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Synchronization of Complex Dynamical Networks with Switching Topology: a Switched System Point of View ⋆

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Abstract: In this paper, we study the synchronization problem for complex dynamical networks with switching topology. The synchronization problem is transformed into the stability problem for a time-varying switched system. We address two basic problems: synchronization under arbitrary switching topology, and synchronization via design of switching within a pre-given collection of topologies when synchronization can not be achieved by using any topology alone in this collection. For the both problems, we first establish synchronization criteria for general connection topology. Then, under the condition of simultaneous triangularization of the connection matrices, a common Lyapunov function and a single Lyapunov function are systematically constructed respectively by those of several lower-dimensional dynamic systems.

1. INTRODUCTION

Complex networks exist in many fields of sciences, engineering and society, and have attracted tremendous attention in recent years (see Boccaletti et al. [2006], Strogatz [2001] and the references therein). As the major collective behavior, synchronization is one of the key issues that have been extensively addressed. A vast number of papers on the topic have appeared (Barahona et al. [2002], Comellas et al. [2007], Hill et al. [2006]).

For a network, when the connection matrix is a constant, symmetric and irreducible, synchronization criteria can be easily given in terms of checking simultaneous stability of several lower-dimensional dynamic systems (Boccaletti et al. [2006]). The classical constant connection topology is of course very restrictive and only reflects a few ideal situations. Time-varying connection topology is more realistic and covers more situations in practice. A number of synchronization criteria and methods are put forward for time-varying connection topology (see for example, Belykh et al. [2004], Boccaletti et al. [2006], Lü et al. [2005] and Stilwell et al. [2006]). In these results the time-varying connection is taken as a "slow-varying" structure, namely, the connection matrix changes continuously with time and usually an upper bound of the "change rate" is known. Often, a nominal value of the connection matrix is taken as a basis and robustness analysis is undertaken to produce synchronization conditions. Commutative topologies are assumed to give simultaneous diagonalization of connection matrices (Boccaletti et al. [2006]).

In the real world, the connection topology of a network may change very quickly–even jumps or switches might occur. Switching connection topology is often due to link failures or new creation in a network. Taking a power grid as an example, when a severe fault happens in a local power system, the transmission line connecting the local power system with the global power grid is automatically cut off by a relay protection device. This makes the connection structure jump suddenly– switch from one topology to another. Switching topology also happens when some nodes in a network are connected or disconnected purposely. Again, consider a power grid, we may cut off or connect a few local power systems for certain purpose. In this case, switching is designable. The switching signal being a designable variable is natural and important in practice. Consider a synchronizable network that has been working for some time. If one or more connection links are removed from a network permanently and thus synchronization is broken, then comes a question: can we still achieve synchronization by adjusting the remainder of links? According to certain physical requirements of the network, some links are not allowed to change. Usually, disconnecting more changeable links permanently can not achieve synchronization. In this case, synchronization could be achieved by suitably switching on and off some links.

Inherently, switching topology is discontinuously "fast-varying" topology and in general can not be handled as general time varying topology. Several methods have been put forward to deal with switching topology in the special network studies. Consensus problems are addressed in Olfati-Saber et al. [2004] by introducing a simple disagreement function for directed networks with switching topology. State consensus problems for discrete-time multiple agent systems with switching topology and delays are discussed in Xiao et al. [2006]. Flocking in switching networks is considered in Tanner et al. [2007]. Hong et al. [2007] gives a Lyapunov-based approach to a multiple

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agent system consisting of double integrators with switching connection. When switching is fast enough, an average model can be applied with certain properties of the original network preserved (Stilwell et al. [2006]). A problem of master-slaver synchronization with switching communication is addressed in Mastellone et al. [2006]. Synchronization in oscillator networks with switching topology is discussed in Papachristodoulou et al. [2005].

In this paper we study the local and global synchronization problem for complex dynamical networks with switching topology from a switched system viewpoint. Two basic problems are addressed: synchronization under arbitrary switching topology, and synchronization via design of a switching signal between pre-given topologies when for each individual topology alone the synchronization problem is not solvable. These two problems are of particular significance for the following reasons. Firstly, just as a switched system may be unstable even all its subsystems are stable, a network with switching topology may not synchronize even each individual connection topology, if put in use alone, can bring synchronization. Therefore, seeking for synchronization criteria for arbitrary switching topology is not trivial. Once synchronization under arbitrary switching topology is assured, collapse can be avoided when interconnection among nodes changes. Therefore, we are able to arbitrarily adjust the interconnection to reach other desirable network performances without worry about destroying synchronization. Secondly, when synchronization is impossible for each individual connection topology, synchronization may still be achieved via switching between these connection topologies. This certainly increases the possibility of synchronizability.

Compared with the vast existing literature on network synchronization, the results of this paper have two distinct features. First of all, a network with switching topology is regarded as a switched system and thus switched system theories and methods may be applied. To the best of our knowledge, no results have appeared using switching between different connection topologies to enhance synchronization. The second one is that unlike non-switching topology and time-varying topology where diagonalizability or simultaneous diagonalizability for different time is a basic assumption, we adopt a mild assumption: simultaneous triangularizability of the connection matrices. This appears not to have been seen in the literature so far.

2. PRELIMINARIES

We consider a dynamical network with switching topology described by:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}^{\sigma(t)}\Gamma x_j, i = 1, 2, \cdots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in R^n$ is the state variable of the i-th node, Γ is the inner-coupling matrix between two connected nodes, $f : R^n \to R^n$ is a continuously differentiable mapping, $\sigma : [0,\infty) \to M = \{1, 2, \cdots, m\}$ is a switching signal. For each $k \in M$, $A_k = (a_{ij}^k)$ is an $N \times N$ matrix representing an outer coupling configuration of the network so totally there are m such outer coupling configurations. Assume $a_{ii}^k = -\sum_{j=1,j\neq i}^{N} a_{ij}^k$. For each $k \in M$

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}^{k}\Gamma x_j, i = 1, 2, \cdots, N \tag{2}$$

is called the k-th subnetwork of the network (1), which is a usual network.

Let $s(t)$ be a solution of each isolated node, that is, $s(t)$ satisfies $\dot{s}(t) = f(s(t))$. Let $\eta_i = x_i - s$. The network (1) is said to synchronize under the switching signal $\sigma(t)$ if $\lim_{t\to\infty} \eta_i = 0, i = 1, 2, \cdots m$. We can easily have

$$\dot{\eta}_i = Df(s(t))\eta_i + c\sum_{j=1}^{N} a_{ij}^{\sigma(t)}\Gamma\eta_j + g_i(t, \eta_i), \tag{3}$$

where $Df(\cdot)$ is the Jocobian of f and

$$g_i(t, \eta_i) = \int_0^1 \left(Df(s(t) + \tau\eta_i) - Df(s(t))\right)\eta_i d\tau.$$

Denote $\eta = \left(\eta_1^T, \eta_2^T, \cdots, \eta_n^T\right)^T$, we can obtain

$$\frac{d}{dt}\eta = \left(I_N \otimes Df(s) + cA_\sigma \otimes \Gamma\right)\eta + g(t, \eta) \tag{4}$$

where I_N is the N-order identity matrix, $g(t, \eta) = (g_1(t, \eta_1)^T, \cdots, g_N(t, \eta_N)^T)^T$, and $\otimes$ is the Kronecker products of matrices.

Dropping $g(t, \eta)$ from (4) produces the linearized network dynamics:

$$\frac{d}{dt}\eta = \left(I_N \otimes Df(s) + cA_\sigma \otimes \Gamma\right)\eta. \tag{5}$$

If A_k are diagonalizable, then there exist non-singular matrices Φ_k such that

$$\Phi_k^{-1} A_k \Phi_k = \mathrm{diag}\{\lambda_1^k, \lambda_2^k, \cdots, \lambda_N^k\}, \tag{6}$$

where λ_i^k are eigenvalues of A_k. If we set

$$w^k = \left((w_1^k)^T, (w_2^k)^T, \cdots, (w_N^k)^T\right)^T = \left(\Phi_k^{-1} \otimes I_N\right)\eta,$$

then, under w^k-coordinates, the k-th subnetwork of (2) is

$$\dot{w}^k = \left(I_N \otimes Df(s(t)) + c(\Phi_k^{-1}A_k\Phi_k) \otimes \Gamma\right)w^k, \tag{7}$$

or equivalently,

$$\dot{w}_i^k = \left(Df(s(t)) + \lambda_i^k c\Gamma\right)w_i^k, i = 1, 2, \cdots, N. \tag{8}$$

Unlike networks with a single connection topology where a similar expression to (6) can be always used to test synchronizability, here (6) is only useful for testing the synchronizability of each individual subnetwork. In order to make use of (6) to study the synchronizability of the network with switching topology, (6) must be valid in the same coordinate frame, that is, $\Phi_k = \Phi, k = 1, \cdots, m$, for some Φ. In other words, A_k must be simultaneously diagonalizable. This is only possible when the matrices A_k are pairwise commutative. Obviously, this is a very strong constraint and most networks do not satisfy this. We will not adopt this constraint in the present paper. Moreover, A_k do not to be diagonalizable.

Giving up this constraint means that we have to work directly on the expression (4) or (5), which is in general very difficult. We consider a weaker constraint than simultaneous diagonizability, namely, simultaneous triangularization.

Proposition 2.1. Suppose that there exists a nonsingular matrix $\Phi = (\phi_{ij})_{N\times N}$ with $\Phi^{-1} = (\psi_{ij})_{N\times N}$ such that

$$\Phi^{-1}A_k\Phi = \begin{pmatrix} b_{11}^k, & b_{12}^k, & b_{13}^k, & \cdots & b_{1N}^k \\ 0, & b_{22}^k, & b_{23}^k, & \cdots & b_{2N}^k \\ \vdots & \vdots & \ddots & & \vdots \\ 0, & 0, & 0, & \cdots & b_{NN}^k \end{pmatrix} = \Lambda_k, \tag{9}$$

with $b_{ii}^k = \lambda_i^k$ the eigenvalues of A_k. Then, (3) can be expressed as

$$\begin{aligned} \dot{w}_1 &= \left(Df + \lambda_1^k c\Gamma\right) w_1 + b_{12}^k c\Gamma w_2 + \cdots \\ &\quad + b_{1N}^k c\Gamma w_N + \psi_{11} g_1(t,\eta_1) + \cdots \\ &\quad + \psi_{1N} g_N(t,\eta_N), \\ \dot{w}_2 &= \left(Df + \lambda_2^k c\Gamma\right) w_2 + b_{23}^k c\Gamma w_3 + \cdots \\ &\quad + b_{2N}^k c\Gamma w_N + \psi_{21} g_1(t,\eta_1) + \cdots \\ &\quad + \psi_{2N} g_N(t,\eta_N), \\ &\cdots \\ \dot{w}_{N-1} &= \left(Df + \lambda_{N-1}^k c\Gamma\right) w_{N-1} \\ &\quad + b_{(N-1)N}^k c\Gamma w_N + \psi_{(N-1)1} g_1(t,\eta_1) + \cdots \\ &\quad + \psi_{(N-1)N} g_N(t,\eta_N), \\ \dot{w}_N &= \left(Df + \lambda_N^k c\Gamma\right) w_N + \psi_{N1} g_1(t,\eta_1) + \cdots \\ &\quad + \psi_{NN} g_N(t,\eta_N). \end{aligned} \tag{10}$$

Proof. Applying the coordinates transformation $w = \left(w_1^T, w_2^T, \cdots, w_N^T\right)^T = \left(\Phi^{-1} \otimes I_N\right)\eta$ gives the results.

Remark 2.2. There are several ways to check simultaneous triangularizability. The most useful method is to test nilpotency of $\{A_k\}_{LA}$, the Lie algebra generated by the connection matrices $\{A_k\}$.

Let $PC_{n\times n}$ ($PC_{n\times n}^1$) be the linear space of the uniformly bounded continuous (continuously differentiable) real matrix-valued functions defined on $[0,\infty)$. For any $P \in PC_{n\times n}$, the norm of P is defined by $\|P\| = \max_{0\le t<\infty}\{\|P(t)\|\}$. A time-varying matrix $Q : [0,\infty) \to R^{n\times n}$ is said to be positive definite (semi-definite), denoted by $Q > 0$ ($Q \ge 0$), if there exists $\alpha > 0$ such that $v^T Q(t) v \ge \alpha\|v\|^2$ ($v^T Q(t) v \ge 0$) for any $v \in R^n$, $t \ge 0$.

3. ARBITRARY SWITCHING

In this section, we consider the case that switching between subnetworks is arbitrary. We will develop conditions under which synchronization is always maintained.

First of all, we can easily have the following condition.

Proposition 3.1. Global synchronization of (1) is achieved under arbitrary switching signal $\sigma(t)$ if there exists a $nN\times nN$ positive definite matrix $P(t) \in PC_{nN\times nN}^1$ satisfying

$$\begin{aligned} &\eta^T(\dot{P} + (I\otimes Df(s(t)) + cA_i\otimes\Gamma)^T P \\ &+P(I\otimes Df(s(t)) + cA_i\otimes\Gamma))\eta \\ &+2\eta^T P g(t,\eta) < 0, \quad \forall t\ge 0, \eta\ne 0, i = 1,2,\cdots,m. \end{aligned} \tag{11}$$

If instead,

$$\begin{aligned} &\dot{P} + (I\otimes Df(s(t)) + cA_i\otimes\Gamma)^T P \\ &+P\left(I\otimes Df(s(t)) + cA_i\otimes\Gamma\right) < 0, i = 1,2,\cdots,m \end{aligned}$$

holds, then the network (1) locally synchronizes under arbitrary switching.

In fact, $\eta^T P\eta$ is a common Lyapunov function for (4). However, to find such a P is very difficult in general. No effective approach is available for the general case. We now consider how to construct such a P from lower-dimensional dynamics in the case that all A_k are simultaneously triangularizable.

Theorem 3.2. Suppose that

(i) there exists an nonsingular constant matrix $\Phi = (\phi_{ij})_{N\times N}$ with $\Phi^{-1} = (\psi_{ij})_{N\times N}$ which makes (9) hold.

(ii) there exist positive definite matrix $P_i(t) \in PC_{n\times n}^1$, constants $\alpha_i > 0$ such that

$$\begin{aligned} &\dot{P}_i(t) + \left(Df(s(t)) + c\lambda_i^k\Gamma\right)^T P_i \\ &+P_i\left(Df(s(t)) + c\lambda_i^k\Gamma\right) \\ &+\alpha_i I < 0, 1\le i\le N, 1\le k\le m. \end{aligned} \tag{12}$$

(iii) there exist a constant $l > 0$ such that

$$\|g_i(t,\eta_i)\| \le l\|\eta_i\|. \tag{13}$$

Define: $\bar{\alpha}_i = \alpha_i - 2l\|P_i\|\sum_{k=1}^N|\psi_{ik}\phi_{ki}|$, $\delta_1 = 1$, $\bar{P}_1 = \delta_1 P_1$,

$$\mu_{1j} = 2\max_{1\le q\le m}\left|b_{1j}^q c\right|\delta_1\|P_1\Gamma\| + 2l\delta_1\|P_1\|\sum_{k=1}^N|\psi_{1k}\phi_{kj}|.$$

If we have δ_{i-1}, $\bar{P}_{i-1}$ and $\mu_{(i-1)j}$, we now define

$$\delta_i = \frac{(N-1)}{2\bar{\alpha}_i}\sum_{p<i}\frac{\mu_{pi}^2}{\bar{\alpha}_p} + 1, \bar{P}_i = \delta_i P_i$$

$$\mu_{ij} = 2\max_{1\le q\le m}\left|b_{ij}^q c\right|\delta_i\|P_i\Gamma\| + 2l\delta_i\|P_i\|\sum_{k=1}^N|\psi_{ik}\phi_{kj}|.$$

If $\bar{\alpha}_i > 0$ and $\bar{\alpha}_i - (N-1)\sum_{j>i}\frac{\mu_{ji}^2}{\bar{\alpha}_j} > 0$ for $i = 1,2,\cdots,N$, then the dynamical network (1) globally synchronizes under arbitrary switching.

Proof. Applying the coordinates transformation $w = (\Phi^{-1}\otimes I_N)\eta$ we have the expression (10). It follows from (13) that

$$\begin{aligned} &\|\psi_{i1}g_1(t,\eta_1) + \cdots + \psi_{iN}g_N(t,\eta_N)\| \\ &\le l\left(|\psi_{i1}|\|\eta_1\| + \cdots + |\psi_{iN}|\|\eta_N\|\right). \end{aligned} \tag{14}$$

Since $\eta_j = \phi_{j1}w_1 + \cdots + \phi_{jN}w_N$ we have

$$\|\eta_j\| \le |\phi_{j1}|\|w_1\| + \cdots + |\phi_{jN}|\|w_N\|.$$

Substituting this into (14) yields

$$\begin{aligned} &\|\psi_{i1}g_1(t,\eta_1) + \cdots + \psi_{iN}g_N(t,\eta_N)\| \\ &\le l\sum_{j=1}^N\left(\sum_{k=1}^N|\psi_{ik}\phi_{kj}|\right)\|w_j\|. \end{aligned} \tag{15}$$

Choose $V_1(w_1) = w_1^T\bar{P}_1 w_1$. Then, when the q-th subnetwork is connected, using (15) we have

$$
\begin{aligned}
\dot{V}_1 &= w_1^T(\dot{P}_1 + (Df + \lambda_1^q c\Gamma)^T P_1 \\
&\quad + P_1(Df + \lambda_1^q c\Gamma))w_1 + 2b_{12}^q c w_1^T P_1 \Gamma w_2 + \cdots \\
&\quad + 2b_{1N}^q c w_1^T P_1 \Gamma w_N + 2 w_1^T P_1 \sum_{p=1}^{N} \psi_{1p} g_p(t, \eta_p) \\
&\le -\alpha_1 \|w_1\|^2 + 2|b_{12}^q c| \|P_1 \Gamma\| \|w_1\| \|w_2\| + \cdots \\
&\quad + 2|b_{1N}^q c| \|P_1 \Gamma\| \|w_1\| \|w_N\| \\
&\quad + 2l\|P_1\| \sum_{j=1}^{N} \left(\sum_{k=1}^{N} |\psi_{1k}\phi_{kj}| \right) \|w_1\| \|w_j\| \\
&= -\left(\alpha_1 - 2l\|P_1\| \sum_{k=1}^{N} |\psi_{1k}\phi_{k1}| \right) \|w_1\|^2 \\
&\quad + 2\sum_{j=2}^{N} \left(|b_{1j}^q c| \|P_1 \Gamma\| + l\|P_1\| \sum_{k=1}^{N} |\psi_{1k}\phi_{kj}| \right) \\
&\quad \times \|w_1\| \|w_j\| \\
&\le -\bar{\alpha}_1 \|w_1\|^2 + \sum_{j=2}^{N} \mu_{1j} \|w_1\| \|w_j\|.
\end{aligned} \tag{16}
$$

According to Young's inequality we have

$$\|w_1\| \|w_j\| \le \frac{\bar{\alpha}_1}{2\mu_{1j}(N-1)} \|w_1\|^2 + \frac{\mu_{1j}(N-1)}{2\bar{\alpha}_1} \|w_j\|^2. \tag{17}$$

Therefore,

$$\dot{V}_1 \le -\frac{1}{2}\bar{\alpha}_1 \|w_1\|^2 + \sum_{j=2}^{N} \frac{\mu_{1j}^2 (N-1)}{2\bar{\alpha}_1} \|w_j\|^2. \tag{18}$$

In general, after having $V_{i-1}(w_{i-1})$, we define $V_i(w_i) = w_i^T P_i w_i$ and thus we have

$$
\begin{aligned}
\dot{V}_i &\le -\left(\frac{(N-1)}{2} \sum_{p<i} \frac{\mu_{pi}^2}{\bar{\alpha}_p} + \frac{\bar{\alpha}_i}{2} \right) \|w_i\|^2 \\
&\quad + \sum_{j=1, j\ne i}^{N} \frac{\mu_{ij}^2 (N-1)}{2\bar{\alpha}_i} \|w_j\|^2.
\end{aligned} \tag{19}
$$

Let $V(w) = \sum_{i=1}^{N} V_i(w_i)$. In view of $\bar{\alpha}_i - (N-1)\sum_{j>i} \frac{\mu_{ji}^2}{\bar{\alpha}_j} > 0$ and $\bar{\alpha}_i > 0$, it holds that

$$\dot{V} \le -\frac{1}{2}\sum_{i=1}^{N} \left(\bar{\alpha}_i - (N-1)\sum_{j>i} \frac{\mu_{ji}^2}{\bar{\alpha}_j} \right) \|w_i\|^2, \tag{20}$$

which completes the proof.

Corollary 3.3. If only conditions (i) and (ii) in Theorem 3.2 hold, then the network (1) locally synchronizes under arbitrary switching.

Proof. Apply Theorem 3.2 to the linearized network (5), we find that $l = 0, \bar{\alpha}_i = \alpha_i > 0$. Note that $b_{ij}^q = 0, \forall q, j > i$, it is clear that $\mu_{ji} = 0, j > i$. Then, the network (1) achieves local synchronization under arbitrary switching.

4. DESIGN OF SWITCHING

In this section, we discuss how to realize synchronization by suitable design of switching between connection topologies.

For simplicity, we only address the problem for the linearized network dynamics (5) and thus give local synchronization results. All results in this section can be easily extended to the global case if certain constraints similar to the condition (iii) in Theorem 3.2 are imposed.

Proposition 4.1. Let $P(t) \in PC_{nN\times nN}^1$ be a positive definite matrix. If the sets

$$
\begin{aligned}
\Omega_k = \{(t,\eta) | \eta^T(\dot{P} + (I \otimes Df(s(t)) + cA_k \otimes \Gamma)^T P \\
+ P(I \otimes Df(s(t)) + cA_k \otimes \Gamma))\eta < 0\}
\end{aligned}
$$

make a partition of $[0,\infty) \times R^{nN}$, i.e., $\bigcup_{k=1}^m \Omega_k = [0,\infty) \times R^{nN}$, then, synchronization of (1) is achieved under the switching law

$$\sigma = \sigma(t,\eta) = i, \text{if } (t,\eta) \in \Omega_i. \tag{21}$$

Proof. It is straightforward to show that $\eta^T P(t) \eta$ is a time-varying Lyapunov function for (5) under the switching law (21).

Proposition 4.1 gives only a general principle to check synchronizability by a single Lyapunov function. The key point is how to fund such a P. In the following, we give a convex combination based method for finding such a P.

Proposition 4.2. Let $\alpha_k \ge 0$ be constants with $\sum_{k=1}^{m} \alpha_k = 1$ and $\bar{A} = \sum_{k=1}^m \alpha_k A_k$. If there exists a positive definite matrix $P(t) \in PC_{nN\times nN}^1$ satisfying

$$
\begin{aligned}
&\dot{P} + \left(I \otimes Df(s(t)) + c\bar{A} \otimes \Gamma\right)^T P \\
&+ P\left(I \otimes Df(s(t)) + c\bar{A} \otimes \Gamma\right) < 0,
\end{aligned} \tag{22}
$$

then, synchronization of (1) is achieved under the switching law

$$
\begin{aligned}
\sigma &= \sigma(t,\eta) \\
&= \arg\min_{1\le k\le m} \{\eta^T(\dot{P} + (I \otimes Df(s(t)) + cA_k \otimes \Gamma)^T P \\
&\quad + P(I \otimes Df(s(t)) + cA_k \otimes \Gamma))\eta\}.
\end{aligned} \tag{23}
$$

Proof. It follows from Proposition 4.1

Unlike Proposition 4.1, Proposition 4.2 is implementable since we only need to solve the matrix inequality (22) for P. Further, we want to construct P from some lower-dimensional dynamics.

Theorem 4.3. Let $\bar{A} = \sum_{k=1}^m \alpha_k A_k$ with $\alpha_k \ge 0$ and $\sum_{k=1}^m \alpha_k = 1$ be triangularizable. Suppose there exist positive definite matrices $P_i(t) \in PC_{n\times n}^1$ satisfying

$$
\begin{aligned}
&\dot{P}_i(t) + (Df(s(t)) + c\lambda_i\Gamma)^T P_i \\
&+ P_i\left(Df(s(t)) + c\lambda_i\Gamma\right) < 0, 1 \le i \le N,
\end{aligned} \tag{24}
$$

where λ_i are eigenvalues of $\bar{A}$. Then, the network (1) synchronizes under some switching law.

Proof. Since $\bar{A}$ is triangularizable, there exists a nonsingular matrix Φ such that

$$\Phi^{-1}\bar{A}\Phi = \begin{pmatrix} b_{11}, & b_{12}, & b_{13}, & \cdots & b_{1N} \\ 0, & b_{22}, & b_{23}, & \cdots & b_{2N} \\ \vdots & \vdots & \ddots & & \vdots \\ 0, & 0, & 0, & \cdots & b_{NN} \end{pmatrix} = \Lambda, \tag{25}$$

with $b_{ii} = \lambda_i$, the eigenvalues of $\bar{A}$. Applying the coordinates transformation $w = \left(w_1^T, w_2^T, \cdots, w_N^T\right)^T = \left(\Phi^{-1} \otimes I_N\right) \eta$ transforms the localized convex combination network $\dot{\eta} = \left(I_N \otimes Df(s) + c\bar{A} \otimes \Gamma\right) \eta$ into

$$\begin{aligned}
\dot{w}_1 &= \left(Df + \lambda_1 c\Gamma\right) w_1 + b_{12} c\Gamma w_2 + \cdots \\
&\quad + b_{1N} c\Gamma w_N, \\
\dot{w}_2 &= \left(Df + \lambda_2 c\Gamma\right) w_2 + b_{23} c\Gamma w_3 + \cdots \\
&\quad + b_{2N} c\Gamma w_N, \\
&\cdots \\
\dot{w}_{N-1} &= \left(Df + \lambda_{N-1} c\Gamma\right) w_{N-1} + b_{(N-1)N} c\Gamma w_N, \\
\dot{w}_N &= \left(Df + \lambda_N c\Gamma\right) w_N.
\end{aligned} \tag{26}$$

Using a process similar to the proof of Theorem 3.2, we can construct $V(w) = \sum_{i=1}^{N} V_i(w_i)$ with $V_i(w_i) = w_i^T \bar{P}_i w_i$ such that $\dot{V} < 0$ for any $w \neq 0$, which is equivalent to

$$\begin{aligned}
&\dot{\bar{P}} + \left(I \otimes Df(s(t)) + c\Lambda \otimes \Gamma\right)^T \bar{P} \\
&+ \bar{P}\left(I \otimes Df(s(t)) + c\Lambda \otimes \Gamma\right) < 0, i = 1, 2, \cdots, m
\end{aligned} \tag{27}$$

with $\bar{P} = \text{diag}\{\bar{P}_1, \bar{P}_2, \cdots, \bar{P}_N\}$. Multiplying both sides of (27) by $(\Phi^T)^{-1} \otimes I_N$ and $\Phi^{-1} \otimes I_N$ respectively shows that (22) is satisfied with $P = \left((\Phi^T)^{-1} \otimes I_N\right) \bar{P} \left(\Phi^{-1} \otimes I_N\right)$.

5. EXAMPLES

In this section, we present two examples showing how to achieve synchronization under arbitrary switching and by design a switching law, respectively.

Example 5.1. Consider the dynamic network

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{2} a_{ij}^{\sigma(t)} x_j, \qquad \sigma \in \{1, 2\}, \tag{28}$$

where $x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}$, $f(x_i) = \begin{pmatrix} -10x_{i1} + \sin(x_{i2}) \\ -10x_{i2} \end{pmatrix}$, $A_1 = (a_{ij}^1)_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$ and $A_2 = (a_{ij}^2)_{2\times 2} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$.

It is easy to see that all the conditions of Theorem 3.2 are satisfied and thus synchronization under arbitrary switchings is guaranteed. Fig.1 gives the state response under the periodic switching of 0.1 second.

Example 5.2. Consider the dynamic network

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{3} a_{ij}^{\sigma} x_j, \qquad \sigma \in \{1, 2\}, \tag{29}$$

where, $x_i = (x_{i1}, x_{i2}, x_{i3})^T$,

$$f(x_i) = \begin{pmatrix} -2x_1 + x_2 + \sin(x_2) \\ -2x_2 + x_3 + \sin(x_3) \\ -2x_3 \end{pmatrix}, \; Df = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -3.5 & 1 & 2.5 \\ 0 & 2.7 & -2.7 \\ 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Obviously, neither subnetwork 1 nor subnetwork 2 synchronizes. We now design a switching law by the single Lyapunov function method to achieve synchronization.

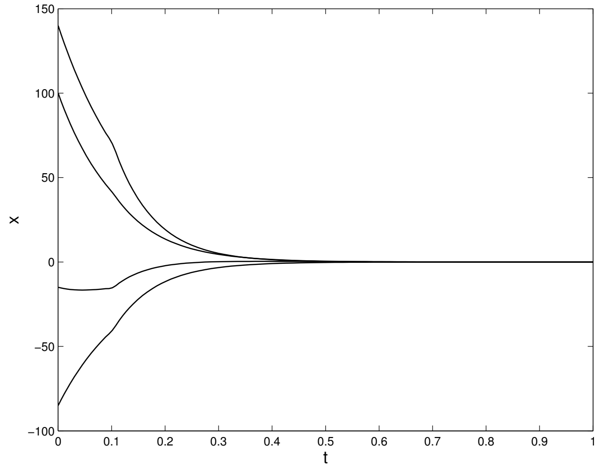

Fig. 1. State response of (28) under an arbitrary switching

Choose $\bar{A} = \frac{1}{2}A_1 + \frac{1}{2}A_2$. Applying Theorem 4.3 gives synchronization. The simulation results with

$x_0 = (0.5, -2, 1, 2, -1.5, -2.5, 2.5, -0.5, 1.5)^T$ are shown in Fig.2.-Fig.5.

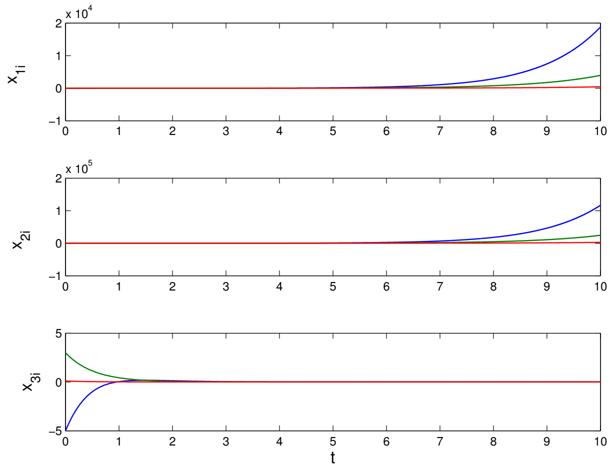

Fig. 2. Synchronization errors of the subnetwork 1 of (29).

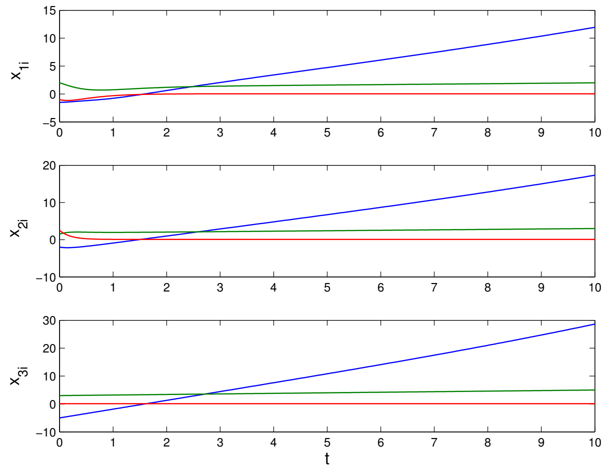

Fig. 3. Synchronization errors of the subnetwork 2 of (29).

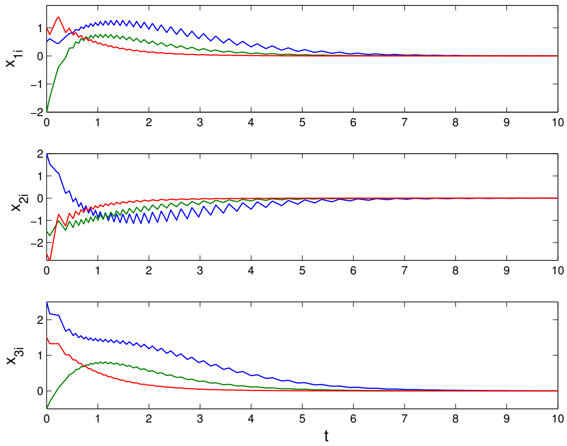

Fig. 4. Synchronization errors of the switched network (29).

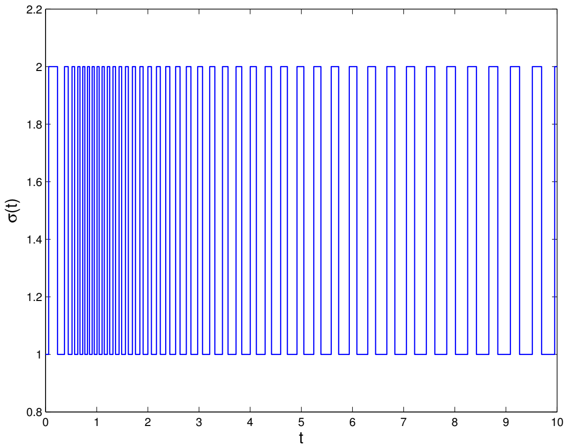

Fig. 5. Switching signal of the switched network (29).

6. CONCLUSIONS

We have established several synchronization criteria and design methods for complex dynamical networks with switching topology. For the case of arbitrary switching topology, synchronization is always preserved under the proposed conditions. When we are given a family of "poor topologies" with each of which the synchronization is impossible, switching in this family of topologies may achieve synchronization. This substantially increases the possibility of synchronizability.

For topological structure of the connection matrices, diagonalizability or even symmetry are no longer assumed. Instead, simultaneous triangularizability is assumed.

REFERENCES

M.Barahona, L.M.Pecora. Synchronization in small-world systems. *Phys Rev Lett*, 89:054101, 2002.

I.Belykh, V.N.Belykh, M.Hasler. Blinking model and synchronization in small-world networks with a time-varying coupling. *Physica D*, 195:188-206, 2004.

S.Boccaletti, V.Latora, Y.Moreno, M.Chavez. Complex networks: structure and dynamics. *Physics Reports*, 424: 175-308, 2006.

S.Boccaletti, D.U.Hwang, M.Chavez, A.Amann, J.Kurths, L.M.Pecora. Synchronization in dynamical networks: evolution along commutative graphs. *Physical Review E*, 74:016102, 2006.

F.Comellas, S.Gago. Synchronization of complex networks. *J.Phys.A*, 40:4483-4492, 2007.

H.Gao, J.Lam, G.Chen. New criteria for synchronization stability of general complex dynamical networks with coupling delays. *Phys Lett. A*, 360:263-273, 2006.

D.J.Hill, G.Chen. Power systems as dynamic networks. *Proc. IEEE International Symposium on Circuits and Systems*, pages: 154-168, 2006.

Y.Hong, L.Gao, D.Cheng, J.Hu. Lyapunov-based approach to multiagent systems with switching jointly connected interconnection. *IEEE Trans. on Automatic Control*, 52:943-948, 2007.

I.F.Ihle, M. Arcakc, T. I. Fossen. Passivity-based designs for synchronized path-following. *Automatica*, 43: (Available on line), 2007.

A.Jadbabaie, J.Lin, A.S.Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automatic Control*, 48: 988-1001, 2003.

X.Li, G.Chen. Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint. *IEEE Trans. Circuit and Systems-I*, 50:1381-1390, 2003.

C.Lorand, P.H.Bauer. On Synchronization Errors in Networked Feedback Systems. *IEEE Trans. Circuits and Systems-I*, 53:2306 - 2317, 2006.

J.Lü, G.Chen. A time-varying complex dynamical network model and its controlled synchronization criteria. *IEEE Trans Automatic control*, 50:841-846, 2005.

S.Mastellone, D.Lee, M.W.Spong. Master-slave synchronization with switching communication through passive model-based control design. *Proc.American Control Conference*, pages: 3203-3208, 2006.

R.Olfati-Saber, R.M.Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Automatic Control*, 49:1520-1533, 2004.

A.Papachristodoulou, A.Jadbabaie. Synchronization in oscillator networks: switching topologies and non-homogeneous delays. *Proc. IEEE Conf. Decision and Control*, pages: 5692-5697, 2005.

L.M.Pecora, T.L.Carroll. Master stability functions for synchronized coupled systems. *Phys Rev Lett*, 80:2109-2112, 1998.

D.J.Stilwell, E.M.Bollt. Sufficient conditions for fast switching synchronization in time-varying network topologies. *SIAM J.Applied Dynamical Systems*, 5:140-156, 2006.

S.H.Strogatz. Exploring complex networks. *Nature*, 410: 268-276, 2001.

H.G.Tanner, A.Jadbabaie, G.J.Pappas. Flocking in fixed and switching networks. *IEEE Trans. Automatic Control*, 52:863-868, 2007.

X.Wang, G.Chen. Complex networks: small-world, scale-free, and beyond. *IEEE Trans Circuits and Systems*, 3: 6-20, 2003.

J.Xiang, G.Chen. On the V-stability of complex dynamical networks. *Automatica*, 43:1049-1057, 2007.

F.Xiao, L.Wang. State consensus for multi-agent systems with switching topologies and time-varying delays. *Int.J.Control*, 79:1277-1284, 2006.

J.Zhao, D.J.Hill. On stability, L_2-gain and H_∞ control for switched systems. *Automatica*, to appear, 2007.