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Multi-loop Feedback Control for Atom Laser Coherence

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Abstract: In this paper we present a multi-loop measurement feedback control scheme to improve atom laser coherence. The first loop (proposed by Thomsen and Wiseman, 2002) aims to cancel the decohering effects of the nonlinear atom-atom interactions via direct measurement feedback. However, there are nonlinear interactions with the optical probe field used in the measurement scheme which may also contribute to a degradation in atom laser performance. Accordingly, we introduce a second feedback loop to implement a LQG controller to reduce these effects. The multi-loop design achieves improved atom laser coherence.

1. INTRODUCTION

Atom lasers are devices that exploit wave properties of matter to form a beam of atoms that is analogous to optical lasers in that it is coherent and intense. While the concept of an atom laser has been experimentally demonstrated (Hagley et al. (1999), Bloch et al. (1999)), atoms lasers are still quite a way from being available for use in applications. Applications for atom lasers include, for example, precision metrology and atom lithography; such applications exploit the fact that the de Broglie wavelength of atoms is much smaller than that of light, and also that atoms are more sensitive to interactions than photons.

Atom lasers are produced from Bose-Einstein Condensates (BEC), which are ensembles of atoms cooled to a very low temperature (nano Kelvin) and have the characteristic property that essentially all the atoms are in the ground state, Metcalf and van der Straten (1999). It is this high occupation of the ground state that is responsible for the coherence properties of atom lasers. However, as atoms are massive objects that interact with one another, the coherence of the condensate and the resulting atom laser beam are degraded.

The use of measurement feedback to counteract the decohering effect of the nonlinear atom-atom interactions was proposed by Thomsen and Wiseman (2002). Their feedback scheme measured light which had interacted with the condensate, the results of which were used in a proportional feedback scheme that varied the condense density by modulating the trap containing the condensate. Appropriate tuning of the feedback gain enables the cancellation of the undesirable nonlinear atom-atom interactions, and may be considered as a quasi feedback linearization scheme. However, there are nonlinear interactions with the optical probe field used in the measurement scheme which may also contribute to a degradation in atom laser performance. Accordingly, we introduce a second feedback loop (used in conjunction with the Thomsen-Wiseman proportional feedback) to implement a classical LQG controller to reduce these effects. We demonstrate that the multi-loop design achieves improved atom laser coherence.

The single mode model for an atom laser we use is described in Section 2. We include in this section the proportional loop design from Thomsen and Wiseman (2002). In Section 3, the LQG loop is designed. The performance of this loop is investigated in Section 4. In Section 5 we introduce a definition of coherence time and give an approximation of the this coherence time for the BEC under the action of the multi-loop control system.

2. MODEL

We consider a single mode model of a BEC represented by a mode operator a satisfying a bosonic commutation relation $[a, a^\dagger] = 1$ (here, $\dagger$ denotes adjoint, and the commutator is defined by $[A, B] = AB - BA$), as shown in Fig. 1. The system is coupled to three quantum field inputs and one classical input. First, an atomic boson vacuum input b_1 with damping rate κ is used to model the out-coupling responsible for the atom laser beam. Second, since the number of atoms in the BEC is reduced after some of them are released as the beam, in order to maintain the number of atoms in the BEC, it is necessary to pump it up by adding atoms from external sources. This is represented by another quantum field input b_2 with pumping rate μ (see the Appendix for further details). The third input is a quantum optical field represented by a mode operator b_3 and the corresponding output is measured by a homodyne detection system (HD). The resulting photocurrent is used in two ways. The first is to produce a proportional feedback signal u_1 with gain k (as in Thomsen and Wiseman (2002)), while the second is used for the LQG feedback signal u_2 designed in this paper. The two feedback signals are combined additively to produce a classical input signal $u = u_1 + u_2$.

Before the feedback is applied, the system is described by a Hamiltonian

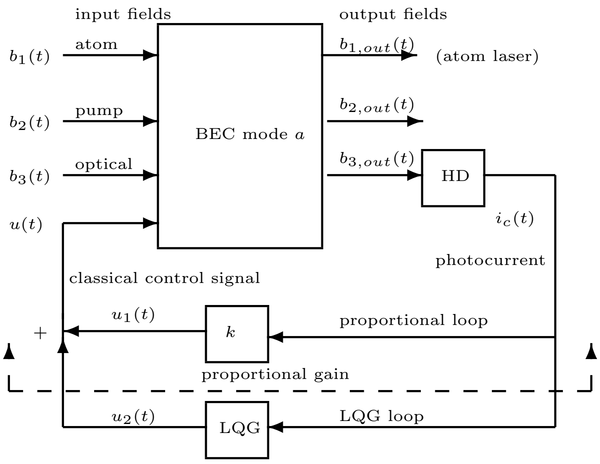

Fig. 1. A schematic representation of the multi-loop atom laser feedback system. The output corresponding to the input b_1 forms an atom laser beam. The optical output, which corresponds to the optical input b_3, is detected by a photon detector (HD) which produces a photocurrent using a standard homodyne scheme. This current is used for proportional feedback $u_1 = ki_c$ and LQG feedback control u_2.

$$H_0 = k_0 a^\dagger a^\dagger aa + (u_1 + u_2)a^\dagger a, \tag{1}$$

and three (respective) field coupling operators

$$L_1 = \sqrt{\kappa}\, a, \tag{2a}$$
$$L_2 = \sqrt{\mu}\, a^\dagger, \tag{2b}$$
$$L_3 = \sqrt{\gamma}\, a^\dagger a. \tag{2c}$$

The first term in the Hamiltonian H_0 represents the atom-atom interaction and the second one is the classical control term. The optical output field $b_{3,out}$ is given by

$$db_{3,out} = L_3 dt + db_3. \tag{3}$$

The output of homodyne detector (HD in Fig. 1) is the real quadrature $z = b_{3,out} + b_{3,out}^\dagger$ which satisfies

$$dz = (L_3 + L_3^\dagger)dt + d(b_3 + b_3^\dagger). \tag{4}$$

Formally, the photocurrent $i_c(t)$ is related to this quadrature by $dz(t) = i_c(t)dt$.

Now consider the system after the proportional feedback of Thomsen and Wiseman (2002) has been applied, but before the LQG feedback is implemented; formally, $u_1(t) = ki_c(t)$. This corresponds to the part above the dashed line in Fig. 1. This system is described by a Hamiltonian

$$H = k_0 a^\dagger a^\dagger aa + u_2 a^\dagger a + \frac{i}{2}(L_3^\dagger L_4 - L_4^\dagger L_3), \tag{5}$$

where the third term is an additional Hamiltonian which results from the proportional feedback , and

$$L_4 = -ika^\dagger a.$$

The three field coupling operators are

$$\boldsymbol{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 + L_4 \end{bmatrix}. \tag{6}$$

The third operator $L_3 + L_4$ is the result of a series (cascade) connection, Thomsen and Wiseman (2002) (see also Gough and James (2007)).

For an arbitrary BEC operator X, the infinitesimal time evolution is given by (Hudson and Parthasarathy (1984), Gardiner and Zoller (2000))

$$dX = \mathcal{L}X dt + [\boldsymbol{L}^\dagger d\boldsymbol{b} - d\boldsymbol{b}^\dagger \boldsymbol{L}, X], \tag{7}$$

where

$$\begin{aligned}\mathcal{L}X = \boldsymbol{L}^\dagger X \boldsymbol{L} + \Big(-\frac{1}{2}\boldsymbol{L}^\dagger \boldsymbol{L} - iH\Big)^\dagger X \\ + X\Big(-\frac{1}{2}\boldsymbol{L}^\dagger \boldsymbol{L} - iH\Big)\end{aligned} \tag{8}$$

and the quantum noise input vector

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{9}$$

The atom laser beam is given by

$$db_{1,out} = L_1 dt + db_1. \tag{10}$$

It follows from (5), the commutation relation $[a, a^\dagger] = 1$, and the relation

$$\frac{i}{2}(L_3^\dagger L_4 - L_4^\dagger L_3) = \sqrt{\gamma}\, ka^\dagger aa^\dagger a \tag{11}$$
$$= \sqrt{\gamma}\, k(a^\dagger a^\dagger aa + a^\dagger a) \tag{12}$$

that the nonlinear atomic interaction term can be cancelled by the choice of proportional gain $k = -k_0/\sqrt{\gamma}$. We shall henceforth assume that this choice has been made. The term $\sqrt{\gamma}\, ka^\dagger a$ corresponds to a shift of the input u, and will be ignored. Note that even though the main nonlinear terms has been cancelled, the dynamics (7) is still nonlinear.

3. LQG LOOP DESIGN

The nonlinearities remaining in the system after the proportional feedback has been implemented may also degrade performance of the atom laser. To analyze this decoherence and design the LQG feedback input for the reduction of the decoherence, we consider linearization. In general, the effectiveness of a linear model may only be limited to a certain period of time, which is determined by the coupling constant of the optical field in this case. The purpose of the estimate feedback is then to extend the effective time of linearization because it is proportional to the coherence time of the atom laser beam. This can be thought of as noise reduction. It will be shown that the noise from the optical field can be effectively reduced by modifying the conditional evolution with the control input u_2.

The measurement through the optical field is providing information about the number of the condensate atoms. As a result, the phase of the BEC mode fluctuates and phase uncertainty increases while the number observable is invariant in time. Fortunately, our control input is given by the rotation operator so that one can expect the fluctuation in the phase can be reduced by rotation control.

For the mode operator a, (7) and (4) can be written as

$$\begin{aligned}da = \Big(-\frac{\kappa}{2} + \frac{\mu}{2} - \frac{|\lambda|^2}{2} + iu_2\Big)adt \\ -\sqrt{\kappa}db_1 + \sqrt{\mu}db_2^\dagger - (\lambda db_3^\dagger - \lambda^* db_3)a,\end{aligned} \tag{13a}$$
$$dz = 2\lambda_r a^\dagger a dt + (db_3 + db_3^\dagger), \tag{13b}$$

where we have defined a constant λ as

$$\lambda = \sqrt{\gamma} + i\frac{k_0}{\sqrt{\gamma}} := \lambda_r + i\lambda_i. \tag{14}$$

Suppose that the BEC mode a is initially in a coherent state whose center is located at $(x_0, 0)$ in the phase space. Let us define new quadrature operators around the center as $a = (x_0+\xi)+i\eta$ and assume that $x_0 \gg 1$, which implies that the number of the trapped atoms is sufficiently large. In this approximation, each quadrature and the output process can be expressed as

$$d\boldsymbol{\xi} = Bu_2 dt + G d\boldsymbol{w}, \tag{15a}$$

$$dz = C\boldsymbol{\xi} dt + D d\boldsymbol{w}. \tag{15b}$$

Here we have introduced

$$B = \begin{bmatrix} 0 \\ x_0 \end{bmatrix}, \quad G = x_0 \begin{bmatrix} 0 & 0 \\ -\lambda_i & \lambda_r \end{bmatrix}, \tag{16a}$$

$$C = [\, \lambda_r x_0 \;\; 0 \,], \quad D = [\, 1 \;\; 0 \,], \tag{16b}$$

and

$$\boldsymbol{\xi} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} w \\ v \end{bmatrix}, \tag{17}$$

where w, v are independent classical Brownian noise processes (that need not commute).

The classical LQG loop is designed to minimize the deviation of the quadratures from the center of the initial coherent state $(x_0, 0)$. We use an infinite horizon LQG cost with integrand

$$\boldsymbol{\xi}^T Q \boldsymbol{\xi} + r u_2^2. \tag{18}$$

The first term represents the weighted deviation of the quadratures, which is what we really want to minimize. The second term is a penalty on the control energy.

Note that ξ of the system (15) is not controllable. In fact, ξ need not be controlled for the atom laser beam coherence. It will be shown that the quadratic cost of ξ is invariant under measurement and control. Hence, the cost matrix Q should be of the form

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}, \tag{19}$$

where $q > 0$.

A solution of this optimal control problem, or an optimal input u_2, is given by the conditional process of the system (15), i.e., the cost functional is minimized by rotating the BEC state corresponding to measurement outcomes. The conditional expectation $\breve{\boldsymbol{\xi}}(t)$ of $\boldsymbol{\xi}(t)$ given the measurement data $z(s)$, $0 \le s \le t$ is given by

$$d\breve{\boldsymbol{\xi}} = Bu_2 dt + (PC^\dagger + GD^\dagger) d\breve{w}, \tag{20a}$$

$$\dot{P} = GG^\dagger - (PC^\dagger + GD^\dagger)(PC^\dagger + GD^\dagger)^\dagger, \tag{20b}$$

where $P := \mathrm{E}[(\boldsymbol{\xi} - \breve{\boldsymbol{\xi}})(\boldsymbol{\xi} - \breve{\boldsymbol{\xi}})^T | z]$ is the conditional covariance matrix of the error and $\breve{w}$ is the innovation process. Note that although we want to control or minimize the cost for η, the output z does not include information about η. Thus, the estimation error of η under this measurement monotonically increases. This is actually a reason that the control of atom laser coherence is difficult.

If the pumping coefficient is accurately tuned to compensate for the decrease in the number of atoms, then by standard LQG methods the optimal control input is given by

$$u_2 = -\sqrt{\frac{q}{r}}\breve{\eta}, \tag{21}$$

where $\breve{\eta}$ is the conditional expectation of η. This is a reasonable form even though the system is singular, because the feedback gain is proportional to the weight q and inversely proportional to r.

4. PERFORMANCE OF THE LQG LOOP

If the weight r is small, a feedback gain of (21) becomes high. This is quite natural because high gain control is generally useful for stabilization. To see the effect of this feedback, let us consider the mean square costs of ξ and η, respectively.

By definition, the variance of the quadratures is decomposed into the variance of the conditional expectations and the mean square error as

$$\mathrm{E}[\|\boldsymbol{\xi}\|^2] = \mathrm{E}[\|\langle \boldsymbol{\xi} \rangle\|^2 + \|\boldsymbol{\xi} - \langle \boldsymbol{\xi} \rangle\|^2], \tag{22}$$

To see the effect of feedback on the quantity (22), let us first calculate each element of P. For the system with the accurately-tuned pumping, they are given by

$$\dot{P}_{11} = -x_0^2 \lambda_r^2 P_{11}^2, \tag{23a}$$

$$\dot{P}_{12} = x_0^2(\lambda_r \lambda_i P_{11} - \lambda_r^2 P_{11} P_{12}), \tag{23b}$$

$$\dot{P}_{22} = x_0^2(\lambda_r^2 + 2\lambda_r\lambda_i P_{12} - \lambda_r^2 P_{12}^2). \tag{23c}$$

From the first equation, the mean square error of ξ is given by

$$P_{11}(t) = \frac{P_{11}(0)}{x_0^2 \lambda_r^2 P_{11}(0) t + 1}. \tag{24}$$

On the other hand, the conditional expectation of ξ is given by

$$d\langle \xi \rangle = x_0 \lambda_r P_{11} d\breve{w}, \tag{25}$$

and therefore

$$\langle \xi(t) \rangle^2 = P_{11}(0) - P_{11}(t). \tag{26}$$

From these, the variance of ξ, or the mean square error from the center of the initial coherent state x_0, is given by

$$\mathrm{E}[\xi(t)^2] = P_{11}(0), \tag{27}$$

which is invariant. This implies that the coherence of the BEC in ξ direction is preserved in time.

The other quadrature η fluctuates due to the interaction with the optical field and needs to be controlled. Since the covariance matrix P is independent of the input u, the second term of (22) cannot be changed and the feedback control is designed to reduce the first term. The conditional expectation η is given by

$$d\langle \eta \rangle = -x_0 \sqrt{\frac{q}{r}} \langle \eta \rangle dt + x_0 \lambda_r \Big(P_{12} - \frac{\lambda_i}{\lambda_r} \Big) d\breve{w}, \tag{28}$$

and the initial condition is the center of the initial state, i.e., $\langle \eta(0) \rangle = 0$. It is easy to see

$$\langle \eta(t) \rangle^2 = \int_0^t e^{-2x_0\sqrt{\frac{q}{r}}(t-s)} \Big[x_0 \lambda_r \Big(P_{12}(s) - \frac{\lambda_i}{\lambda_r} \Big) \Big]^2 ds, \tag{29}$$

where

$$P_{12}(t) = \frac{\lambda_i}{\lambda_r} \Big(1 - \frac{1}{1 + (\lambda_r x_0)^2 P_{11}(0) t} \Big). \tag{30}$$

Note that $P_{12}(t) > 0$ for $t > 0$. On the other hand, P_{22} can be represented as

$$\dot{P}_{22}(t) = x_0^2\Big[|\lambda|^2 - \lambda_r^2\Big(P_{12} - \frac{\lambda_i}{\lambda_r}\Big)^2\Big] \tag{31}$$

When the feedback control is applied, the variance of η from the center of the initial state is given by

$$\begin{aligned} \mathrm{E}[\eta(t)^2] &= P_{22}(0) + x_0^2|\lambda|^2 t \\ &+ \int_0^t ds\,\Big[e^{-2x_0\sqrt{\frac{q}{r}}(t-s)} - 1\Big] \\ &\times\Big[x_0\lambda_r\Big(P_{12}(s) - \frac{\lambda_i}{\lambda_r}\Big)\Big]^2. \end{aligned} \tag{32}$$

If the gain of the feedback is sufficiently large, the fluctuation in the conditional expectation of η can be reduced and the variance is approximately represented as

$$\mathrm{E}[\eta(t)^2] \sim P_{22}(0) + x_0^2|\lambda|^2 t - \frac{\lambda_i P_{12}(t)}{\lambda_r P_{11}(0)}. \tag{33}$$

On the other hand, if $u = 0$, it is given by

$$\mathrm{E}[\eta(t)^2] = P_{22}(0) + x_0^2|\lambda|^2 t. \tag{34}$$

Thus, the variance is reduced by the amount of

$$\frac{\lambda_i P_{12}(t)}{\lambda_r P_{11}(0)} > 0 \tag{35}$$

subject to high gain feedback control. Note that this quantity is sensitive to the coupling constant with the optical field γ. If γ is small, so is the fluctuation in the BEC state and it is easy to reduce the variance by feedback. If γ is large, the fluctuation from the optical field is very strong and the control input u can hardly reduce the noise. In this sense, the coupling constant γ can be thought of as a parameter representing the degree of controllability in the BEC system. A numerical example of the variances is given in Fig. 2.

5. COHERENCE TIME

The analysis of the feedback effect in the previous section is based on the linear model. If the control input u_2 efficiently reduces the fluctuation of the optical field, the BEC state can keeps its initial coherence for a longer time and the linear model is more effective to describe the system. Thus, the coherence time of the atom laser beams is represented by the effective time of the linear model.

In the linear model, the variance of ξ is invariant under measurement and feedback whereas that of η monotonically increases. So the coherence of the BEC state is destroyed as the initial coherent state is stretched out along the line $\xi = 0$ in time. In the full model (7) with accurately-tuned pumping, since the nonlinear interaction is represented by the number operator, the number of atoms is preserved and the phase uncertainty increases along the curve $(x_0 + \xi)^2 + \eta^2 = x_0^2$ on the phase space. And the stationary solution of (7) is a mixture of number states with the mean number x_0^2 and the variance $P_{11}(0)$. Thus, the coherent time, or the effective time of the linear model, can be measured by the magnitude of the variance $\mathrm{E}[\eta^2]$.

The coherence time is usually related to the normalized first-order coherent function. If the phase distribution is

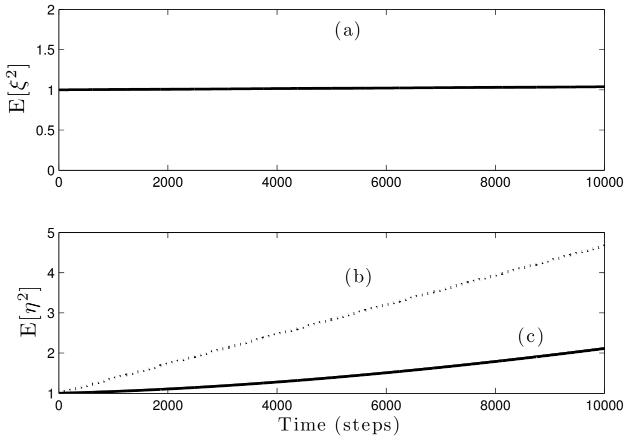

Fig. 2. Numerical results for the linear model with $x_0 = 10$, $k_0 = 1$ and $\gamma = 0.3$. (a) The variance of ξ of the BEC state under the two feedback controls. As expected from the linear model (15), the variance is invariant in time. (b) The variance of η of the same state for $u_2n = 0$. The variance increases linearly in time. Note that the proportional feedback is applied, i.e., $u_1 \neq 0$. (c) The variance of η when an optimal control (21) is applied. This feedback suppresses fluctuation noise and keeps the initial coherence for a longer time.

characterized by the variance $\Delta\phi^2$, the first-order coherence function can be approximately written as

$$g^{(1)}(t) = e^{-\Delta\phi^2(t)}, \tag{36}$$

and then, the coherence time T is given by

$$\tau = \int_0^\infty e^{-\Delta\phi^2(t)}dt. \tag{37}$$

In the linear model, one can give another interpretation to the definition of the coherence time. Let us first consider the case of $u = 0$. The phase distribution can be related to the variance of the quadrature as

$$(x_0\Delta\phi(\tau))^2 = \mathrm{E}[\eta(t)^2]. \tag{38}$$

From (34), we have

$$\Delta\phi(t)^2 = \frac{P_{22}(0)}{x_0^2} + |\lambda|^2 t. \tag{39}$$

Thus, (37) yields

$$\tau = e^{-\frac{P_{22}(0)}{x_0^2}}\frac{1}{|\lambda|^2} \sim \Big(1 - \frac{P_{22}(0)}{x_0^2}\Big)\frac{1}{|\lambda|^2}. \tag{40}$$

On the other hand, $\mathrm{E}[\eta^2] = x_0^2$ is satisfied if

$$t = \Big(1 - \frac{P_{22}(0)}{x_0^2}\Big)\frac{1}{|\lambda|^2}. \tag{41}$$

Thus, the coherence time T is approximately described by a time when the BEC state spreads to order x_0^2.

This consideration allows us to introduce the coherence time in a different way. We first notice that the equality $\mathrm{E}[\eta^2] = x_0^2$ would overestimate the coherence time. Secondly, the coherence time should be dependent on the mean number x_0 because the influence of the decoherence from the optical field is inversely proportional to the mean number. Thus, it is reasonable to think that the linear model is effective until when $\mathrm{E}[\eta^2]$ is approximately less than x_0, i.e., here we define the coherent time τ of the atom laser beam as

$$\mathrm{E}[\eta(\tau)^2] = x_0. \tag{42}$$

If $u_2 = 0$ (no LQG loop, proportional loop only), (34) is used to yield

$$\tau_0 = \left(1 - \frac{1}{x_0}\right)\frac{1}{x_0|\lambda|^2}, \tag{43}$$

where we have assumed that the system is initially prepared in a coherent state so that $P_{11}(0) = P_{22}(0) = 1$. If high gain feedback control is applied (both LQG and proportional loops), one may use (33) to obtain

$$\tau_f \sim \frac{1}{x_0|\lambda|^2}\left[1 + \sqrt{1 + \frac{4}{x_0}\left\{1 + \left(\frac{\lambda_i}{\lambda_r}\right)^2\right\}}\right] \tag{44}$$

Note that the coherence time is also sensitive to the coupling constant with the optical field γ. In particular, the difference of the coherence time between (43) and (44) is determined by

$$\frac{\lambda_i}{\lambda_r} = \frac{k_0}{\gamma}. \tag{45}$$

If γ is small, the coherence time is improved better.

This is a natural consequence from a physical point of view. The decoherence of the BEC state is mainly caused by the interaction with the optical field. As stated earlier, if the interaction between the BEC and optical field is weak, the coherence of the BEC state is not seriously destroyed by the interaction so that it is easy to recover the coherence and the atom laser beam can hold its initial coherence for a sufficiently long time. Figure 3 illustrates the dependance of the coherence times on the coupling strength γ.

It is also worth noting that the coherence time is proportional to

$$\frac{1}{|\lambda|^2} = \frac{\gamma}{\gamma^2 + k_0^2}. \tag{46}$$

This factor indicates that there is an optimal γ to maximize the coherence time, as shown in Fig. 3. The optimal coupling constant is approximately given as $\gamma \sim k_0$.

6. CONCLUSION

This paper addresses the possibility of feedback control for stabilizing quantum systems with nonlinear dynamics by designing a multi-loop feedback strategy to keep the system in the linear domain as long as possible. The two feedback loops work in different ways. The proportional loop is designed to cancel the nonlinear terms in the Hamiltonian which represent the atom-atom interactions in the atom laser, and the LQG loop is designed to reduce the fluctuation noise from the optical field. To apply the first loop, the interaction with the optical field is necessary to extract information from the system for the purpose of eliminating the nonlinear effect of the atom-atom interactions via feedback. However, this interaction induces another nonlinearity into the system. The second loop is then necessary to reduce the fluctuations from the optical field. Combining these two loops, the performance of the atom laser is improved and the system holds its coherent properties longer than if only the proportional loop was used.

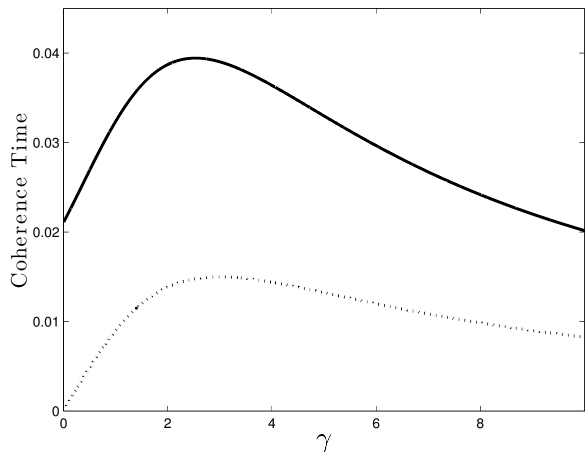

Fig. 3. Coherence time τ_f (solid line) and τ_0 (dashed line) as a function of γ.

Appendix A. PUMPING

The atom laser beam is formed when the condensate atoms lead out from the trapped BEC. As a result, the number of atoms in the system is reduced, which is represented by the decay of the BEC mode a due to the linear coupling to the input b_1. The pumping input b_2 is then introduced to compensate for the decrease in the number of atoms.

It can be experimentally realized by evaporative cooling of the uncondensed atoms and mathematically modeled by the coupling to the excited states of a trapped field. After physically relevant approximations, the pumping effect is expressed as [Thomsen and Wiseman (2002)]

$$\mu_0 \mathcal{A}^{-1}[a^\dagger]\mathcal{D}[a^\dagger]X, \tag{A.1}$$

where, for arbitrary operators X and L,

$$\mathcal{A}[L]X := \frac{1}{2}(L^\dagger L X + X L^\dagger L), \tag{A.2a}$$

$$\mathcal{D}[L]X := L^\dagger X L - \mathcal{A}[L]X. \tag{A.2b}$$

Let us further simplify the pumping to obtain the model used in this paper. Recall that the quadrature operators are approximately order of $\|\xi\|^2, \|\eta\|^2 \lesssim x_0$. In this approximation, for example, we can express as $aa^\dagger \sim x_0^2 + 2x_0\xi$.

Let us set $\mathcal{A}[a^\dagger]^{-1}Z := Y$ for some operator Z. By definition, this can be rewritten as

$$Z - \frac{1}{2}aa^\dagger Y - \frac{1}{2}Yaa^\dagger = 0. \tag{A.3}$$

Note that this is an algebraic Lyapunov equation in the infinite dimensional space. Since the number of atoms is always positive, $-aa^\dagger$ can be thought of as a stable operator in the sense that $-\langle\phi|aa^\dagger|\phi\rangle < 0$ for any ϕ. As a result, the equation above has a solution of the form

$$Y = \int_0^\infty e^{-aa^\dagger t/2} Z e^{-aa^\dagger t/2} dt. \tag{A.4}$$

For an eigenstate of the quadrature operator ξ, $\xi|\tilde{\xi}\rangle = \tilde{\xi}|\tilde{\xi}\rangle$, the matrix element of $\langle\tilde{\xi}'|Y|\tilde{\xi}\rangle$ is given by

$$\langle\tilde{\xi}'|Y|\tilde{\xi}\rangle = \frac{1}{x_0^2 + x_0(\tilde{\xi} + \tilde{\xi}')}\langle\tilde{\xi}'|Z|\tilde{\xi}\rangle \tag{A.5}$$

$$\sim \langle\tilde{\xi}'|\frac{1}{x_0^2}\Big(Z - \frac{\xi Z + Z\xi}{x_0}\Big)|\tilde{\xi}\rangle. \tag{A.6}$$

Consequently, the solution can be approximated as

$$Y \sim \frac{1}{x_0^2}\Big(Z - \frac{\xi Z + Z\xi}{x_0}\Big). \tag{A.7}$$

In the lowest order approximation, the pumping effect can be expressed as

$$\mu_0 \mathcal{A}[a^\dagger]^{-1}\mathcal{D}[a^\dagger]X \sim \mu\mathcal{D}[a^\dagger]X, \tag{A.8}$$

where we have defined as $\mu = \mu_0/x_0^2$. This expression of the pumping is equivalent to the coupling operator L_2 defined in (2).

REFERENCES

I. Bloch, T.W. Hänsch, and T. Esslinger. Atom Laser with a cw Output Coupler. *Phys. Rev. Lett.*, 82:3008–3011, 1999.

C.W. Gardiner and P. Zoller. *Quantum Noise.* Springer, Berlin, 2000.

J. Gough, M.R. James. The Series Product and Its Application to Quantum Feedforward and Feedback Networks. quant-ph/0707.0048v1 2007.

E.W. Hagley, L. Deng, M. Kozuma, J. Wen, K. Helmerson, S.L. Rolston, W.D. Phillips. A Well-Collimated Quasi-Continuous Atom Laser. *Science*, 283:1706–1709, 1999.

R.L. Hudson and K.R. Parthasarathy. Quantum Ito's formula and stochastic evolutions. *Commun. Math. Phys.*, 93:301–323, 1984.

H.J. Metcalf and P. van der Straten. Laser Cooling and Trapping. Springer, New York, 1999.

L.K. Thomsen, H.M. Wiseman. Atom-laser coherence and its control via feedback. *Phys. Rev. A*, 65:063607 2002.