa^0(x) - \kappa^{\mathrm{NP}}(x)\|_2 \le \zeta^{\mathrm{NP}} = \|\gamma\|_2\, d_H(\mathcal{X}, \mathcal{X}_\nu),\ \forall x \in \mathcal{X}$$

ii) The bound ζ^{NP} converges to zero:

$$\lim_{\nu\to\infty} \zeta^{\mathrm{NP}} = 0$$

Proof. See Canale et al. (2007a).

As regards the computation of the Lipschitz constants $\gamma = [\gamma_1, \ldots \gamma_m]$, which are needed to compute the approximation error bound ζ^{NP}, estimates $\widehat{\gamma}_i, i = 1, \ldots, m$ can be derived as follows:

$$\widehat{\gamma}_i = \inf\left(\tilde{\gamma}_i : \tilde{u}_i^h + \tilde{\gamma}_i\|\tilde{x}^h - \tilde{x}^k\|_2 \ge \tilde{u}_i^k,\ \forall k, h = 1, \ldots, \nu\right) \tag{17}$$

The next result shows that $\widehat{\gamma}_i$ is convergent to $\gamma_i, i = 1, \ldots, m$.

Theorem 2.

$$\lim_{\nu\to\infty} \widehat{\gamma}_i = \gamma_i,\ \forall i = 1, \ldots, m$$

Proof. See Canale et al. (2007a).

4. STABILITY AND PERFORMANCE ANALYSIS OF NP APPROXIMATED PREDICTIVE CONTROL LAWS

A key point in the use of approximating functions in NMPC implementation is the stability and input and state constraint satisfaction of the resulting controlled system. In this Section, such issues will be addressed. In particular, it will be shown how the approximated control law κ^{NP} obtained with NP technique is able to keep the state trajectories inside a given compact set and to make them converge to an arbitrarily small neighborhood of the origin. Besides, an analysis of the capability of κ^{NP} of satisfying input and state constraints will be carried out. Finally, performance degradation issues will be considered using the distance between the state trajectory of the nominal control system and the one obtained employing the approximated control law, given the same initial condition x_0.

4.1 Problem settings

The analysis of the stabilizing properties of function κ^{NP} will be carried out considering the compact set $\mathcal{X}$ over which κ^{NP} is defined. Indeed, since $\mathcal{X}$ and $\mathbb{U}$ are both compact, continuity of function f in (1) over $\mathbb{R}^n \times \mathbb{R}^m$ implies that f is Lipschitz continuous over $\mathcal{X} \times \mathbb{U}$ with Lipschitz constant γ_f, i.e.:

$$\|f(w^1) - f(w^2)\|_2 \le \gamma_f\|w^1 - w^2\|_2,\ \forall w^1, w^2 \in \mathcal{X}\times\mathbb{U} \tag{18}$$

where $w = (x^T, u^T)^T$. Since f is known, γ_f can be numerically or analytically computed.

Note that, due to the Lipschitz properties (10) and (18) of the control law $\kappa^0(x)$ and of the system model f respectively, function $F^0(x)$ defined in (4) is Lipschitz continuous too, with Lipschitz constant: $L_F = \gamma_f\sqrt{1 + \|\gamma\|_2^2}$.

The use of $\kappa^{\mathrm{NP}}(x)$ instead of $\kappa^0(x)$ in the feedback control loop gives rise to the following discrete time nonlinear autonomous system:

$$x_{t+1}^{\mathrm{NP}} = f(x_t^{\mathrm{NP}}, \kappa^{\mathrm{NP}}(x_t^{\mathrm{NP}})) = F^{\mathrm{NP}}(x_t^{\mathrm{NP}}) \tag{19}$$

whose state trajectory at time instant t with initial condition x_0 is indicated as:

$$\phi^{\mathrm{NP}}(t, x_0) = \underbrace{F^{\mathrm{NP}}(F^{\mathrm{NP}}(\ldots F^{\mathrm{NP}}}_{t \text{ times}}(x_0)\ldots))$$

It is useful to compute an upper bound on the Euclidean norm of the one-step state trajectory perturbation induced by the use of control function κ^{NP} instead of κ^0. Considering any initial state $x_0 \in \mathcal{X}$, such perturbation is computed as:

$$x_1^{\mathrm{NP}} - x_1 = f(x_0, \kappa^{\mathrm{NP}}(x_0)) - f(x_0, \kappa^0(x_0))$$

Therefore, at time instant t the following equation is obtained:

$$x_{t+1}^{\mathrm{NP}} = F^0(x_t^{\mathrm{NP}}) + e(x_t^{\mathrm{NP}}) \tag{20}$$

with

$$e(x) = f(x, \kappa^{\mathrm{NP}}(x)) - f(x, \kappa^0(x)) \tag{21}$$

Since $\kappa^0(x)$ is not known in general, $e(x)$ cannot be explicitly computed, but a bound μ on its Euclidean norm $\forall x \in \mathcal{X}$ can be derived from (13) and (21):

$$\begin{aligned}
&\|e(x)\|_2^2 = \|f(x_0, \kappa^{\mathrm{NP}}(x_0)) - f(x_0, \kappa^0(x_0))\|_2^2 \le \\
&\le \gamma_f^2\, \|(x_0, \kappa^{\mathrm{NP}}(x_0)) - (x_0, \kappa^0(x_0))\|_2^2 = \\
&= \gamma_f^2\, \left(\|(x_0 - x_0\|_2^2 + \|\kappa^{\mathrm{NP}}(x_0) - \kappa^0(x_0))\|_2^2\right) \le \\
&\le \gamma_f^2\, \zeta^{\mathrm{NP}}(\nu)^2 \\
&\qquad \Rightarrow \|e(x)\|_2 \le \gamma_f\, \zeta^{\mathrm{NP}}(\nu) = \mu(\nu)
\end{aligned} \tag{22}$$

Thus the value of $\mu(\nu)$ depends on the number ν of exact solutions of (3) considered for the approximation of κ^0. On the basis of property (14) it can be noted that:

$$\lim_{\nu\to\infty} \mu(\nu) = 0 \tag{23}$$

Thus it is always possible to choose a suitable value of ν which guarantees a given one-step perturbation upper bound $\mu(\nu)$. Given these preliminary considerations, the aims of this Section are:

i) to find sufficient conditions on μ (and, consequently, on ν) which guarantee that the state trajectory $\phi^{\mathrm{NP}}(t, x_0)$ is kept inside the compact set $\mathcal{X}$, over which the approximation is carried out, and converge to an arbitrarily small neighborhood of the origin, for any $t \ge 0$ and any $x_0 \in \mathcal{G} \subset \mathcal{X}$, where $\mathcal{G}$ is a positively invariant set with respect to system (4):

$$\exists \mathcal{G} \subset \mathcal{X} : \phi^0(t, x_0) \in \mathcal{G},\ \forall x_0 \in \mathcal{G},\ \forall t \ge 0 \tag{24}$$

Note that the set $\mathcal{X}$ has to be chosen such that it contains in its interior a set $\mathcal{G}$ satisfying (24). An indication on the existence of such a set $\mathcal{G}$ is given by property (7). In fact, suppose that the state constraint set $\mathbb{X}$ is bounded and that the feasibility set $\mathcal{F}$ is such that $\mathbb{X} \subset \mathcal{F}$, then, any set $\mathcal{G}$ such that $\mathbb{X} \subseteq \mathcal{G} \subset F$ is positively invariant with respect to system (4). Moreover, note that $\{0\} \in \mathcal{G}$, since the origin is a stable fixed point for the nominal system (4).

ii) To evaluate the capability of κ^{NP} of keeping the state and input variables inside the subsets $\mathbb{X}$ and $\mathbb{U}$ respectively, i.e.:

$$F^{\mathrm{NP}}(x) \in \mathbb{X}$$
$$\kappa^{\mathrm{NP}}(x) \in \mathbb{U}$$

thus satisfying the constraints. As regards the input constraints, they are trivially satisfied due to property (12).

iii) To estimate an upper bound $\Delta(\nu)$ of the distance

$$d(t, x_0) = \|\phi^{\mathrm{NP}}(t, x_0) - \phi^0(t, x_0)\|_2 \qquad (25)$$

between the nominal and FMPC controlled state trajectories:

$$d(t, x_0) \le \Delta(\nu),\ \forall x_0 \in \mathcal{G},\ \forall t \ge 0$$

such that

$$\lim_{\nu \to \infty} \Delta(\nu) = 0$$

Δ is regarded as a measure of performance degradation of system (19) with respect to system (4).

4.2 Main results

In order to introduce convergence properties of the approximated controller κ^{NP} under the assumption of uniform asymptotic stability in the origin of system (4) for any $x_0 \in \mathcal{X}$, the following candidate Lyapunov function $V : \mathcal{X} \to \mathbb{R}^+$ for system (4) is defined:

$$V(x) = \sum_{j=0}^{\widehat{T}-1} \|\phi^0(j, x)\|_2 \qquad (26)$$

where:

$$\widehat{T} \ge T$$
$$T = \inf_{x \in \mathcal{X}} \left(T \in \mathbb{N} : \|\phi^0(t+T, x)\|_2 < \|x\|_2,\ \forall t \ge 0\right)$$

The following inequalities hold:

$$\|x\|_2 \le V(x) = \frac{V(x)}{\|x\|_2}\|x\|_2 \le b\,\|x\|_2,\ \forall x \in \mathcal{X} \qquad (27)$$

where

$$b = \sup_{x \in \mathcal{X}} \frac{V(x)}{\|x\|_2}$$

and

$$\begin{aligned} &V(F^0(x)) - V(x) = \Delta V(x) = \\ &= -\frac{\|x\|_2 - \|\phi^0(\widehat{T}, x)\|_2}{\|x\|_2}\|x\|_2 \le -K\|x\|_2,\ \forall x \in \mathcal{X} \end{aligned} \qquad (28)$$

with

$$K = \inf_{x \in \mathcal{X}} \frac{\|x\|_2 - \|\phi^0(\widehat{T}, x)\|_2}{\|x\|_2},\ 0 < K < 1$$

Thus $V(x)$ is a Lyapunov function for system (4) over $\mathcal{X}$. Moreover, it can be easily showed that $V(x)$ is Lipschitz continuous, with Lipschitz constant $\tilde{L}_V$:

$$|V(x^1) - V(x^2)| \le \tilde{L}_V\|x^1 - x^2\|_2,\ \forall x^1, x^2 \in \mathcal{X} \qquad (29)$$

with

$$\tilde{L}_V = \sum_{j=0}^{\widehat{T}-1}(L_F)^j \qquad (30)$$

thus the following inequality holds:

$$\begin{aligned} &\forall x \in \mathcal{X},\ \forall e : (F^0(x) + e) \in \mathcal{X} \\ &V(F^0(x) + e) \le V(F^0(x)) + \tilde{L}_V\mu \end{aligned} \qquad (31)$$

Note that constant $\tilde{L}_V$ as defined in (30) is not in general the lowest constant such that (29) holds. From a practical point of view, a less conservative estimate $\hat{L}_V$ of the "best" constant L_V can be computed by considering a number ξ of values of $\tilde{x}^h \in \mathcal{X}, h = 1, \ldots, \xi$, such that the set $\mathcal{X}_\xi = \{\tilde{x}^h \in \mathcal{X}, h = 1, \ldots, \xi\}$ satisfies the property:

$$\lim_{\xi \to \infty} d_H(\mathcal{X}, \mathcal{X}_\xi) = 0$$

and applying the following:

$$\widehat{L}_V = \inf(\tilde{L}_V : V(\tilde{x}^h) + \tilde{L}_V\|\tilde{x}^h - x^k\| \ge V(x^k),\ \forall x^k, x^h \in \mathcal{X}_\xi)$$

A proof similar to that of Theorem 2 can be used to show that

$$\lim_{\xi \to \infty} \widehat{L}_V = L_V$$

In the following, the $\|\cdot\|_2$-ball set centered in x is denoted as:

$$\mathbb{B}(x, r) = \{\widehat{x} \in \mathbb{R}^n\ :\ \|\widehat{x} - x\|_2 \le r,\ \}$$

and notation $\mathbb{B}(\mathcal{A}, r),\ \mathcal{A} \subseteq \mathbb{R}^n$ is used to indicate the set:

$$\mathbb{B}(\mathcal{A}, r) = \bigcup_{x \in \mathcal{A}} \mathbb{B}(x, r)$$

Theorem 3. Let κ^0 be the exact nonlinear MPC control law computed according to (3), such that (5), (6) and (9) hold. Let κ^{NP} be the NP approximation of κ^0 computed using a number ν of exact off-line solutions such that (11) and (24) hold. Then, it is always possible to find a suitable value of ν such that there exist a finite value $\Delta \in \mathbb{R}^+$ with the following properties:

i) the distance $d(t, x_0)$ in (25) is bounded by Δ:

$$d(t, x_0) \le \Delta,\ \forall x_0 \in \mathcal{G},\ \forall t \ge 0 \qquad (32)$$

ii) Δ can be explicitly computed as:

$$\Delta = \sup_{t \ge 0}\ \min(\Delta_1(t, \mu), \Delta_2(t, \mu))$$

where:

$$\Delta_1(t, \mu) = \sum_{k=0}^{t-1}(L_F)^k\mu \qquad (33)$$

$$\Delta_2(t, \mu) = 2\,\eta^t \sup_{x_0 \in \mathcal{G}} V(x_0) + \frac{b}{K}\,L_V\,\mu \qquad (34)$$

with $\eta = \left(1 - \dfrac{K}{b}\right)$, $0 < \eta < 1$.

iii) $\Delta(\nu)$ converges to 0:

$$\lim_{\nu \to \infty} \Delta(\nu) = 0 \qquad (35)$$

iv) the state trajectory of system (19) is kept inside the set $\mathbb{B}(\mathcal{G}, \Delta)$ for any $x_0 \in \mathcal{G}$:

$$\phi^{\mathrm{NP}}(t, x_0) \in \mathbb{B}(\mathcal{G}, \Delta),\ \forall x_0 \in \mathcal{G},\ \forall t \ge 0$$

v) the set $\mathbb{B}(\mathcal{G}, \Delta)$ is contained in $\mathcal{X}$

$$\mathbb{B}(\mathcal{G}, \Delta) \subseteq \mathcal{X}$$

vi) the state trajectories of system (19) asymptotically converge to the set $\mathbb{B}(0, q)$:

$$\lim_{t \to \infty} \|\phi^{\mathrm{NP}}(t, x_0)\|_2 \le q,\ \forall x_0 \in \mathcal{G}$$

with

$$q = \frac{b}{K}\,L_V\,\mu \le \Delta \qquad (36)$$

.

Proof. See Canale et al. (2007a).

Remark 1. If $L_F < 1$ (i.e. F^0 is a contraction operator), a simplified formulation for bound Δ is obtained. In fact, Lyapunov function (26) can be chosen as $V(x) = \|x\|_2$, with $b = 1$ in (27), $K = (1 - L_F)$ in (28) and $L_V = 1$ in (29). Thus the bound $\Delta_2(t, \mu)$ in (34) is computed as:

$$\Delta_2(t, \mu) = 2(L_F)^t \sup_{x_0 \in \mathcal{G}} \|x_0\|_2 + \frac{1}{1 - L_F}\mu$$

and q in (36) is:

$$q = \frac{1}{1 - L_F}\,\mu$$

On the other hand the bound $\Delta_1(t, \mu)$ in (33) is such that:

$$\Delta_1(t, \mu) \le \frac{1}{1 - L_F}\mu,\ \forall t \ge 0$$

therefore a simpler formulation for Δ is obtained:

$$\Delta = \sup_{t \ge 0}\ \min(\Delta_1(t, \mu), \Delta_2(t, \mu)) = \frac{1}{1 - L_F}\mu$$

The main consequence of Theorem 3 is that, with the proper value of ν, for any initial condition $x_0 \in \mathcal{G}$ it is guaranteed that the state trajectory ϕ^{NP} is kept inside the set $\mathcal{X}$ and converges to the set $\mathbb{B}(0, q)$, which can be arbitrarily small since q linearly depends on μ:

$$\lim_{\nu\to\infty} q = \frac{b}{K} L_V \lim_{\nu\to\infty} \mu(\nu) = 0$$

Therefore, it is possible to "tune" the number ν of state values considered for the approximation of the control law κ^0 in order to guarantee the desired regulation precision. Moreover, on the basis of (32) and (35) it can be noted that for any $\epsilon > 0$ it is always possible to find a suitable value of ν such that

$$d(t, x_0) < \epsilon, \ \forall x_0 \in \mathcal{G}, \ \forall t \geq 0$$

i.e. it is possible to obtain the desired upper bound on the distance between the state trajectories $\phi^{\mathrm{NP}}(t, x_0)$ and $\phi^0(t, x_0)$. Moreover, as $\nu \to \infty$ the performances of control system F^{NP} match with those of F^0, as it would be expected since $\zeta^{\mathrm{NP}}(\nu) \to 0$. As a consequence, during the computation of the approximating function the value of ν can be chosen in order to set a compromise between system performances (with respect to the nominal MPC control law) and memory usage, which increases as ν does. Note that with NP approximation the computational time does not increase with increasing ν values, as pointed out in the example of Section 5.

4.3 State constraint satisfaction

Theorem 3 does not address explicitly the problem of state constraint satisfaction for the FMPC controlled system (19), i.e.:

$$\phi^{\mathrm{NP}}(t, x) \in \mathbb{X}, \forall x \in \mathcal{G}, \forall t \geq 1$$

However, as a consequence of Theorem 3, it is possible to choose ν such that there exist a finite number $\overline{T}$ of time steps after which the state trajectory ϕ^{NP} is kept inside the constraint set $\mathbb{X}$, for any initial condition $x_0 \in \mathcal{G}$. Moreover the value of $\overline{T}$ decreases as ν increases. In fact, using (32) it follows that

$$\begin{aligned} &\forall x_0 \in \mathcal{G}, \ \forall t \geq 0 \\ &\|\phi^{\mathrm{NP}}(t, x_0)\|_2 \leq \|\phi^0(t, x_0)\|_2 + \Delta(\nu) \end{aligned} \tag{37}$$

Then, considering a value of ν such that:

$$\mathbb{B}(0, \epsilon + \Delta(\nu)) \subset \mathbb{X} \tag{38}$$

with $\epsilon > 0$ "small" enough, on the basis of the uniform asymptotic stability assumption (5), it is always possible to find $\overline{T} < \infty$ such that:

$$\|\phi^0(t + \overline{T}, x_0)\|_2 < \epsilon, \ \forall x_0 \in \mathcal{G}, \ \forall t \geq 0$$

Using (37) it can be noted that:

$$\begin{aligned} \|\phi^{\mathrm{NP}}(t + \overline{T}, x_0)\|_2 &\leq \|\phi^0(t + \overline{T}, x_0)\|_2 + \Delta(\nu) < \\ &< \epsilon + \Delta(\nu), \ \forall x_0 \in \mathcal{G}, \ \forall t \geq 0 \\ \Rightarrow \phi^{\mathrm{NP}}(t + \overline{T}, x_0) &\in \mathbb{B}(0, \epsilon + \Delta(\nu)), \ \forall x_0 \in \mathcal{G}, \ \forall t \geq 0 \end{aligned}$$

and, on the basis of (38):

$$\phi^{\mathrm{NP}}(t + \overline{T}, x_0) \in \mathbb{X}, \ \forall x_0 \in \mathcal{G}, \ \forall t \geq 0$$

thus after a finite number $\overline{T}$ of time steps there is the guarantee that state constraints are satisfied. Note that in general the higher is ϵ in (38), the lower is $\overline{T}$. Since the maximum value of ϵ such that (38) holds is higher as $\Delta(\nu)$ decreases, $\overline{T}$ in general decreases as $\Delta(\nu)$ does, i.e. as ν increases.

5. SIMULATION EXAMPLE

Consider the two-dimensional nonlinear oscillator obtained from the Duffing equation (see e.g. Jordan and Smith (1987)):

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) - 0.6\, x_2(t) - x_1(t)^3 - x_1(t) \end{aligned} \tag{39}$$

where the input constraint set $\mathbb{U}$ is:

$$\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 5\}$$

The following discrete time model to be used in the nominal MPC design has been obtained by forward difference approximation:

$$x_{t+1} = \begin{bmatrix} 1 & T_s \\ -T_s & (1 - 0.6\, T_s) \end{bmatrix} x_t + \begin{bmatrix} 0 \\ T_s \end{bmatrix} u_t + \begin{bmatrix} 0 & 0 \\ -T_s & 0 \end{bmatrix} x_t^3$$

with sampling time $T_s = 0.05$ s. The nominal MPC controller κ^0 is designed according to (3) with horizons $N_p = 100$, $N_c = 5$ and the following functions L and Φ:

$$L(x, u) = x^T Q x + u^T R u, \ \Phi = 0$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ R = 0.5$$

The following linear state inequality constraints define the considered set $\mathbb{X}$:

$$\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 5\}$$

The state prediction has been performed setting $u_{t+k|t} = u_{t+N_c-1|t}, \ k = N_c, ..., N_p - 1$. The optimization problem (3) employed to compute $\kappa^0(x)$ has been solved using a sequential constrained Gauss-Newton quadratic programming algorithm (see e.g. Nocedal and Wright (2006)), where the underlying quadratic programs have been solved using the MatLab® function `quadprog`.
Fig. 1 shows the obtained feasibility set $\mathcal{F}$ and the set $\mathcal{X}$ considered for the approximation, together with the constraint set $\mathbb{X}$ and the trajectories of the nominal and FMPC controlled systems with initial condition $x_0 = [-2.2, -3.4]^T$ and $\nu = 1.4\,10^4$. The starting point is outside the state constraint set and near to the boundary of the feasibility set. The state trajectories obtained with online optimization and NP approximation are practically superimposed, and their distance in (25) is such that $d(t, x_0) < 0.012$. The time courses of the state variables are reported in Fig. 2.(a) and 2.(b), while Fig. 2.(c) shows the courses of input variable u: note that input and state constraints are never violated with NP control law. If the achieved performances are not satisfactory, better results can be obtained, according to Theorem 3, by increasing the number ν of off-line computed values. Moreover the regulation precision can

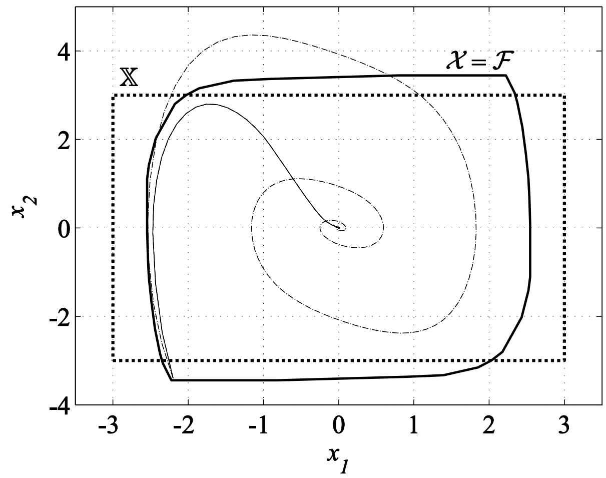

Fig. 1. Duffing oscillator example: sets $\mathcal{F}$ and $\mathcal{X}$ (thick solid line), constraint set $\mathbb{X}$ (thick dotted line) and nominal (solid), approximated (dashed) and uncontrolled (dash-dotted) state trajectories with initial condition $x_0 = [-2.2, -3.4]^T$. NP approximation carried out with $\nu = 1.4\,10^4$

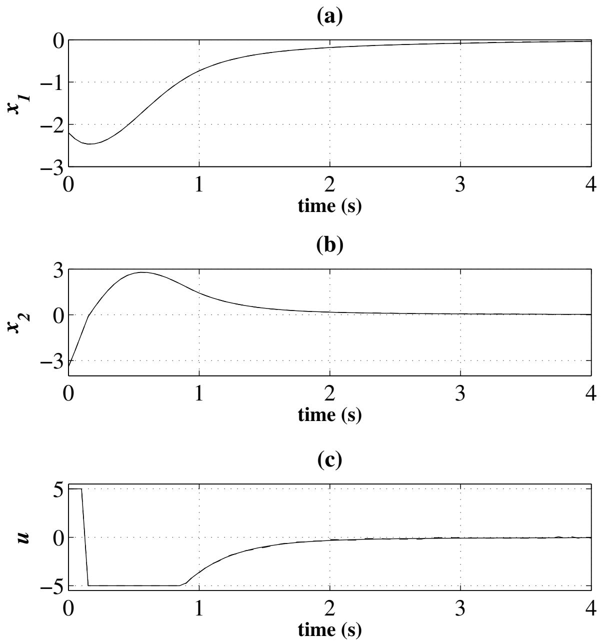

Fig. 2. Duffing oscillator: nominal (solid) and approximated (dashed) (a), (b) state courses and (c) control input with initial condition $x_0 = [-2.2, -3.4]^T$. NP approximation carried out with $\nu = 1.4\,10^4$.

be also improved by considering a more dense gridding near the origin in the computation of the off-line exact solutions. As regards the computation efforts, the mean computational time obtained with the nominal controller over more than 200 simulations considering different starting conditions is equal to 0.06 s. Table 1 shows the mean computational time $\bar{t}$ obtained with NP approximation with increasing values of ν (which means better regulation precision and lower performance degradation). A mean computation time of about 10^{-5} s is obtained independently on ν. Indeed, such computational times depend on the employed calculator: in this case MatLab® 7 and an AMD Athlon(tm) 64 3200+ with 1 GB RAM have been used. Note that the NP computational do not depend on the prediction and control horizons and in general on the nominal controller, which could be more complex (i.e. with higher required computational effort) than the one considered in this example.

Table 1. Duffing oscillator example: mean computational times of NP approximation.

	$\nu \simeq 10^6$	$\nu \simeq 10^5$	$\nu \simeq 10^4$	$\nu \simeq 10^3$
$\bar{t}$	$3\,10^{-5}$ s	$3.5\,10^{-5}$ s	$4\,10^{-5}$ s	$2.1\,10^{-5}$ s

6. CONCLUSIONS

In this paper the application of SM function approximation methodologies has been investigated in the implementation of a given predictive control law for nonlinear systems. Such methodologies rely on the off-line computation of a finite number ν of exact values of the nominal predictive control law: the obtained approximation error converges to zero as ν increases. In particular, a "Nearest Point" approach has been introduced, whose computation times are independent on ν, which can be used even in presence of quite fast plant dynamics. Conditions on the approximating function have been given in order to guarantee closed loop stability, performance and state constraint satisfaction properties. Such control computation is simply reduced to the evaluation of a static non linear function, independent on the prediction and control horizons, the computational time can be significantly reduced leading to a fast implementation of the predictive control law. This way, it may overcome the problems related to the constraints satisfaction in neural networks approximation approaches. The effectiveness of the proposed methodology has been shown by the application to nonlinear oscillator example. Note that approaches employing SM approximation techniques for the implementation of predictive controllers has also been successfully applied to practical control problems, such as semi-active suspension control and energy generation using tethered airfoils (see Canale et al. (2006) and Canale et al. (2007c)).

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