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Optimal Controller Design for Networked Control Systems

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Abstract: This paper addresses the problem of optimal control system design for networked control systems. We focus on a situation where the plant is single-input single-output and the communication link between the controller and the plant is signal-to-noise ratio constrained. In this setting, we characterize the controllers that minimize the tracking error variance, while respecting the channel signal-to-noise ratio constraint. We also provide a description of the optimal tradeoff curve in the performance *versus* signal-to-noise ratio plane and, as a byproduct, we establish easily computable bounds on the achievable performance. We illustrate our results with a numerical example based on a bit rate limited channel.

Keywords: Control over networks, control under communication constraint.

1. INTRODUCTION

In networked control systems (NCS's) loops are closed over communication channels that cannot be regarded as transparent. These systems have received much attention over the last years, as witnessed by the special issue edited by Antsaklis and Baillieul (2007), the handbook by Hristu-Varsakelis and Levine (Eds.) (2005), and the many references therein. Typical channel artifacts include bit rate limits, random data dropouts and random delays (see, e.g., the survey paper by Hespanha et al. (2007)). In some cases, the impact of these characteristics is so severe, that the communication channels become bottlenecks on the achievable performance. In these cases, sensible NCS designs should take the channel characteristics explicitly into account.

Many interesting networked design techniques have been reported in the literature for channels with random delays (e.g., Nilsson (1998); Hespanha et al. (2007)), data dropouts (e.g., Ling and Lemmon (2004); Schenato et al. (2007); Seiler and Sengupta (2005); Něsić and Teel (2004)) and quantization (e.g., Nair et al. (2007); Xiao et al. (2003); Fu and Xie (2005)). In the latter case, most results focus on quantifying the minimal data rates that allow one to stabilize a given plant (see Nair and Evans (2004); Tatikonda and Mitter (2004)) or to achieve a certain level of performance (Savkin (2006)). In relation to the last body of work referred to above, an interesting alternative viewpoint has been proposed by Braslavsky et al. (2007). That work uses a power constrained additive noise channel model and derives explicit expressions for the minimal signal-to-noise ratio that allows one to stabilize a single-input single-output (SISO) linear system. Using elementary information theory concepts, this result can be interpreted in terms of minimal rates for stabilization, recovering (in some cases) the results in Nair and Evans (2004).

In the present work, we also use an additive noise channel model for the link between the controller and the plant. However, unlike Braslavsky et al. (2007), we consider a signal-to-noise ratio constraint and adopt a more performance oriented viewpoint. We characterize (in terms of a single scalar parameter) the controller that minimizes the tracking error variance, while respecting the channel signal-to-noise ratio constraint. As a byproduct, we also establish easily computable bounds on the achievable performance and provide a characterization of the optimal tradeoff curve in the performance *versus* signal-to-noise ratio plane. The present work complements our recent contributions documented in Silva et al. (2007a); Goodwin et al. (2008); Silva et al. (2007b), where coding system design for NCS's has been explored.

The remainder of this paper is organized as follows: Section 2 introduces the notation used throughout the paper and recalls some basic results. Section 3 describes the NCS architecture of interest here. Section 4 presents analysis guidelines, while Section 5 studies performance limits. Section 6 derives the main results of this paper. We illustrate our findings with an example in Section 7. Concluding remarks and directions for future work are given in Section 8.

2. NOTATION AND PRELIMINARIES

We use standard vector space notation for signals, i.e., x denotes $\{x(k)\}_{k\in\mathbb{N}_0}$. We also use z as both the argument of the z-transform and as the forward shift operator, where the meaning is clear from the context.

The set of all real rational SISO transfer functions is denoted by $\mathcal{R}$. We also define the sets $\mathcal{U}_\infty \subset \mathcal{RH}_\infty \subset \mathcal{R}_p \subset \mathcal{R}$, whose distinctive features are as follows: $\mathcal{R}_p$ contains proper transfer functions, $\mathcal{RH}_\infty$ contains stable and proper transfer functions, and $\mathcal{U}_\infty$ contains transfer functions in $\mathcal{RH}_\infty$ that have inverses in $\mathcal{RH}_\infty$. Every

$A(z) \in \mathcal{R}$ having no poles on the unit circle belongs to $\mathcal{L}_2$. If this is the case, then we define the 2−norm of $A(z)$ via

$$||A(z)||_2^2 \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|A(e^{j\omega})\right|^2 d\omega,$$

where $|\cdot|$ denotes magnitude.

For a standard 1-degree of freedom (dof) control loop (i.e., the loop in Fig. 1 with no channel), having SISO plant $G(z)$ and SISO controller $C(z)$, we define the following closed loop transfer functions:

$$S(z) \triangleq (1 + G(z)C(z))^{-1}, \quad T(z) \triangleq 1 - S(z),$$
$$S_i(z) \triangleq G(z)S(z), \quad S_u(z) \triangleq C(z)S(z) = G(z)^{-1}T(z).$$

The well known Youla parameterization states that every 1-dof admissible controller[1] for $G(z) \in \mathcal{R}_p$ can be written as (see, e.g., Francis (1987))

$$C(z) = (X(z) + M(z)Q(z))\,(Y(z) - N(z)Q(z))^{-1}, \quad (1)$$

where $M(z), N(z) \in \mathcal{RH}_\infty$ are coprime and $G(z) = N(z)M(z)^{-1}$, $X(z), Y(z) \in \mathcal{RH}_\infty$ are such that $N(z)X(z) + M(z)Y(z) = 1$, and $Q(z) \in \mathcal{RH}_\infty$ is a free parameter (the Youla parameter). With this parameterization for $C(z)$, every closed loop transfer function defined above can be written as an affine function of $Q(z)$.

3. NCS ARCHITECTURE

In this paper we consider the NCS architecture depicted in Fig. 1. In that figure, $G(z) \in \mathcal{R}_p$ is the plant model, $C(z) \in \mathcal{R}_p$ is the controller transfer function, r is a wide sense stationary process that models the reference,[2] and y is the plant output. Without loss of generality, we will restrict attention to reference sequences having power spectral density functions that admit spectral factors $\Omega_r(z) \in \mathcal{U}_\infty$.

In contrast to standard (i.e., non networked) control systems, the communication link between the controller and the plant in Fig. 1 is not transparent: it comprises a non ideal channel. We will focus on an additive noise channel, i.e., we will model the relationship between the channel output w and the channel input v via

$$w = v + n,$$

where n is the channel noise. The noise sequence n is assumed to be a zero mean white noise sequence, uncorrelated with the reference r, having variance $0 < \sigma_n^2 < \infty$ and power spectral density

$$\Phi_n(e^{j\omega}) = \sigma_n^2, \ \forall\omega \in [-\pi, \pi].$$

A key feature of our model is that the channel has a fixed and given signal-to-noise ratio. This means that σ_n^2 is proportional to the variance of the input of the channel (i.e., proportional to σ_v^2) and is not a given constant. We define the channel signal-to-noise ratio as γ:

$$\gamma \triangleq \frac{\sigma_v^2}{\sigma_n^2} \in \mathbb{R}_0^+. \quad (2)$$

The model described above has been widely used in the signal processing literature to model bit rate limited channels (see, e.g., Jayant and Noll (1984); Schreier and Temes (2004)). It has also found application in the study of NCS architectures, as described in Xiao et al. (2003); Goodwin et al. (2008). A simplified version of this model, that assumes given noise statistics, has been employed by Braslavsky et al. (2007).

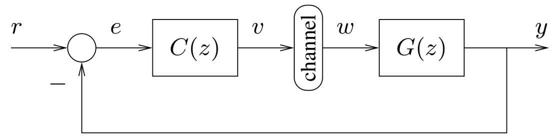

Fig. 1. Considered networked architecture.

The channel description and signal assumptions made above will be used implicitly in the remainder of this paper.

4. ANALYSIS

In this section we provide analysis guidelines for the NCS described in Section 3.

From Fig. 1 it follows that the tracking error, e, defined as

$$e \triangleq r - y,$$

satisfies (recall the definitions in Section 2)

$$e = S(z)r - S_i(z)n. \quad (3)$$

We are interested in NCS performance and, thus, we will focus on the variance of e, namely σ_e^2. From (3) it follows that

$$\sigma_e^2 = ||S(z)\Omega_r(z)||_2^2 + \sigma_n^2\,||S_i(z)||_2^2. \quad (4)$$

In view of (2), σ_n^2 is not an independent constant. It depends on v and this signal, in turn, depends on the reference sequence and the channel noise. Indeed, it is easy to see from Fig. 1 that

$$\sigma_v^2 = ||S_u(z)\Omega_r(z)||_2^2 + \sigma_n^2\,||T(z)||_2^2. \quad (5)$$

We note that (5) establishes a relationship between σ_v^2 and σ_n^2 that arises from the system architecture. Indeed, from (5) we have that the signal-to-noise ratio imposed by the system architecture is given by

$$\frac{\sigma_v^2}{\sigma_n^2} = \frac{1}{\sigma_n^2}\,||S_u(z)\Omega_r(z)||_2^2 + ||T(z)||_2^2. \quad (6)$$

This relationship may not be consistent with (2), a relationship imposed by the channel. Indeed, one can prove the following (see also Braslavsky et al. (2007)):

Lemma 1. (Bound on γ). Consider the NCS in Fig. 1 with the reference and channel models described in Section 3. Then, (6) can be made consistent with (2) if and only if

$$||T(z)||_2^2 < \gamma. \quad (7)$$

Proof. Immediate from (6). ■

Motivated by Lemma 1, we can use the Youla parameterization to define

$$\gamma_{\inf} \triangleq \min_{Q(z)\in\mathcal{RH}_\infty} ||T(z)||_2^2. \quad (8)$$

It thus becomes clear that $\gamma_{\inf}$ is the largest lower bound on the channel signal-to-noise ratio that allows one to stabilize the NCS in Fig. 1 (within the model of Section

[1] i.e., a proper stabilizing controller that defines a well possed 1-dof control loop (Zhou et al. (1996)). By extension, a closed loop transfer function is said admissible if and only if it is associated with an admissible controller.

[2] In our framework, the consideration of plant disturbances and measurement noise presents no additional technical difficulties.

3). Explicit analytic expressions for γ_{inf} are presented in Braslavsky et al. (2007).

The following immediate corollary to Lemma 1 is also informative:

Corollary 2. (Behavior as $\gamma \to \gamma_{\text{inf}}$). Consider the conditions of Lemma 1 and assume that (7) holds. Then:

(1) If $G(z)$ is stable, then $\gamma_{\text{inf}} = 0$ and
$$\lim_{\gamma \to \gamma_{\text{inf}}} \sigma_e^2 = \infty,$$
unless the plant is left in open loop (i.e., unless $S_u(z) = 0$). In that case, $\sigma_e^2 = ||\Omega_r(z)||_2^2$ for all γ.

(2) If $G(z)$ is unstable, then $\gamma_{\text{inf}} > 0$ and
$$\lim_{\gamma \to \gamma_{\text{inf}}} \sigma_e^2 = \infty$$
for every admissible controller.

Proof. Since (7) holds, we have from (2), (4) and (6) that
$$\sigma_e^2 = ||S(z)\Omega_r(z)||_2^2 + \frac{||S_u(z)\Omega_r(z)||_2^2 \, ||S_i(z)||_2^2}{\gamma - ||T(z)||_2^2}$$
$$\geq ||S(z)\Omega_r(z)||_2^2 + \frac{||S_u(z)\Omega_r(z)||_2^2 \, ||S_i(z)||_2^2}{\gamma - \gamma_{\text{inf}}}.$$
The result follows upon noting that for stable plants $T(z) = 0$ ($\Leftrightarrow S_u(z) = 0$) is admissible, and that for unstable plants $T(z)$ must be non zero. ■

Remark 3. If $C(z)$ were fixed (and non-zero for stable plants), then $\sigma_e^2 \to \infty$ for every $\gamma \to ||T(z)||_2^2$. In our setting, however, we can choose the controller $C(z)$ so as to avoid $||T(z)||_2^2$ being close to γ. The only case where this is not possible is when $\gamma \to \gamma_{\text{inf}}$. □

Corollary 2 shows that if $\gamma \to \gamma_{\text{inf}}$, then both the stability and performance of the NCS under study will be heavily compromised (see also Baillieul (2002)). This motivates the question of what is the best achievable performance, as measured by σ_e^2, for any $\gamma > \gamma_{\text{inf}}$, and how to design a controller that achieves that performance. To give an answer to this question, we start studying in Section 5 performance limits for the considered networked control system setup. These results are then exploited in Section 6 to give an answer to the question raised above.

5. PERFORMANCE LIMITS

This section explores performance limits for the considered NCS setup. To that end, we will assume throughout this section that σ_n is a given positive number. This will allow us to elucidate *optimal trade off curves* in the performance *versus* signal-to-noise ratio plane (see, e.g., Section 4.7 in Boyd and Vandenberghe (2004)). As a byproduct, we will also obtain bounds on the achievable performance for the NCS under study.

We use the Youla parameterization (see Section 2) to define
$$J_{\sigma_n}(Q(z)) \triangleq ||S(z)\Omega_r(z)||_2^2 + \sigma_n^2 \, ||S_i(z)||_2^2 \,, \qquad (9)$$
$$R_{\sigma_n}(Q(z)) \triangleq \sigma_n^{-2} \, ||S_u(z)\Omega_r(z)||_2^2 + ||T(z)||_2^2 \,. \qquad (10)$$
$J_{\sigma_n}(Q(z))$ is the tracking error variance, as a function of $Q(z) \in \mathcal{RH}_\infty$, when the noise channel variance equals σ_n^2; $R_{\sigma_n}(Q(z))$ is the corresponding channel signal-to-noise ratio.

The set of all achievable pairs $(J_{\sigma_n}, R_{\sigma_n})$ is given by
$$\mathcal{F}_{\sigma_n} \triangleq \big\{(\alpha_e, \alpha_\gamma) \in \mathbb{R}^2 : \; J_{\sigma_n}(Q(z)) \leq \alpha_e \text{ and } R_{\sigma_n}(Q(z)) \leq \alpha_\gamma \text{ for some } Q(z) \in \mathcal{RH}_\infty \big\}.$$
We note that, given the fact that we consider $\sigma_n > 0$, the constraint (7) is implicit in the definition of $\mathcal{F}_{\sigma_n}$.

It is clear that J_{σ_n} and R_{σ_n} are competing objectives, i.e., one cannot simultaneously minimize both J_{σ_n} and R_{σ_n}. As a consequence, the set $\mathcal{F}_{\sigma_n}$ has no optimal point, i.e., there exist no $(\alpha_e, \alpha_\gamma) \in \mathcal{F}_{\sigma_n}$ such that $\alpha_e \leq J_{\sigma_n}(Q(z))$ and, simultaneously, $\alpha_\gamma \leq R_{\sigma_n}(Q(z))$ for every $Q(z) \in \mathcal{RH}_\infty$. Nevertheless, we can consider the set $\mathcal{P}_{\sigma_n} \subset \mathcal{F}_{\sigma_n}$ containing the points in the $(J_{\sigma_n}, R_{\sigma_n})$ plane that provide the *best* tradeoff in the following sense: $(\theta_e, \theta_\gamma) \in \mathcal{P}_{\sigma_n}$ if and only if, for every $(\alpha_e, \alpha_\gamma) \in \mathcal{F}_{\sigma_n}$, $\alpha_e \leq \theta_e$ and $\alpha_\gamma \leq \theta_\gamma$ implies $\alpha_e = \theta_e$ and $\alpha_\gamma = \theta_\gamma$. Roughly speaking, $\mathcal{P}_{\sigma_n}$ contains the points in $\mathcal{F}_{\sigma_n}$ that *cannot be improved in both components simultaneously.*

Paraphrasing Boyd and Vandenberghe (2004), we conclude that if one is interested in *good* solutions, then one should focus on $\mathcal{P}_{\sigma_n}$. Indeed, there exist Youla parameters that achieve less tracking error variance at the expense of a smaller signal-to-noise ratio for every point in $\mathcal{F}_{\sigma_n}$ that does not belong to $\mathcal{P}_{\sigma_n}$. The set $\mathcal{P}_{\sigma_n}$ cointains the so-called Pareto optimal points for the problem of simultaneously minimizing J_{σ_n} and R_{σ_n}. In our case, $\mathcal{P}_{\sigma_n}$ defines a curve in $\mathbb{R}^2$, the optimal tradeoff curve. The next lemma characterizes $\mathcal{P}_{\sigma_n}$.

Lemma 4. (Characterization of $\mathcal{P}_{\sigma_n}$). Consider $\epsilon \in \mathbb{R}$ and define
$$L_{\sigma_n,\epsilon}(Q(z)) \triangleq \epsilon J_{\sigma_n}(Q(z)) + (1-\epsilon) R_{\sigma_n}(Q(z)), \qquad (11)$$
$$Q_{\sigma_n,\epsilon}(z) \triangleq \arg \min_{Q(z) \in \mathcal{RH}_\infty} L_{\sigma_n,\epsilon}(Q(z)). \qquad (12)$$

The following holds:

(1) $(J_{\sigma_n}(Q_{\sigma_n,\epsilon}(z)), R_{\sigma_n}(Q_{\sigma_n,\epsilon}(z))) \in \mathcal{P}_{\sigma_n}$ if and only if $\epsilon \in [0, 1]$.

(2) For every $\sigma_n > 0$,
$$\min_{\substack{Q(z) \in \mathcal{RH}_\infty \\ (J_{\sigma_n}(Q(z)), R_{\sigma_n}(Q(z))) \in \mathcal{F}_{\sigma_n}}} J_{\sigma_n}(Q(z)) = J_{\sigma_n}(Q_{\sigma_n,1}(z))$$

Proof.

(1) A standard result (see, e.g., Boyd and Vandenberghe (2004)) states that, if $\lambda_1, \lambda_2 > 0$, then $Q_p(z)$, defined by
$$Q_p(z) \triangleq \arg \min_{Q(z) \in \mathcal{RH}_\infty} \lambda_1 J_{\sigma_n}(Q(z)) + \lambda_2 R_{\sigma_n}(Q(z)),$$
defines a point in $\mathcal{P}_{\sigma_n}$, i.e., $Q_p(z)$ is such that $(J_{\sigma_n}(Q_p(z)), R_{\sigma_n}(Q_p(z))) \in \mathcal{P}_{\sigma_n}$. Moreover, since both J_{σ_n} and R_{σ_n} are convex in $Q(z)$, we have that for every $(\theta_e, \theta_\gamma) \in \mathcal{P}_{\sigma_n}$, there exist $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 > 0$, such that $Q_p(z)$ defines the point $(\theta_e, \theta_\gamma)$. It is easy to verify that, in our case, $Q_p(z)$ defines points in $\mathcal{P}_{\sigma_n}$ for $\lambda_1 = 0$ and for $\lambda_2 = 0$ (as long as $\lambda_1 + \lambda_2 \neq 0$). Therefore, $Q_p(z)$ defines points in $\mathcal{P}_{\sigma_n}$ for every $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 > 0$. To complete the proof, we define $M \triangleq \lambda_1 + \lambda_2$ $(M \neq 0)$ and note that
$$Q_p(z) = \arg \min_{Q(z) \in \mathcal{RH}_\infty} \frac{\lambda_1}{M} J_{\sigma_n}(Q(z)) + \frac{\lambda_2}{M} R_{\sigma_n}(Q(z)).$$

The result follows making $\epsilon \triangleq \lambda_1/M$ (which lies in $[0,1]$) and using the definitions in (11) and (12).

(2) By definition of $Q_{\sigma_n,\epsilon}(z)$ and $L_{\sigma_n,\epsilon}$ we have that

$$\begin{aligned} Q_{\sigma_n,1}(z) &= \arg \min_{Q(z)\in\mathcal{RH}_\infty} L_{\sigma_n,1}(Q(z)) \\ &= \arg \min_{Q(z)\in\mathcal{RH}_\infty} J_{\sigma_n}(Q(z)). \end{aligned} \qquad (13)$$

Since minimization of $L_{\sigma_n,\epsilon}$ (for $\epsilon \in [0,1]$) restricts the search to $\mathcal{P}_{\sigma_n}$ (see Part 1 of this Lemma), we conclude that (13) also holds if we restrict the optimization problem to all $(J_{\sigma_n}(Q(z)), \ R_{\sigma_n}(Q(z))) \in \mathcal{P}_{\sigma_n}$. The result follows upon noting that, by definition of Pareto optimal points (i.e., those in $\mathcal{P}_{\sigma_n}$), $J_{\sigma_n}(Q_{\sigma_n,1}(z))$ is actually the minimum of J_{σ_n} on $\mathcal{F}_{\sigma_n}$. ■

With the aid of Lemma 4 we can find, for every $\sigma_n > 0$, an optimal tradeoff curve in the tracking error variance *versus* channel signal-to-noise ratio plane. Each of these curves can be found by solving a series of convex optimization problems, namely by finding

$$\arg \min_{Q(z)\in\mathcal{RH}_\infty} \epsilon J_{\sigma_n}(Q(z)) + (1-\epsilon)R_{\sigma_n}(Q(z))$$

for $\epsilon \in [0,1]$.

It is straightforward to use Lemma 4 to derive an upper bound on the minimal tracking error variance when the channel has a given (admissible) signal-to-noise ratio γ. If we denote that minimum variance by $[\sigma_e^2]_{opt}^\gamma$, then it is immediate to see that, for every $\gamma > \gamma_{\inf}$,

$$[\sigma_e^2]_{opt}^\gamma \le \mathcal{B}_u(\gamma) \triangleq J_{\sigma_n^*}(Q_{\sigma_n^*,1}(z)),$$

where σ_n^* is such that $R_{\sigma_n^*}(Q_{\sigma_n^*,1}(z)) = \gamma$. In the case of stable plants, we can tighten the bound for small γ. To that end, we define

$$\Sigma_1 \triangleq \left\{\sigma_n > 0 : J_{\sigma_n}(Q_{\sigma_n,1}(z)) > ||\Omega_r(z)||_2^2\right\},$$

$$R_1 \triangleq \max\left\{R_{\sigma_n}(Q_{\sigma_n,1}(z)) : \sigma_n \in \Sigma_1\right\}.$$

Then, we can conclude that for stable plant models, and for every $\gamma \ge 0$,

$$[\sigma_e^2]_{opt}^\gamma \le \mathcal{B}_u^s(\gamma),$$

where

$$\mathcal{B}_u^s(\gamma) \triangleq \begin{cases} ||\Omega_r(z)||_2^2 & \text{, if } \gamma < R_1 \\ J_{\sigma_n^*}(Q_{\sigma_n^*,1}(z)) & \text{, if } \gamma \ge R_1 \text{, where } \sigma_n^* \text{ is such that } R_{\sigma_n^*}(Q_{\sigma_n^*,1}(z)) = \gamma. \end{cases}$$

A trivial lower bound on the best achievable performance can be obtained by noting that

$$[\sigma_e^2]_{opt}^\gamma \ge \mathcal{B}_l \triangleq \min_{Q(z)\in\mathcal{RH}_\infty} ||S(z)\Omega_r(z)||_2^2.$$

$\mathcal{B}_l$ is the minimal achievable tracking error variance when the channel is transparent, i.e., in the non networked case.

These bounds can be used to initialize numerical algorithms capable of estimating the actual best achievable performance, as outlined in Section 6.

6. OPTIMAL CONTROLLER DESIGN

In this section we return to the question of our interest. We give a characterization of the controller that minimizes σ_e^2, while respecting the channel signal-to-noise ratio constraint.

If $\gamma > \gamma_{\inf}$, then $R_{\sigma_n}(Q(z)) = \gamma$ defines σ_n^2 in terms of $Q(z)$ (recall (10)). In that cases, it makes sense to define

$$Q_{opt}^\gamma(z) \triangleq \arg \min_{\substack{Q(z)\in\mathcal{RH}_\infty \\ R_{\sigma_n}(Q(z))=\gamma}} \sigma_e^2.$$

$Q_{opt}^\gamma(z)$ is the Youla parameter associated with the controller that minimizes the tracking error variance subject to the fact that the channel has a signal-to-noise ratio equal to γ.[3] Consistent with the notation introduced in Section 5, we have that

$$[\sigma_e^2]_{opt}^\gamma \triangleq \min_{\substack{Q(z)\in\mathcal{RH}_\infty \\ R_{\sigma_n}(Q(z))=\gamma}} \sigma_e^2$$

is the associated minimum tracking error variance.

The next theorem, whose proof relies on the results in Section 5, gives a characterization of $Q_{opt}^\gamma(z)$:

Theorem 5. (Optimal Youla parameter). Consider $\gamma > \gamma_{\inf}$. Define the set

$$\Sigma \triangleq \{\sigma_n > 0 : \exists\epsilon \in [0,1] \text{ such that } \gamma = R_{\sigma_n}(Q_{\sigma_n,\epsilon}(z))\}$$

and consider the function $f : \Sigma \to [0,1]$ implicitly defined via $\gamma = R_{\sigma_n}(Q_{\sigma_n,f(\sigma_n)}(z))$. Then,

$$Q_{opt}^\gamma(z) = Q_{\sigma_{opt},f(\sigma_{opt})}(z),$$

$$[\sigma_e^2]_{opt}^\gamma = J_{\sigma_{opt}}(Q_{opt}^\gamma(z)),$$

where

$$\sigma_{opt} \triangleq \arg \min_{\sigma_n\in\Sigma} J_{\sigma_n}(Q_{\sigma_n,f(\sigma_n)}). \qquad (14)$$

Proof. Consider $\gamma > \gamma_{\inf}$ and fix $\sigma_n \in \Sigma$. In these conditions, σ_e^2 depends only on $Q(z)$ and, indeed, equals J_{σ_n} (see (9)). Accordingly, we can define the auxiliary problem of finding

$$Q_{\sigma_n}(z) \triangleq \arg \min_{\substack{Q(z)\in\mathcal{RH}_\infty \\ R_{\sigma_n}(Q(z))\le\gamma}} J_{\sigma_n}(Q(z)). \qquad (15)$$

The well known KKT optimality conditions (see, e.g., Luenberger (1969); Boyd and Vandenberghe (2004)) state that $Q_{\sigma_n}(z)$ must be a stationary point of

$$L(Q(z)) \triangleq \lambda_1 J_{\sigma_n}(Q(z)) + \lambda_2 R_{\sigma_n}(Q(z)),$$

for some non-negative λ_1 and λ_2, satisfying $\lambda_1 + \lambda_2 > 0$ and $\lambda_2\left(R(Q_{\sigma_n}(z)) - \gamma\right) = 0$. Proceeding as in the proof of Lemma 4, Part 1, we conclude that $Q_{\sigma_n}(z)$ must be a stationary point of $L_{\sigma_n,\epsilon}$ for some $\epsilon \in [0,1]$. Moreover, since $L_{\sigma_n,\epsilon}$ is convex in $Q(z)$, it has only one stationary point at $Q_{\sigma_n,\epsilon}(z)$ (see (12)). Therefore, $Q_{\sigma_n}(z) = Q_{\sigma_n,\epsilon}(z)$ for some $\epsilon \in [0,1]$ such that $(1-\epsilon)\left(R(Q_{\sigma_n,\epsilon}(z)) - \gamma\right) = 0$.

If $\epsilon \neq 1$, then it is immediate to see that the inequality constraint in (15) is active at the optimum. Using the definition of Pareto optimality, one can conclude that this also holds when $\epsilon = 1$. As a consequence, $\epsilon = f(\sigma)$ (recall the definition of f). The previous discussion implies that

$$Q_{\sigma_n}(z) = Q_{\sigma_n,f(\sigma_n)}(z) = \arg \min_{\substack{Q(z)\in\mathcal{R}_\infty \\ R_{\sigma_n}(Q(z))=\gamma}} J_{\sigma_n}(Q(z)).$$

The previous analysis holds for every $\sigma_n \in \Sigma$. It thus suffices to pick the value of σ_n that minimizes σ_e^2, when evaluated at $Q_{\sigma_n}(z)$ (or, equivalently, at $Q_{\sigma_n,f(\sigma_n)}(z)$). This is achieved using (14). ■

[3] The optimal controller can be recovered directly from (1).

The characterization of $Q^{\gamma}_{opt}(z)$ (and the corresponding minimal tracking error variance) given by Theorem 5, although analytical, is not explicit. Nevertheless, it can be used as the basis of very simple numerical algorithms to approximately find the optimal Youla parameter and the corresponding minimal cost. Indeed, one can evaluate $J_{\sigma_n}(Q_{\sigma_n,f(\sigma_n)}(z))$ quite easily: For any $\sigma_n \in \Sigma$, standard bisection allows one to find $\epsilon \in [0,1]$ such $f(\sigma_n) = \epsilon$. Then, minimization of $L_{\sigma_n,f(\sigma_n)}$ yields $Q_{\sigma_n,f(\sigma_n)}(z)$ and consequently, $J_{\sigma_n}(Q_{\sigma_n,f(\sigma_n)}(z))$. Once one is able to evaluate $J_{\sigma_n}(Q_{\sigma_n,f(\sigma_n)}(z))$, it is possible to use elementary optimization procedures to find σ_{opt} (for example, one can use the so-called golden section optimization procedure; see, e.g., Press et al. (1988)). Once this is accomplished, finding $Q^{\gamma}_{opt}(z)$ is straightforward.

The only point that needs clarification in the numerical procedure described above is how to calculate $Q_{\sigma_n,f(\sigma_n)}(z)$ or, more generally, $Q_{\sigma_n,\epsilon}(z)$. This problem can be tackled using Parseval's relation to interpret $L_{\sigma_n,\epsilon}$ in the time domain and, in that domain, using standard LQR tools (this approach is illustrated in (Goodwin et al., 2001, Section 22.6)). One can also use standard model matching techniques (see, e.g., Francis (1987)), as described next. Recall the notation in Section 2 and define

$$\xi_N(z) \triangleq z^r \prod_{i=1}^{n_c} \frac{1 - z\bar{c}_i}{z - c_i}, \quad \xi_M(z) \triangleq \prod_{i=1}^{n_p} \frac{1 - z\bar{p}_i}{z - p_i},$$
$$\tilde{N}(z) \triangleq \xi_N(z)N(z), \quad \tilde{M}(z) \triangleq \xi_M(z)M(z),$$

where r is the relative degree of $G(z)$ and $\{c_i\}_{i=1,\cdots,n_c}$ (resp. $\{p_i\}_{i=1,\cdots,n_p}$) denotes the set of zeros (resp. poles) of $G(z)$ in $|z| > 1$.

Theorem 6. (Closed form expression for $Q_{\sigma_n,\epsilon}(z)$). Assume that $\Omega_r(z) \in \mathcal{U}_\infty$ and that $G(z)$ is free of unstable hidden modes. Define

$$P(z) \triangleq \begin{bmatrix} \sqrt{\epsilon}\,\tilde{M}(z)\tilde{N}(z)\Omega_r(z) \\ \sigma_n\sqrt{\epsilon}\,\tilde{N}(z)^2 \\ -\sigma_n^{-1}\sqrt{1-\epsilon}\,\tilde{M}(z)^2\Omega_r(z) \\ -\sqrt{1-\epsilon}\,\tilde{M}(z)\tilde{N}(z) \end{bmatrix}$$

and consider an inner-outer factorization of $P(z)$ given by $P(z) = P_i(z)P_o(z)$, where $P_i(z)$ is inner and $P_o(z)$ is outer (see, e.g., Francis (1987)). Also define

$$A(z) \triangleq \begin{bmatrix} \sqrt{\epsilon}\,\tilde{M}\xi_N(z)Y(z)\Omega_r(z) \\ \sigma_n\sqrt{\epsilon}\,\tilde{N}(z)\xi_N(z)Y(z) \\ \sigma_n^{-1}\sqrt{1-\epsilon}\,\tilde{M}(z)\xi_M(z)X(z)\Omega_r(z) \\ \sqrt{1-\epsilon}\,\tilde{N}(z)\xi_M(z)X(z) \end{bmatrix}.$$

Then, for every $\sigma_n > 0$ and every $\epsilon \in (0,1)$,[4]

$$Q_{\sigma_n,\epsilon}(z) = P_o(z)^{-1}\Big(\big\{\big[P_i(z^{-1})^T A(z)\big]_\perp\big\}\big|_{z=0} + \big[P_i(z^{-1})^T A(z)\big]_\infty\Big), \quad (16)$$

where $[M(z)]_\infty$ denotes the portion of $M(z)$ in $\mathcal{RH}_\infty$, $[M(z)]_\perp \triangleq M(z) - [M(z)]_\infty$, and $\{M(z)\}|_{z=0} \triangleq M(0)$.

If, in addition, $G(z)$ has no poles or zeros on the unit circle, then (16) also holds for $\epsilon \in \{0,1\}$.

Proof. For brevity, we omit the proof (it proceeds along the lines of the proofs in, e.g., Chen et al. (2003)). ■

[4] $(\cdot)^T$ denotes transposition.

Remark 7. (The case of $\epsilon \in \{0,1\}$). If $G(z)$ has poles or zeros on the unit circle, and σ_n is such that $f(\sigma_n) \in \{0,1\}$, then we can still use (16) to calculate $J_{\sigma_n}\left(Q_{\sigma_n,f(\sigma_n)}(z)\right)$. We note, however, that in that case the "min" operator in the definition of $L_{\sigma_n,\epsilon}$ needs to be replaced by an "inf". Nevertheless, $f(\sigma_n) \in \{0,1\} \Leftrightarrow \sigma_n \in \{0,\infty\}$ and these situations arise only in degenerate cases where $\gamma = 0$ or $\gamma \to \infty$. □

The crucial point in calculating $Q_{\sigma_n,\epsilon}(z)$ is the inner-outer factorization of $P(z)$. This can be made with the aid of the algorithms described in, e.g., Oară (2005) and the references therein.

7. EXAMPLE

In this section we illustrate the results in this paper with an example where the communication channel is bit-rate limited (i.e., not strictly signal-to-noise ratio constrained). To enforce the bit rate limit, we will use an uniform quantizer to quantize v prior transmission. As is usual in the signal processing literature (see, e.g., Jayant and Noll (1984); Schreier and Temes (2004)), we will assume, for the purpose of design, that the quantization noise sequence is zero mean white noise, uncorrelated to the reference. If, in addition, we assume that overload is rare and that v is Gaussian, then we can use the model described in Section 3 with a signal-to-noise ratio given by

$$\gamma = \frac{3}{\alpha^2}(2^b - 1)^2. \quad (17)$$

In (17), b is the number of bits of the quantizer and α is the overload factor. We will use $\alpha = 4$ (see Jayant and Noll (1984) for further details).

We consider the plant and reference models

$$G(z) = \frac{0.3}{(z-0.8)}, \quad \Omega_r(z) = \frac{0.1}{(z-0.9)},$$

and take the sampling interval as $1[s]$. For this situation, we use the results in Section 5 to determine upper and lower bounds on the best achievable performance, for $b \in [0,8]$. In a second step, we employ Theorem 5 to establish the actual best achievable performance for $b \in \{0,1,\cdots,8\}$. The results are summarized in Fig. 2, where "Upper Bound", "Lower Bound" and "Optimal" refer to the situations described above. We also included two additional curves: "MV" refers to the performance achieved when using the minimum variance controller, i.e., the controller associated with the lower bound $\mathcal{B}_l$, in the quantized situation; "Empirical" refers to simulation results obtained using an *actual uniform quantizer.*[5] It is apparent that as $b \to \infty$ (equivalently, as $\gamma \to \infty$), the channel becomes transparent, no matter what the controller design is. Nevertheless, for low signal-to-noise ratios (i.e., small bit rates) improved performance is achieved by using the optimal controller studied in this paper. Indeed, for $b \in \{0,1\}$ the minimum variance controller does not satisfy (7) and cannot be used in the networked situation. It is also worth noting that the agreement[6] between simulation data and our theoretical analysis is remarkable for

[5] For every $b \in \{0,\cdots,8\}$, the results correspond to the average of 200 simulations (with different reference realizations and considering 10^5 samples).

[6] Recall that the bit rate limited channel is not strictly signal-to-noise ratio constrained.

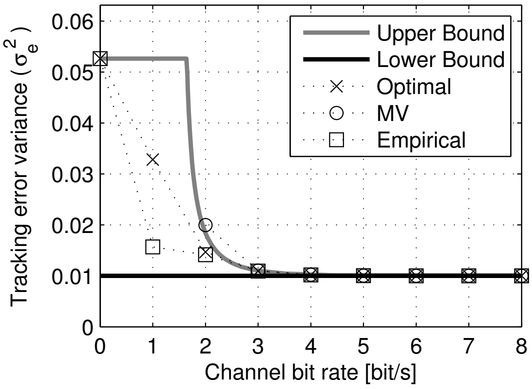

Fig. 2. Tracking error variance as a function of the channel bit rate.

$b \geq 2$ (there is a 3.8% mismatch for $b = 2$ and less than 0.4% mismatch for $b \geq 3$). For $b = 1$, the mismatch is about 54%. However, the qualitative behavior of the loop is as predicted by our analysis.

8. CONCLUSION

This paper has studied the optimal design of scalar NCS's where the communication channel has a signal-to-noise ratio constraint. In particular, we have derived a closed form expression for the optimal controller and have characterized the set of admissible signal-to-noise ratios and associated performance specifications. An interesting direction for future work would be to combine the results in this paper with those in Goodwin et al. (2008). This would provide optimal NCS designs that exploit the possibility of simultaneously synthesizing controllers and coding systems. Another extension would be the consideration of unreliable signal-to-noise ratio constrained channels, as those studied in, e.g., Silva et al. (2007a).

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