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A Delay-Partitioning Projection Approach to Stability Analysis of Neutral Systems ⋆

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Abstract: This paper introduces a new effective approach to study the stability of neutral systems. By employing a special Lyapunov-Krasovskii functional form based on delay partitioning, delay-dependent stability criteria are established for the nominal and the uncertain case (polytopic type) in terms of linear matrix inequalities (LMI). Numerical examples are employed to illustrate that the delay-partitioning projection approach can be applied to estimate the upper bounds for the delays for the system to maintain stability. Judging from these numerical results, the stability criteria obtained are less conservative than those of existing methods.

Keywords: Delay system; delay partitioning; neutral system; stability; uncertainty.

1. INTRODUCTION

Time delays are often attributed as the major sources of instability in various engineering systems. Consequently, a vast amount of effort has recently been devoted to deriving stability criteria for delay systems. For many physical examples in practice, their system models can be described by functional differential equations of the neutral type, in which the models depend not only on the state delay but also on the state derivatives (Chen and Zheng [2007] and Hu and Liu [2007]). A number of methods aiming at reducing the conservatism of these stability criteria (that is, less conservative upper bound of the delay for the system to remain stable) have been proposed. One approach is to take an appropriate and equivalent model transformation for the original systems. Fridman and Shaked [2003] summarized four main model transformations and showed different sources for conservatism under different model transformations. Under such transformations, the conservatism is mainly due to the bounding of the cross product terms which appear in the derivative of the Lyapunov-Krasovskii functional. To reduce the number of such terms and employ tighter bounds on them would certainly lead to better results. Therefore, Park proposed a new upper bound of a vector cross-product with cross- as well as inner-products in Park [1999]. Moon et al. [2001] provided another inequality by reducing the limitation in Park [1999] so that certain matrix variables are no longer required to satisfy a specific structure. Recently, a free-weighting matrix method was proposed in Wu et al. [2004b] and Xu et al. [2005] to investigate the delay-dependent stability, in which the bounding techniques on some cross product terms are not involved. This treatment produces better results, which is often associated with an increase in variables. Another approach is the construction of new Lyapunov-Krasovskii functionals with a proper distribution of the time delay (see Kolmanovskii and Richard [1999]). Gu et al. [2003] introduced LMI stability conditions via complete and discretized Lyapunov-Krasovskii functional which leads to results close to analytical ones in some examples. In addition, a Lyapunov-Krasovskii functional augmented with the delayed state were used by He et al. [2005] and Parlakçi [2007] to derive improved the delay dependent stability criteria for neutral systems.

Robust stability of neutral systems with mixed delays and time-varying structured uncertainties have been considered by He et al. [2004]. Based on a descriptor model transformation and the decomposition of a discrete delay term matrix, Han [2004] has addressed the problem of the robust stability of neutral systems with nonlinear parameter perturbations. In this paper, the nominal system and the uncertain system with polytopic-type uncertainties will be considered for stability investigation. We provide an improvement as well as a generalization of the results in Lam et al. [2007] to the case with a constant delay. Through the use of a new Lyapunov-Krasovskii functional form based on the idea of 'delay partitioning,' the results obtained have turned out to be less conservative than existing methods.

The paper is organized as follows. In Section 2, we derive a general stability result based on delay partitioning and projection for neutral systems. The nominal and the polytopic uncertain cases are considered with different discrete delays in the state and its derivative. A result is also provided when these delays are equal. In Section 3, two detailed numerical examples are used to illustrate that the proposed approach improves existing methods and gives better (less conservative) upper bounds on the delay for stability than those reported earlier.

Notation: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). 0 is a null matrix with an

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appropriate dimension. The superscript "T" represents the transpose of the matrix. col$\{\cdot\}$ denotes a matrix column with blocks given by the matrices in $\{\cdot\}$. A block diagonal matrix with diagonal blocks $A_1, A_2, \ldots, A_r$ will be denoted by diag$\{A_1, A_2, \ldots, A_r\}$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations. For a given real matrix B, the orthogonal complement $B^{\perp}$ (possibly non-unique), if exists, is defined as the matrix with maximum column rank that satisfies $BB^{\perp} = 0$ and $B^{\perp T}B^{\perp} > 0$.

2. PROBLEM FORMULATION

Consider the following linear neutral system with constant delays,

$$\Sigma: \quad \dot{x}(t) - C\dot{x}(t-g) = Ax(t) + A_d x(t-h)$$
$$x(t) = \phi(t), \quad t \in [-\beta, 0], \quad \beta = \max\{g, h\}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\phi(t)$ is the initial function, $h > 0$ and $g > 0$ are constant delays in the state and its derivative, respectively (h and g will be referred to as the *retarded* delay and *neutral* delay, respectively).

By employing a delay-partitioning approach, we aim at extending the methodology used in Lam et al. [2007] for retarded systems with two delay components via a new form of Lyapunov-Krasovskii functionals to obtain less conservative results. In order to estimate an upper bound of the discrete delay for stability, we partition h into several components, that is, $h = \sum_{i=1}^{r} h_i$ where r is a positive integer. To facilitate development, define $\sigma_j = \sum_{i=1}^{j} h_i$ with $\sigma_0 = 0$ in the boundary expression of the summation. Therefore, h_i $(i = 1, \ldots, r)$ represent a partition of the lumped time-invariant delay σ_r.The delay-dependent stability conditions to be obtained are based on the following lemmas.

Lemma 1. Let $Y \in \mathbb{R}^{n\times n}$ and the bi-diagonal upper triangular block matrix

$$J_k(Y) \triangleq \begin{pmatrix} I_n & -Y & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -Y \\ 0 & & & I_n \end{pmatrix} \in \mathbb{R}^{kn\times kn}$$

If $Z = (\, J_k(Y) \;\; S \,) \in \mathbb{R}^{kn\times(kn+m)}$ where $S = \begin{pmatrix} S_1 \\ \vdots \\ S_k \end{pmatrix} \in \mathbb{R}^{kn\times m}$ with $S_i \in \mathbb{R}^{n\times m}$ $(i = 1, \ldots, k)$, then

$$Z^{\perp} = \mathrm{col}\left\{ -\sum_{i=1}^{k} Y^{i-1}S_i, -\sum_{i=2}^{k} Y^{i-2}S_i, \ldots, -S_k, I_m \right\}.$$

Proof. First note that when W is invertible, we have $(\, W \;\; S \,)^{\perp} = \begin{pmatrix} -W^{-1}S \\ I_m \end{pmatrix}$. The result given in the lemma follows from the fact that

$$J_k^{-1}(Y) = \begin{pmatrix} I_n & Y & \cdots & Y^{k-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Y \\ 0 & \cdots & 0 & I_n \end{pmatrix}.$$

Lemma 2. (Finsler's Lemma). Consider real matrices B and M such that B has full row rank and $M = M^T$. Then the following statements are equivalent:

(1) There exists a scalar ℓ such that

$$\ell B^T B - M > 0$$

(2) The following condition holds:

$$B^{\perp T} M B^{\perp} < 0$$

To guarantee robustness of the results with respect to small changes of delay, we assume throughout this paper that the difference equation $x(t) - Cx(t-g) = 0$ is asymptotically stable for all values of g or that system matrix C is a Schur-Cohn stable matrix, that is, all the eigenvalues of C are inside the unit circle.

3. MAIN RESULTS

In the sequel, we will establish general stability results by the delay-partitioning projection approach for neutral systems for the nominal system as well as the uncertain (polytopic) system.

3.1 Nominal Systems

Theorem 3. Neutral system Σ is asymptotically stable if there exist matrices $P > 0$, $R > 0$, $X > 0$, and $\begin{pmatrix} Q_i & M_i \\ * & N_i \end{pmatrix} > 0 \, (i = 1, \ldots, r)$ satisfying

$$B^{\perp T} \begin{pmatrix} \Omega_1 + \Omega_2 & \Omega_3 & 0 \\ \Omega_3^T & \Omega_4 & 0 \\ 0 & 0 & \Omega_5 \end{pmatrix} B^{\perp} < 0 \qquad (1)$$

where $B^{\perp} \in \mathbb{R}^{(3r+2)n\times(2r+2)n}$ is the orthogonal complement of $B = (\, J_r(I_n) \;\; S \,)$,

$$S = \begin{pmatrix} 0 & 0 \ldots 0 & -I_n & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ -I_n & 0 \ldots 0 & 0 & & & -I_n \end{pmatrix} \in \mathbb{R}^{rn\times(2r+2)n}$$

and

$$\Omega_1 = \begin{pmatrix} A^T(P+M_1) + (P+M_1)A + Q_1 & 0 & \ldots & 0 & M_1A_d + PA_d \\ * & Q_2 - Q_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \ldots & Q_r - Q_{r-1} & 0 \\ * & * & \ldots & * & -Q_r \end{pmatrix}$$

$$\Omega_2 = (\, A \;\; 0 \;\ldots\; 0 \;\; A_d \,)^T (\sigma_r X + R + N_1) (\, A \;\; 0 \;\ldots\; 0 \;\; A_d \,)$$

$$\Omega_3 = \begin{pmatrix} 0 & \ldots & 0 & 0 & A^T(\sigma_r X + R + N_1)C + PC + M_1C \\ M_2 - M_1 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & M_r - M_{r-1} & 0 & 0 \\ 0 & \ldots & 0 & -M_r & A_d^T(\sigma_r X + R + N_1)C \end{pmatrix}$$

$$\Omega_4 = \begin{pmatrix} N_2 - N_1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \dots & N_r - N_{r-1} & 0 & \vdots \\ * & \dots & * & -N_r & 0 \\ * & \dots & * & * & \begin{matrix} C^T(\sigma_r X + R \\ + N_1)C - R \end{matrix} \end{pmatrix}$$

$$\Omega_5 = \mathrm{diag}\left\{-h_1^{-1}X, \dots, -h_r^{-1}X\right\}$$

Proof. Construct the Lyapunov-Krasovskii functional $V = V_1 + V_2 + V_3 + V_4$ where

$$V_1 = x^T(t)Px(t)$$

$$V_2 = \int_{t-g}^{t} \dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}s$$

$$V_3 = \int_{-\sigma_r}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)X\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$

$$V_4 = \sum_{i=1}^{r}\int_{t-\sigma_i}^{t-\sigma_{i-1}} \begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix}^T \begin{pmatrix} Q_i & M_i \\ * & N_i \end{pmatrix} \begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix} \mathrm{d}s$$

Now, consider the derivative of V along the solution of system Σ with respect to t, we require:

$$\begin{aligned} \dot{V}_1 &= 2x^T(t)P\left(C\dot{x}(t-g) + Ax(t) + A_d x(t-h)\right) \\ &= x^T(PA + A^TP)x(t) + 2x^T(t)PC\dot{x}(t-g) \\ &\quad + 2x^T(t)PA_d x(t-\sigma_r) \end{aligned} \quad (2)$$

$$\dot{V}_2 = \dot{x}^T(t)R\dot{x}(t) - \dot{x}^T(t-g)R\dot{x}(t-g) \quad (3)$$

$$\begin{aligned} \dot{V}_3 &= \int_{-\sigma_r}^{0} \left[\dot{x}^T(t)X\dot{x}(t) - \dot{x}^T(t+\theta)X\dot{x}(t+\theta)\right]\mathrm{d}\theta \\ &= \sigma_r\dot{x}^T(t)X\dot{x}(t) - \int_{t-\sigma_r}^{t} \dot{x}^T(s)X\dot{x}(s)\,\mathrm{d}s \\ &= \sigma_r\dot{x}^T(t)X\dot{x}(t) - \sum_{i=1}^{r}\int_{t-\sigma_i}^{t-\sigma_{i-1}} \dot{x}^T(s)X\dot{x}(s)\,\mathrm{d}s \\ &\le \sigma_r\dot{x}^T(t)X\dot{x}(t) - \sum_{i=1}^{r}\left[h_i^{-1}\left(\int_{t-\sigma_i}^{t-\sigma_{i-1}} \dot{x}(s)\,\mathrm{d}s\right)^T \right. \\ &\quad \left. \cdot X\left(\int_{t-\sigma}^{t-\sigma_{i-1}} \dot{x}(s)\,\mathrm{d}s\right)\right] \end{aligned} \quad (4)$$

and also

$$\begin{aligned} \dot{V}_4 &= \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}^T \begin{pmatrix} Q_1 & M_1 \\ * & N_1 \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} - \sum_{j=1}^{r-1}\left(\begin{pmatrix} x(t-\sigma_j) \\ \dot{x}(t-\sigma_j) \end{pmatrix}^T \right. \\ &\quad \left. \cdot \begin{pmatrix} Q_j - Q_{j+1} & M_j - M_{j+1} \\ * & N_j - N_{j+1} \end{pmatrix} \begin{pmatrix} x(t-\sigma_j) \\ \dot{x}(t-\sigma_j) \end{pmatrix}\right) \\ &\quad - \begin{pmatrix} x(t-\sigma_r) \\ \dot{x}(t-\sigma_r) \end{pmatrix}^T \begin{pmatrix} Q_r & M_r \\ * & N_r \end{pmatrix} \begin{pmatrix} x(t-\sigma_r) \\ \dot{x}(t-\sigma_r) \end{pmatrix} \end{aligned} \quad (5)$$

Note that, in the above derivations, Jensen's integral inequality has been used. Combining (2)-(5), it follows that

$$\dot{V} \le \xi^T(t) \begin{pmatrix} \Omega_1 + \Omega_2 & \Omega_3 & 0 \\ \Omega_3^T & \Omega_4 & 0 \\ 0 & 0 & \Omega_5 \end{pmatrix} \xi(t)$$

where

$$\begin{aligned} \zeta(t) &= \mathrm{col}\{\, x(t-\sigma_1) \ \dots \ x(t-\sigma_r) \,\} \\ \xi(t) &= \mathrm{col}\{\, x(t) \ \zeta(t) \ \dot{\zeta}(t) \ \dot{x}(t-g) \\ &\quad \int_{t-\sigma_1}^{t} \dot{x}(s)\,\mathrm{d}s \ \dots \int_{t-\sigma_r}^{t-\sigma_{r-1}} \dot{x}(s)\,\mathrm{d}s\} \end{aligned}$$

and Ω_i $(i = 1, \dots, 5)$ are given in (1). By the Newton-Leibniz formula, we have

$$x(t) - x(t-\sigma_1) - \int_{t-\sigma_1}^{t} \dot{x}(s)\,\mathrm{d}s = 0$$

$$\vdots$$

$$x(t-\sigma_{r-1}) - x(t-\sigma_r) - \int_{t-\sigma_r}^{t-\sigma_{r-1}} \dot{x}(s)\,\mathrm{d}s = 0$$

That is,

$$B\xi(t) = (\, J_r(I_n) \ \ S \,)\,\xi(t) = 0$$

The full column rank matrix representation of the right orthogonal complement of $B \in \mathbb{R}^{rn \times (3r+2)n}$ is denoted by $B^{\perp}$, and a computation method is offered in Lemma 1. By Lemma 2, $\dot{V}$ is negative as long as

$$\xi^T(t)\left(B^TB - \begin{pmatrix} \Omega_1 + \Omega_2 & \Omega_3 & 0 \\ \Omega_3^T & \Omega_4 & 0 \\ 0 & 0 & \Omega_5 \end{pmatrix}\right)\xi(t) > 0 \quad (6)$$

holds, which is equivalent to inequality (1). This implies that system Σ is asymptotically stable. Hence, the proof completes.

Remark 4. The augmented Lyapunov functional introduced in V_4 is applicable for neutral systems with $C \neq 0$. For the retarded type (that is, $C = 0$), the augmented state vector $(x^T(s), \dot{x}^T(s))$ contains redundant information such that M_i and N_i $(i = 1, \dots, r)$ may be omitted. By removing R, M_i, and N_i from V_2 and V_4, respectively, together with the removal of $\dot{\zeta}(t)$ and $\dot{x}(t-g)$ from $\xi(t)$, a delay-dependent stability criterion corresponding to retarded systems will result.

3.2 Uncertain Systems

In what follows, polytopic-type uncertainties are considered in system Σ, that is, the system matrices satisfy the real convex polytopic constraint

$$\begin{aligned} &(\, A \ A_d \ C \,) \in \Omega, \quad \Omega \triangleq \Big\{ (\, A(\alpha) \ A_d(\alpha) \ C(\alpha) \,) \\ &= \sum_{j=1}^{p} \alpha_j \,(\, A_j \ A_{dj} \ C_j \,), \sum_{j=1}^{p} \alpha_j = 1, \alpha_j \ge 0 \Big\} \end{aligned} \quad (7)$$

where A_j, A_{dj} and C_j $(j = 1, \ldots, p)$ are constant matrices with appropriate dimensions and α_j $(j = 1, \ldots, p)$ are time-invariant uncertainties.

For system Σ with the polytopic uncertain domain defined in (7), we consider an alternative equivalent version of Theorem 3 for the subsequent use with parameter-dependent Lyapunov functional. The idea is to have a LMI stability characterization with no product terms involving the Lyapunov matrices and the system matrices. To achieve this, we choose

$$\bar{\xi}(t) = \mathrm{col}\{\, x(t)\ \zeta(t)\ \dot{x}(t)\ \dot{\zeta}(t)\ \dot{x}(t-g)\ \int_{t-\sigma_1}^{t} \dot{x}(s)\,\mathrm{d}s \ldots \int_{t-\sigma_r}^{t-\sigma_{r-1}} \dot{x}(s)\,\mathrm{d}s\}$$

and insert the following identity

$$2\left(x^T(t)W_1 + \dot{x}^T(t)W_2\right) \cdot (\dot{x}(t) - C\dot{x}(t-g) - Ax(t) - A_d x(t-\sigma_r)) = 0 \quad (8)$$

into the derivative of the Lyapunov functional V defined in the proof of Theorem 3, the slack parameter-*independent* matrices W_1 and W_2. Therefore, a corresponding condition of inequality (6) is given by

$$\bar{\xi}^T(t)\left(\bar{B}^T\bar{B} - \begin{pmatrix} \bar{\Omega}_1 & \bar{\Omega}_2 & 0 \\ \bar{\Omega}_2^T & \bar{\Omega}_3 & 0 \\ 0 & 0 & \Omega_5 \end{pmatrix}\right)\bar{\xi}(t) > 0 \quad (9)$$

where

$$\bar{B} = \begin{pmatrix} J_r(I_n) & \bar{S} \end{pmatrix} \in \mathbb{R}^{rn\times(3r+3)n}$$

$$\bar{S} = \begin{pmatrix} 0 & 0 \ldots 0 & -I_n & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ -I_n & 0 \ldots 0 & 0 & & & -I_n \end{pmatrix} \in \mathbb{R}^{rn\times(2r+3)n}$$

$$\bar{\Omega}_1 = \begin{pmatrix} Q_1 - W_1A - A^TW_1^T & 0 & \ldots & 0 & -W_1A_d \\ * & Q_2 - Q_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \ldots & Q_r - Q_{r-1} & 0 \\ * & * & \ldots & * & -Q_r \end{pmatrix}$$

$$\bar{\Omega}_2 = \begin{pmatrix} P - A^TW_2^T + W_1 + M_1 & 0 & \ldots & 0 & 0 & -W_1C \\ 0 & M_2 - M_1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & M_r - M_{r-1} & 0 & \vdots \\ -A_d^TW_2^T & 0 & \ldots & * & -M_r & 0 \end{pmatrix}$$

$$\bar{\Omega}_3 = \begin{pmatrix} R + N_1 + W_2 + W_2^T + (\sigma_r)X & 0 & \ldots & 0 & 0 & -W_2C \\ * & N_2 - N_1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \ldots & N_r - N_{r-1} & 0 & \vdots \\ * & * & \ldots & * & -N_r & 0 \\ * & * & \ldots & * & * & -R \end{pmatrix}$$

Under such a treatment, the stability result to be derived contains no product term of the Lyapunov matrices and the system matrices which is suitable for determining the stability of neutral system with polytopic-type uncertainty. An LMI-based delay-dependent robust stability condition for uncertain neutral system Σ is given by the following theorem.

Theorem 5. Neutral system Σ with polytopic-type uncertainties (7) is robustly stable if there exist matrices $P_j > 0$, $R_j > 0$, $X_j > 0$, $\begin{pmatrix} Q_{ij} & M_{ij} \\ * & N_{ij} \end{pmatrix} > 0\,(i = 1, \ldots, r)$ and W_1, W_2 such that

$$\bar{B}^{\perp T}\begin{pmatrix} \bar{\Omega}_{1j} & \bar{\Omega}_{2j} & 0 \\ \bar{\Omega}_{2j}^T & \bar{\Omega}_{3j} & 0 \\ 0 & 0 & \Omega_{5j} \end{pmatrix}\bar{B}^{\perp} < 0 \quad (10)$$

holds for $j = 1, \ldots, p$, where $\bar{B}^{\perp} \in \mathbb{R}^{(3r+3)n\times(2r+3)n}$ is the right orthogonal complement of $\bar{B} = \begin{pmatrix} J_r(I_n) & \bar{S} \end{pmatrix}$ in (9).

$$\bar{\Omega}_{1j} = \begin{pmatrix} Q_{1j} - W_1A_j - A_j^TW_1^T & 0 & \ldots & 0 & -W_1A_{dj} \\ * & Q_{2j} - Q_{1j} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \ldots & Q_{rj} - Q_{(r-1)j} & 0 \\ * & * & \ldots & * & -Q_{rj} \end{pmatrix}$$

$$\bar{\Omega}_{2j} = \begin{pmatrix} P_j + W_1 - A_j^TW_2^T + M_{1j} & 0 & \ldots & 0 & 0 & -W_1C_j \\ 0 & M_{2j} - M_{1j} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & M_{rj} - M_{(r-1)j} & 0 & \vdots \\ -A_{dj}^TW_2^T & 0 & \ldots & * & -M_{rj} & 0 \end{pmatrix}$$

$$\bar{\Omega}_{3j} = \begin{pmatrix} W_2 + W_2^T + R_j + N_{1j} + (\sigma_r)X_j & 0 & \ldots & 0 & 0 & -W_2C_j \\ * & N_{2j} - N_{1j} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \ldots & N_{rj} - N_{(r-1)j} & 0 & \vdots \\ * & * & \ldots & * & -N_{rj} & 0 \\ & & & & & -R_j \end{pmatrix}$$

$$\Omega_{5j} = \mathrm{diag}\left\{-h_1^{-1}X_j, \ldots, -h_r^{-1}X_j\right\}$$

Proof. Construct a Lyapunov-Krasovskii functional candidate $\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3 + \bar{V}_4$ with

$$\bar{V}_1 = \sum_{j=1}^{p} x^T(t)\alpha_j P_j x(t)$$

$$\bar{V}_2 = \sum_{j=1}^{p}\int_{t-g}^{t} \dot{x}^T(s)\alpha_j R_j \dot{x}(s)\,\mathrm{d}s$$

$$\bar{V}_3 = \sum_{j=1}^{p}\int_{-\sigma_r}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\alpha_j X\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$

$$\bar{V}_4 = \sum_{j=1}^{p}\sum_{j=1}^{r}\int_{t-\sigma_j}^{t-\sigma_{j-1}} \begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix}^T \begin{pmatrix} Q_i & M_i \\ * & N_i \end{pmatrix}\begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix}\mathrm{d}s$$

Proceeding as in the proof of Theorem 1, a sufficient condition for $\dot{\bar{V}} < 0$ along the solutions of uncertain neutral system Σ is given by

$$\bar{\xi}^T(t)\left(\bar{B}^T\bar{B} - \begin{pmatrix} \bar{\Omega}_{1j} & \bar{\Omega}_{2j} & 0 \\ \bar{\Omega}_{2j}^T & \bar{\Omega}_{3j} & 0 \\ 0 & 0 & \Omega_{5j} \end{pmatrix}\right)\bar{\xi}(t) > 0 \quad (11)$$

with the same free weighting matrices W_1, W_2 in (8). For each inequality in (11), the corresponding orthogonal complement matrix $\bar{B}^{\perp}$ does not change due to the same Newton-Leibniz formula. The proof can then be established by following a similar line of arguments as that in Theorem 3.

It should be pointed out that the results in Theorems 3 and 5 are neutral-delay-independent and retarded-delay-dependent. When the effect of the neutral delay is involved in the analysis, a less conservative stability criterion can be obtained. Instead, we study the stability of uncertain neutral system Σ with the identical neutral delay and retarded delay(that is, $g = h$). By observing the structure of the Lyapunov-Krasovskii functional used in the proof of Theorem 5, it is obvious that the stability condition for uncertain systems can be easily obtained by employing $\bar{V} = \bar{V}_1 + \bar{V}_3 + \bar{V}_4$ and omitting the term $\dot{x}(t-g)$ from $\bar{\xi}(t)$.

Corollary 6. When $g = h$, neutral system Σ with polytopic-type uncertainties (7) is robustly stable if there exist matrices $P_j > 0$, $X_j > 0$, $\begin{pmatrix} Q_{ij} & M_{ij} \\ * & N_{ij} \end{pmatrix} > 0$, $(i = 1, \ldots, r)$, and W_1, W_2 such that

$$B^{\perp T} \begin{pmatrix} \bar{\Omega}_{1j} & \bar{\Omega}_{2j} & 0 \\ \bar{\Omega}_{2j}^T & \bar{\Omega}_{3j} & 0 \\ 0 & 0 & \Omega_{5j} \end{pmatrix} B^{\perp} < 0 \tag{12}$$

holds for $j = 1, \ldots, p$, where $B^{\perp} \in \mathbb{R}^{(3r+2)n \times (2r+2)n}$ is the orthogonal complement of $B = (\, J_r(I_n) \;\; S \,)$ with the same matrix S in Theorem 3,

$$\bar{\Omega}_{2j} = \begin{pmatrix} \begin{matrix} P_j - A_j^T W_2^T \\ + W_1 + M_{1j} \end{matrix} & 0 & \ldots & 0 & -W_1 C_j \\ 0 & M_{2j} - M_{1j} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & M_{rj} - M_{(r-1)j} & 0 \\ -A_{dj}^T W_2^T & 0 & \ldots & * & -M_{rj} \end{pmatrix}$$

$$\bar{\Omega}_{3j} = \begin{pmatrix} \begin{matrix} (\sigma_r)\, X_j + N_{1j} \\ + W_2 + W_2^T \end{matrix} & 0 & \ldots & 0 & -W_1 C_j \\ * & N_{2j} - N_{1j} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \ldots & N_{rj} - N_{(r-1)j} & 0 \\ * & * & \ldots & * & -N_{rj} \end{pmatrix}$$

and $\bar{\Omega}_{1j}$, Ω_{5j} are defined as in Theorem 5.

Remark 7. For retarded delay components with equal width $(h_1 = h_2 = \cdots = h_r)$, their influence on the maximal delay bound for stability is identical and indistinguishable from the effects due to the retarded delay. Consequently, by maximizing the delay bound of each component, we can compute an overall stability bound on the effective time delay. For the retarded delay h in system Σ partitioned into r identical delay components, that is $h = rh_1$, nominal system Σ is asymptotically stable if there exist matrices $P > 0$, $R > 0$, $X > 0$, and $\begin{pmatrix} Q_i & M_i \\ * & N_i \end{pmatrix} > 0$, $(i = 1, \ldots, r)$ satisfying inequality (1) where all h_i $(i = 1, \ldots, r)$ are replaced by h_1. Similarly, the robust stability condition of system Σ with polytopic-type uncertainties can also be obtained by the above partitioning procedure from Theorem 5.

4. NUMERICAL EXAMPLES

In this section, two examples are provided to demonstrate that stability conditions proposed in this paper are less conservative.

Example 1 Consider the linear neutral system Σ with the following polytopic system matrices

$$A = \begin{pmatrix} 0 & -0.12 + 12\rho \\ 1 & -0.465 - \rho \end{pmatrix},$$
$$A_d = \begin{pmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix},$$

where $|\rho| \leq 0.035$. If we let $\rho_m = 0.035$ and set

$$A_1 = \begin{pmatrix} 0 & -0.12 + 12\rho_m \\ 1 & -0.465 - \rho_m \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -0.12 - 12\rho_m \\ 1 & -0.465 + \rho_m \end{pmatrix},$$
$$A_{d1} = A_{d2} = A_d = \begin{pmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{pmatrix},$$

system Σ has the polytopic-type uncertainties described by (7). According to the method mentioned in Theorem 5 and Corollary 6 with $r = 10$, the upper bounds on the time delay obtained to the system are listed in Table 1 for different two cases $c = 0$ and $c = 0.05$. For the retarded system $(C = 0)$, the upper bounds of delay given in Fridman and Shaked [2003] and Suplin et al. [2006] are 0.782 and 0.863, respectively. When $c = 0.05$, a maximum value of h in Suplin et al. [2006] with $g = h$ is found to be 0.462 which is less than those given by Theorem 5 (0.4637) and Corollary 6 (0.6176) of this paper.

Table 1. Comparison on upper bound of delay

c	0	0.05
Fridman and Shaked [2003]	0.782	/
Suplin et al. [2006]	0.863	0.462
Theorem 5	0.8682	0.4637
Corollary 6	0.8682	0.6176

Example 2 Consider the uncertain neutral system Σ with

$$A = \begin{pmatrix} -2.0 + \delta & 0 \\ 0 & -0.9 + \delta \end{pmatrix},$$
$$A_d = \begin{pmatrix} -1.0 + \delta & 0 \\ -1.0 & -1.0 + \delta \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

where $|\delta| \leq a$. The retarded delay h is considered to be partitioned into a number of identical delay components based on the idea in remark 7. By maximizing the width of the delay components, the overall stability bound on delay is the sum of each part. Two cases are taken into account in this example.

Case 1: $a \equiv 0$. Let us consider the nominal form of the neutral system in Table 2 which presents the allowable maximum values of the time delay h for different values of the parameter c. It is clearly seen from Table 2 that the stability criteria proposed in this paper yield much improved bounds than those in other papers for nominal systems.

Case 2: $a \neq 0$. Choosing $c = 0.1$, the delay bounds obtained for different value of a, in the sense of the

Table 2. Comparison on upper bound of delay with $a = 0$ and $g = h$

c	0.1	0.3	0.5	0.7	0.9
Fridman and Shaked [2003]	3.49	2.06	1.14	0.54	0.13
Xu et al. [2005]	3.58	2.30	1.46	0.86	0.32
Wu et al. [2004a]	4.35	4.13	3.67	2.87	1.41
He et al. [2005]	4.42	4.17	3.69	2.87	1.41
Parlakçi [2007]	4.5747	4.2910	3.7575	2.8835	1.4142
Theorem 3 ($r = 5$)	5.9670	5.4897	4.6939	3.4823	1.5467
Theorem 3 ($r = 10$)	6.0196	5.5341	4.7275	3.5023	1.5510

Table 3. Comparison on upper bound of delay with $c = 0.1$ for different a

				Theorem 5		Corollary 6	
a	Han [2002]	Wu et al. [2004a]	Parlakçi [2007]	$r = 5$	$r = 10$	$r = 5$	$r = 10$
0.05	3.61	3.64	3.7934	4.5286	4.5662	5.8419	5.8938
0.10	2.90	3.06	3.1690	4.4128	4.4498	5.7246	5.7760
0.15	2.19	2.60	2.6745	4.3041	4.3406	5.6145	5.6652
0.20	1.48	2.24	2.2817	4.2021	4.2380	5.5107	5.5609
0.25	0.77	1.94	1.9666	4.1059	4.1411	5.4128	5.4624

feasibility of the corresponding criteria, are summarized in Tables 3 along with those given in Han [2002], Wu et al. [2004a] and Parlakçi [2007]. Tables 3 shows that the proposed methodology of this paper gives better results for uncertain neutral systems.

5. CONCLUSION

Stability analysis based on a new form of Lyapunov-Krasovskii functional has been provided for linear neutral systems. By employing more partitioning components in the retarded delay, improved stability bounds for the retarded delay have been obtained. Based on the numerical examples, these stability criteria for neutral systems are less conservative those those in the literature. Furthermore, the delay-partitioning projection approach proposed in this paper can be extended to solve many problems, such as exponential stability analysis, H_2/H_∞ control, guaranteed cost control and so on, for (descriptor)/(neutral) delay systems.

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