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# On Convergence Rate of Second-Order Sliding Mode Control Algorithms

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**Abstract:** Transient processes in the systems controlled by second-order sliding mode (SOSM) algorithms are analyzed in the frequency domain. A methodology of the frequency-domain analysis of transient processes that feature a decaying oscillation of variable frequency is developed. A simple criterion of the existence of finite-time convergence is proposed. It is shown that the convergence rate in a system controlled by a relay or SOSM controller depends on the angle between the high-frequency asymptote of the Nyquist plot of the plant and the low-amplitude asymptote of the negative reciprocal of the describing function of the controller, which is called the *phase deficit*. Examples of analysis are given.

## 1. INTRODUCTION

The problems of convergence rate in the analysis of conventional sliding-mode (SM) and second-order sliding mode (SOSM) control systems are of great importance. A lot of efforts are required to provide a proof that a certain algorithm can provide a finite-time convergence to be legitimately called a SOSM. There are a number of SOSM algorithms available now, which have advantages and disadvantages with respect to each other. The most popular are "twisting", "super-twisting", "twisting-as-a-filter" (Levant, 1993; 2001), "sub-optimal" (Bartolini *et al.*, 1998; 1999), and a number of other algorithms (Shtessel *et al.*, 2003). The problem of convergence rate is a valid problem in the conventional SM control and "terminal SM" (Man *et al.*, 1994; Yu *et al.*, 1999) control too. Therefore, some common approach to the problem of the convergence rate assessment, including qualitative (finite-time or asymptotic) and quantitative assessment, is of high importance.

The frequency-domain approach that also involves the describing function (DF) method (Atherton, 1975) would provide a number of advantages over the direct solution/estimates of the system differential equations. The first and most important one would be the unification of the treatment of all the algorithms based on some frequency-domain characteristics. This in turn may lead to the formulation of some criteria that should be satisfied for a SOSM algorithm to provide a finite-time convergence, and relatively simple rules that would allow one to develop new SOSM algorithms.

The paper is devoted to that problem and organized as follows. At first a system comprising a second-order plant and an asymptotic SOSM (relay) controller is analysed with the use of the approach proposed. The instantaneous frequency and amplitude of the oscillations as functions of time are derived. Then a system comprising the twisting SOSM controller and a second-order plant is analysed via the proposed approach. Again, the frequency and the amplitude of the oscillations are found as functions of time. Finally, an approach to analysis of the type of convergence based on the frequency-domain characteristics is considered.

## 2. ANALYSIS OF CONVERGENCE RATE – FREQUENCY-DOMAIN APPROACH

Carry out a frequency-domain analysis of the transient process in the asymptotic second-order SM controlled system, which is the relay feedback system that has the plant of relative degree two (in particular, it is a second-order plant). The time-domain analysis of such system was done by Anosov (1959). Let the system be given as follows:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}, \qquad (1)$$

$$u = -c \cdot \operatorname{sgn} y , \qquad (2)$$

where the linear part is given by (1), and the relay nonlinearity is given by (2). We shall consider only the case of the second-order system with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$, $\mathbf{B} = [0 \quad b_2]^{\mathrm{T}}$, $\mathbf{C} = [1 \quad 0]$, $a_1 \ge 0, a_2 > 0, b_2 > 0$. We shall also use the transfer function of the linear part determined as follows: $W_l(s) = b_2 /(s^2 + a_2 s + a_1)$.

Apply the describing function method (DF) in the following modified form to analysis of system (1), (2). Assume that the linear part is a low-pass filter, and $y(t)$ is a damped oscillation of variable frequency, so that $a(t)$ is the instantaneous amplitude and $\Omega(t)$ is the instantaneous frequency of oscillations of $y(t)$. Replace the nonlinearity in equation (2) with its DF:

$$u = N(a) \cdot y , \qquad (3)$$

where $N(a) = 4c/(\pi a)$ (4)

is the DF (Atherton, 1975). Obtain the transfer function of the closed-loop system (1), (3) using the DF (4):

$$W_{cl}(s) = \frac{N(a)b_2}{s^2 + a_2 s + a_1 + N(a)b_2}.$$

The characteristic equation of the closed-loop system is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0, \tag{5}$$

where $\omega_n = \sqrt{a_1 + N(a)b_2}$, $\xi = 0.5a_2 / \sqrt{a_1 + N(a)b_2}$. Assuming that $\xi < 1$ (which always holds at least starting from a certain instant – due to the growth of $N(a)$) we can write an analytical solution. Introduce the notions of *instantaneous amplitude*, *instantaneous frequency* and *instantaneous phase angle*. We shall consider that for every time *t* the solution of (5) provides instantaneous values of the parameters of the oscillations, so that the solution of (1), (3) can be written as follows:

$$y(t) = a_0\, e^{\sigma t} \sin \Psi(t), \tag{6}$$

where $\sigma = -\xi\omega_n = -0.5a_2$ is the decay constant, $\Psi(t)$ is the instantaneous phase, $\Psi(t) = \int_0^t \Omega(\tau)d\tau + \phi$, $\phi$ is selected to satisfy initial conditions, $\Omega = 0.5\sqrt{4(a_1 + N(a)b_2) - a_2^2}$ is the instantaneous frequency, $a_0$ is the initial amplitude, and

$$a(t) = a_0\, e^{\sigma t} = a_0\, e^{-0.5a_2 t} \tag{7}$$

is the instantaneous amplitude.

It follows from formula (7) that the amplitude of the oscillation decreases exponentially, with constant decay $\sigma$. Therefore, *an asymptotic convergence of* $y(t)$ *to zero takes place*, because the oscillations have non-zero amplitude at any finite time. The frequency of the oscillations grows according to the following formula:

$$\Omega(t) = 0.5\sqrt{4\left(a_1 + \frac{4cb_2}{\pi a(t)}\right) - a_2^2}. \tag{8}$$

An example of analysis of the system with the linear plant $W_l(s) = 1/(s^2 + s + 1)$ is given in Fig. 1. The theoretical analysis is given along with the simulations. The simulations show that the theoretical value of the decay is a little smaller than the actual one, which is a result of using the approximate DF method. On the other hand, one can see that the relationship between the frequency and the amplitude of the oscillations (formula (8)) gives quite a satisfactory result, so that if the damping value is slightly adjusted then the match between the analytical results and the simulations becomes very good (see Fig. 1).

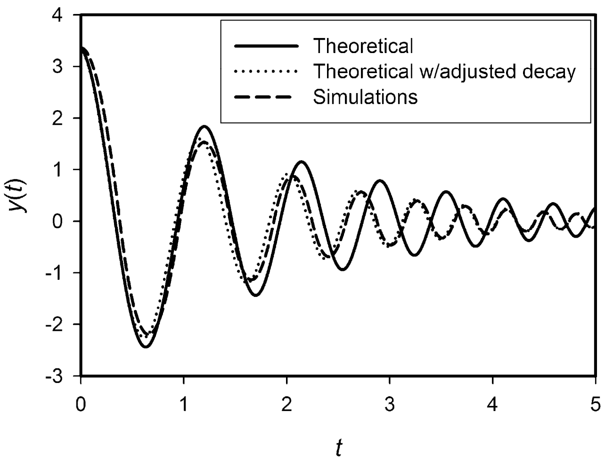


Fig. 1. Example of analysis of asymptotic SOSM controlled system

Now let us carry out similar analysis for a SOSM control system. Assume as before that the linear part is given by (1). Let the controller be the "twisting" SOSM controller (Levant, 1993) given as follows:

$$u = -c_1 \cdot \operatorname{sgn} y - c_2 \cdot \operatorname{sgn} \dot{y}, \tag{9}$$

Apply the DF analysis to this system. Since the twisting algorithm includes two relay nonlinearities apply to them two describing functions – like in (Boiko *et al.*, 2004). For the first relay

$$N_1(a) = \frac{4c_1}{\pi a}, \tag{10}$$

and for the second relay

$$N_2(a^*) = \frac{4c_2}{\pi a^*}, \tag{11}$$

where $a^*$ is the instantaneous amplitude of $\dot{y}(t)$, which still needs to be obtained. In the case of the twisting SOSM controller, the decay will not be constant any longer (it is shown below). For that reason, a different representation of the system output is used. We shall consider the following output formula:

$$y(t) = a(t) \sin \Psi(t), \tag{12}$$

where the instantaneous phase $\Psi(t)$ is given by the same formula as above, $a(t)$ is the instantaneous amplitude. Find amplitude $a^*$ via differentiating (12).

$$\dot{y}(t) = a(t)[\sigma(t)\sin\Psi(t) + \Omega(t)\cos\Psi(t)], \tag{13}$$

where $\sigma(t)$ and $\Omega(t)$ are instantaneous decay and frequency respectively. Therefore, $a^* = a\sqrt{\sigma^2(t) + \Omega^2(t)}$, and the DF for the second relay can be rewritten as

$$N_2(a) = \frac{4c_2}{\pi a\sqrt{\sigma^2 + \Omega^2}}\,. \tag{14}$$

In the same way as for the asymptotic SOSM controlled system, write the formula for the closed-loop system for the case of the twisting controller

$$W_{cl}(s) = \frac{(N_1(a) + sN_2(a))b_2}{s^2 + (a_2 + N_2(a)b_2)s + a_1 + N_1(a)b_2}\,.$$

The characteristic equation of the closed-loop system is, therefore,

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = 0\,, \tag{15}$$

where $\xi = 0.5(a_2 + N_2(a)b_2)/\sqrt{a_1 + N_1(a)b_2}$, $\omega_n = \sqrt{a_1 + N_1(a)b_2}$. The decay is now time-varying:

$$\sigma(t) = -\xi\omega_n = -0.5(a_2 + N_2(a)b_2) \tag{16}$$

and (16) denotes an instantaneous value. Since the physical meaning of formula (16) is oscillations decay, it provides the instantaneous rate of the amplitude change, and is not related to the initial amplitude (unlike formula (7)): $\sigma(t) = \dot{a}(t)/a(t)$. The instantaneous amplitude can be found via solving the following differential equation:

$$\dot{a}(t) = a(t)\sigma(t), \quad a(0) = a_0\,. \tag{17}$$

The formulas for the instantaneous decay and instantaneous frequency are as follows.

$$\sigma(t) = -0.5\left(a_2 + \frac{4c_2b_2}{\pi a(t)\sqrt{\sigma^2(t) + \Omega^2(t)}}\right), \tag{18}$$

$$\Omega(t) = 0.5\sqrt{4\left(a_1 + \frac{4c_1b_2}{\pi a(t)}\right) - \left(a_2 + \frac{4c_2b_2}{\pi a(t)\sqrt{\sigma^2(t) + \Omega^2(t)}}\right)^2}\,. \tag{19}$$

The formulas for the instantaneous amplitude (17), instantaneous decay (18) and instantaneous frequency (19) make a set of one differential and two algebraic equations. The proposed solution algorithm is as follows. Express $\Omega$ from (18):

$$\Omega = \sqrt{\frac{16c_2{}^2b_2{}^2}{\pi^2a^2(2\sigma + a_2)^2} - \sigma^2} \tag{20}$$

and substitute the expression in formula (19). Solve this equation for $\sigma$.

$$\sigma = -\frac{2c_2b_2}{\sqrt{\pi^2a^2a_1 + 4\pi ac_1b_2}} - \frac{a_2}{2} \tag{21}$$

Substitution of (21) in (17) yields the following differential equation for $a(t)$.

$$\dot{a} = -\frac{2c_2b_2a}{\sqrt{\pi^2a^2a_1 + 4\pi ac_1b_2}} - \frac{a_2}{2}a, \quad a(0) = a_0\,. \tag{22}$$

Formula (22) is a first-order nonlinear differential equation of the type:

$$\dot{z} = -\lambda z - g(z), \quad z(0) = z_0 > 0\,, \tag{23}$$

where $g(z) = \dfrac{\alpha}{\sqrt{1 + \beta/z}}$, $\alpha = \dfrac{2c_2b_2}{\pi\sqrt{a_1}}$, $\beta = \dfrac{4c_1b_2}{\pi a_1}$, $\lambda = \dfrac{a_2}{2}$, $z=a$. The nonlinear function $g(z)$ has infinite derivative at $z = 0$, which makes the finite-time convergence of the process given by (22) possible. Prove it and assess the convergence time. At first prove an auxiliary lemma

*Lemma 1.* For the first-order nonlinear differential equation

$$\dot{z} = -g(z)\,, \tag{24}$$

where $g(z) > 0$ for all $z > 0$, and $g(0) = 0$, and the initial condition $z(0) = z_0 > 0$ the following holds. If the solution of equation (24) is $z(t)$, such that a finite-time convergence to zero takes place $z(T_g) = 0$, $z(t) \in [0; z_0]$, and there exists a function $h(z)$ such that $h(z) \le g(z)$ for all $z \in [0; z_0]$, $h(z) > 0$ for all $z > 0$, and $h(0) = 0$, then the finite-time convergence to zero of the solution of equation

$$\dot{z} = -h(z) \tag{25}$$

takes place too, with the convergence time $T_h$ ($z(T_h) = 0$), so that $T_h \ge T_g$.

*Proof.* Transform equation (24) into an equation with $z$ being an independent variable and time being a dependent variable: $dt/dz = -1/g(z)$, from which the time can be found as $t(z) = -\int_{z_0}^{z} \frac{1}{g(z)}dz$, and the convergence time as:

$$T_g = -\int_{z_0}^{0} \frac{1}{g(z)}dz = \int_{0}^{z_0} \frac{1}{g(z)}dz\,.$$

Now since $h(z) \le g(z)$ and $g(z) > 0$, $h(z) > 0$ for all $z > 0$

$$T_h = -\int_{z_0}^{0} \frac{1}{h(z)}dz = \int_{0}^{z_0} \frac{1}{h(z)}dz \ge T_g \quad \blacksquare$$

*Theorem 1.* The process of conversion of the amplitude described by (22) from the initial value $a_0$ provides finite-time conversion with the conversion time not exceeding

$$T^* = \frac{2}{\lambda}\left( \ln\left( \lambda\sqrt{z_0} + \frac{\alpha}{\sqrt{z_0+\beta}} \right) - \ln\frac{\alpha}{\sqrt{z_0+\beta}} \right). \qquad (26)$$

*Proof.* Consider equation (23), which is a reformulated (22). Replace nonlinearity $g(z)$ in it with another nonlinearity $h(z)$ such that $h(z) \le g(z),\quad z \in [0; z_0]$, for which the finite-time conversion property can be (has been) proved. Select $h(z)$ to be $h(z) = \rho\sqrt{z},\ \ \rho > 0$. Select parameter $\rho = \frac{\alpha}{\sqrt{z_0+\beta}}$ to satisfy the requirement $h(z) \le g(z),\quad z \in [0; z_0]$. Also, note that $h(z_0) = g(z_0)$. Therefore, since $g^2(z) = \frac{\alpha^2 z}{z+\beta}$ and $h^2(z) = \frac{\alpha^2 z}{z_0+\beta}$, $g^2(z) \ge h^2(z)$ for all $z \in [0; z_0]$. Via the substitute $z_1 = \sqrt{z}$, and respectively $\dot z = 2z_1\dot z_1$, equation containing the square root function is transformed into a linear equation: $\dot z_1 = -0.5\rho z_1 - 0.5\lambda$, which has a solution $z_1(t) = -\frac{\rho}{\lambda}(1-e^{-0.5\lambda t}) + z_1(0)e^{-0.5\lambda t}$. By solving the equation $z_1(T^*) = 0$ find $T^*$ as given by (26). ■

The nonlinear functions $g(z)$ and $h(z)$ for $c_1 = 50$, $c_2 = 5$ and other parameters of the above example are presented in Fig. 2. The considered first-order system with the square root nonlinearity (assuming also symmetric properties of the square root function for negative $z$) is known as having a *terminal sliding mode* (or *power-fractional sliding mode*) (Man *et al.*, 1994; Yu *et al.*, 1999), which has finite-time convergence. Since $h(z) \le g(z),\quad z \in [0; z_0]$, according to Lemma 1, the system (23) provides a faster convergence than the system with the square root nonlinearity. Time $T^*$ serves as a higher estimate of the convergence time in system (22).

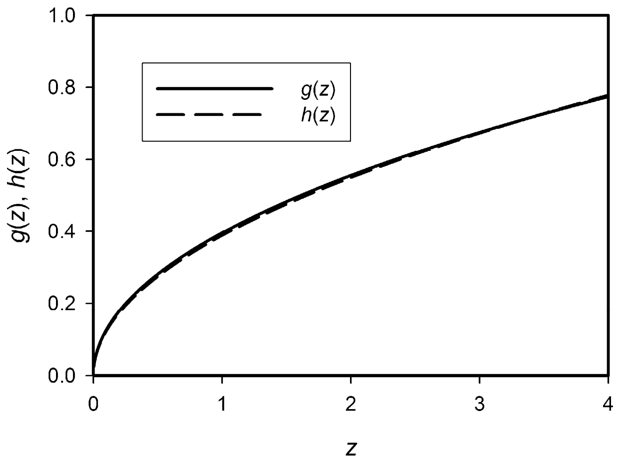


Fig. 2. Functions $g(z)$ and $h(z)$ of differential equation for amplitude

An example of analysis of the system with the linear plant $W_l(s) = 1/(s^2+s+1)$ and the twisting controller with $c_1 = 50$, $c_2 = 5$ is given in Fig. 3. The theoretical value of the higher estimate of the convergence time is $T^* = 4.85$, which is close to the theoretical convergence time due to closeness of functions $g(z)$ and $h(z)$ (Fig. 2). The theoretical analysis is given along with the simulations. The simulations show that the theoretical value of the decay is again a little smaller than the actual one. Yet, the proposed approach provides a good estimate of the SOSM transient dynamics.

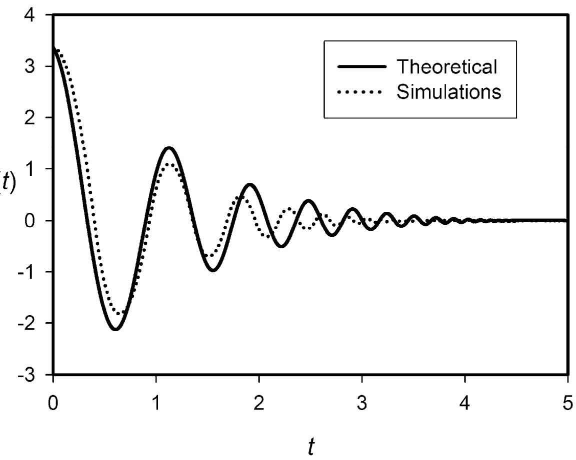


Fig. 3. Example of analysis of twisting SOSM controlled system

## 3. FREQUENCY-DOMAIN CHARACTERISTICS AND CONVERGENCE RATE

A method of analysis of decaying oscillations of growing frequency was proposed above. It is based on a modification of the DF method that involves consideration of instantaneous amplitude, frequency and decay rate instead of respective constant values. Now, with those methodology and results available, let us look at the problem of the existence of periodic motions, asymptotic decay and finite time convergence considering the harmonic balance in the system.

Consider application of the concepts of the DF method to analysis of possible periodic motions. Periodic motions can exist in the system if the Nyquist plot of the linear part $W(j\omega)$ intersects the negative reciprocal of the DF $-N^{-1}(a)$ (Fig. 4). In Fig. 4, two Nyquist plots corresponding to the second- $W_1(j\omega)$ and third-order $W_2(j\omega)$ linear parts and two negative reciprocal DF corresponding to the relay control $-N_1^{-1}(a)$ and to the twisting algorithm $-N_2^{-1}(a)$ (Boiko *et al.*, 2004) are depicted. Intersection of $W_2(j\omega)$ and either of the DFs provides a periodic solution (points A or B) of finite frequency and amplitude. Plot $W_1(j\omega)$ does not have any points of intersection with either $-N_1^{-1}(a)$ or $-N_2^{-1}(a)$

except the origin. However, the character of the process in the system is different – depending on whether the control is a conventional ideal relay (plot $-N_1^{-1}(a)$ ) or the SOSM control (plot $-N_2^{-1}(a)$ ). In the former case the convergence is asymptotic, in the second one – it is finite-time.

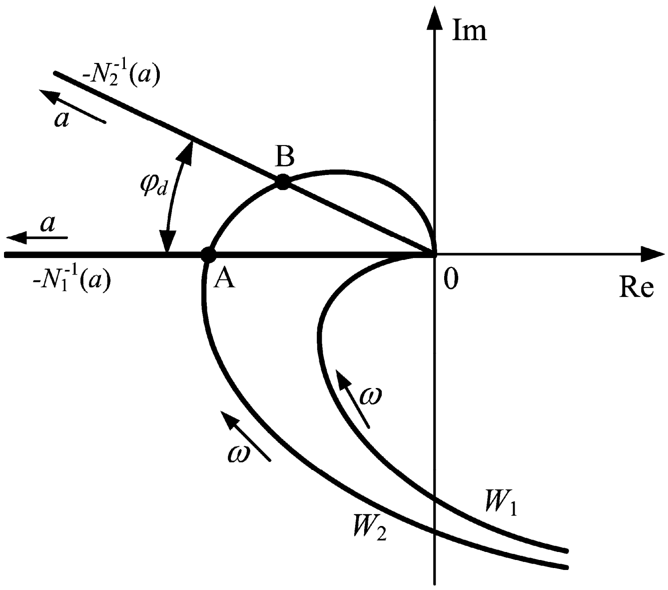


Fig. 4. Determination of periodic motions and decaying oscillations

Let us consider the condition of the phase balance that is a part of the harmonic balance condition. For a periodic motion to occur in the system the following condition must hold:

$$\varphi_l(\Omega) + \arg N(a) = -\pi , \quad (27)$$

where $\Omega$ is the frequency and $a$ is the amplitude of the self-excited periodic motion, $\varphi_l(\omega) = \arg W_l(j\omega)$ is the phase characteristic of the linear part. Considering linear part $W_1(j\omega)$ we should note that there is a significant difference between the controls with $-N_1^{-1}(a)$ and $-N_2^{-1}(a)$. In the first case, formally speaking, there is a frequency at which the phase balance condition (27) holds. This frequency is $\Omega = \infty$. Therefore, we might say that in the system with $W_1(j\omega)$ and $-N_1^{-1}(a)$, a periodic motion of infinite frequency occurs. As for the second option, a periodic motion cannot occur at any frequency (including $\Omega = \infty$ ). There is a condition that we shall further refer as a *phase deficit*. Quantitatively, let us call the *phase deficit* the minimum phase value that needs to be added (with the negative sign) to the phase characteristic of the linear part to make the phase balance condition hold at some frequency (including the case of $\Omega = \infty$ ). Note: we do not consider now the case of possibly non-monotone frequency characteristics. The *phase deficit* is depicted in Fig. 4 as $\varphi_d$. Therefore,

$$\varphi_l(\Omega) - \varphi_d + \arg N(a) = -\pi , \quad (28)$$

assuming that $\varphi_d \ge 0$ and $\arg N(a) \ge 0$ for SOSM.

Now consider controllers that include a nonlinearity with infinite derivative in zero. For this type of nonlinearity, the DF $N(a) \to \infty$ if $a \to 0$ and, therefore, $-N^{-1}(a) \to 0$ if $a \to 0$. Also, assume that $-N^{-1}(a)$ is a straight line on the complex plane (other types of $-N^{-1}(a)$ will be considered below). Formulate the following theorem.

*Theorem 2*. For the second-order linear part given by (1) and the controller containing at least one ideal relay function, and having the describing function $N(a)$ of the controller such that the ratio $\frac{\operatorname{Im} N(a)}{\operatorname{Re} N(a)} = \text{const}$ (the negative reciprocal DF of the controller is a straight line on the complex plane), the following three modes of oscillations can occur. A. A periodic motion occurs if the phase deficit value is negative. B. An oscillation having asymptotic convergence of the amplitude to zero (periodic process of infinite frequency and zero amplitude) occurs if the phase deficit value is zero. C. An oscillation having finite-time convergence of the amplitude to zero occurs if the phase deficit value is positive.

*Proof*. A. If the phase deficit is negative there always exists a point of intersection of the Nyquist plot of the linear part and of the negative reciprocal of the DF of the controller (follows from the definition of the phase deficit). Therefore, there is a solution of the harmonic balance equation (Atherton, 1975), and a self-excited periodic motion occurs.

C. It follows from the definition of the nonlinearity of the controller that

$$N(a) = \frac{k_1}{r(a)} + j\frac{k_2}{r(a)}, \quad (29)$$

where $k_1 > 0$, $k_2 > 0$ are constant coefficients, $r(a)$ is an increasing function of the amplitude $a$: $\frac{dr(a)}{da} > 0$ for all $a \in [0;\infty)$, such that $r(0) = 0$ (examples of this function can be $r(a) = a$, $r(a) = \sqrt{a}$, etc.). The negative reciprocal of (29) becomes

$$-N^{-1}(a) = -\frac{r(a)}{k_1^2 + k_2^2}(k_1 - jk_2)$$

Therefore, the *phase deficit* for this system is $\varphi_d = \arctan(k_2/k_1)$. Considering that $y(t) = a(t)\sin\Psi(t)$ and $\dot{y}(t) = a(t)[\sigma(t)\sin\Psi(t) + \Omega(t)\cos\Psi(t)]$ represent the response of the nonlinear controller to signal $y(t)$ as an expansion in the basis of functions $y(t)$, $\dot{y}(t)$ (weighted sum), the following holds:

$$u(t) \approx -(p_1 y(t) + \frac{p_2}{\Omega}\dot{y}(t))$$
$$= -a\left((p_1 + p_2\frac{\sigma}{\Omega})\sin\Psi + p_2\cos\Psi\right),$$

where the sign "-" is attributed to the negative feedback, the "approximate equality" is due to the use of the approximate DF method. (Note: that basis would be an orthogonal one if the decay were zero.) Weight $p_2 = k_2 / r(a)$; weight $p_1$ can be determined for a particular controller. It reduces to $p_1 = k_1 / r(a)$ when $\sigma = 0$. Therefore, the controller output can be represented as follows:

$$u(t) \approx -\left( p_1 + \frac{p_2}{\Omega} s \right) y(t),$$

where $s = \frac{d}{dt}$. Similar to (16), we can write the following formula for the instantaneous decay:

$$\sigma = -0.5\left( a_2 + \frac{p_2(a)}{\Omega} b_2 \right) = -0.5\left( a_2 + \frac{k_2}{r(a)\Omega} b_2 \right) \qquad (30)$$

and instantaneous frequency (similar to (18), (19)):

$$\Omega = 0.5\sqrt{4(a_1 + p_1(a)b_2) - \left( a_2 + \frac{k_2}{r(a)\Omega} b_2 \right)^2}.$$

As an auxiliary result, find the following from the last formula:

$$\lim_{a \to 0} r(a)\Omega = \lim_{a \to 0} 0.5 r(a)\sqrt{4(a_1 + p_1(a)b_2) - \left( a_2 + \frac{k_2}{r(a)\Omega} b_2 \right)^2} = 0,$$

considering that $\Omega \to \infty$ when $a \to 0$ and $r(0) = 0$. Therefore, considering the differential equation for the amplitude

$$\dot{a} = a\sigma = -\frac{k_2}{2r(a)\Omega} b_2 a - \frac{a_2}{2} a, \qquad (31)$$

one can see that the nonlinearity present in this equation is the one with $g(0) = 0$ and infinite derivative at $a = 0$:

$$g(a) = \frac{k_2 b_2 a}{2r(a)\Omega}, \quad \lim_{a \to 0} g(a) = 0 \ \text{(follows from } \frac{dr(a)}{da} > 0\text{)},$$

$$g'(a) = \frac{k_2 b_2}{2}\left( \frac{1}{r(a)\Omega} - \frac{ar'(a)}{r^2(a)\Omega} - \frac{a\, {d\Omega}/{da}}{r(a)\Omega^2} \right), \quad \lim_{a \to 0} g'(a) = \infty$$

(due to the first term in the brackets; considering also boundedness on the second term, and $d\Omega/da < 0$). These dynamics have a terminal sliding mode and finite convergence time (Yu *et. al.*, 1999). This completes the proof for option C.

B. For this option, coefficient $k_2$ in (29) is zero. As follows from (31) $\dot{a} = -0.5 a_2 a$, thus, providing exponential (asymptotic) convergence. ■

It follows from (26) and (31) that within the finite-time convergence option, the convergence time depends on $k_2$ and, therefore, on the *phase deficit* value. It also follows from (26) that $\lim_{c_2 \to 0} T^* = \infty$, i.e. when the *phase deficit* approaches *zero* the convergence time becomes infinite.

## 4. EXTENTION TO OTHER TYPES OF CONTROLLERS AND LINEAR PARTS

The relationship between the DF of the controller and the possibility of a particular mode of the transient process to occur was established above. However, this was done for a particular type of controllers – namely the one that satisfies the condition $\operatorname{Im} N(a)/\operatorname{Re} N(a) = \text{const}$. This is true for the twisting controller (Levant, 1993), the sub-optimal algorithm (Bartolini, 1998; 1999), and possibly some other controllers/algorithm that can be designed in the future. Yet, it is not a condition that is always satisfied. However, the fact of finite-time convergence depends on the shape of $-N^{-1}(a)$ only in the vicinity of the origin (in the complex plane), so that if the process starts from a certain amplitude, only the amplitudes in the range from the initial one to zero will be attained. Furthermore, if the process starts at some finite amplitude the time over which it reaches another smaller finite amplitude will always be finite. Therefore, to establish the fact of finite time convergence one needs to analyse only a very small vicinity of the origin, and consequently, what is important in that respect is the location of the low amplitude asymptote of the plot $-N^{-1}(a)$.

Let us reformulate the definition of the *phase deficit*. Let us define the *phase deficit* as the minimum phase value that needs to be added (with the negative sign) to the phase characteristic of the linear part to make the high-frequency asymptote of the Nyquist plot of the linear part to coincide with the low-amplitude asymptote of the negative reciprocal DF of the controller. The noted property can also be extended to linear parts of higher order and relative degree two. Due to the effect of the "order collapse" a higher-order system would exhibit the properties of the corresponding second-order system at high enough frequencies. A rigorous proof can be developed with the methodology presented above.

Let us illustrate this statement with analysis of two different controllers. Let the linear part be the one that was considered in the above examples, and the control be

$$u = -c_1 \cdot \operatorname{sgn} y - c_2 \cdot f(\dot{y}), \qquad (32)$$

with nonlinear function $f(\dot{y})$ being in the first case $f(\dot{y}) = f_I(\dot{y}) = \operatorname{sgn}(\dot{y})$ and in the second case $f(\dot{y}) = f_{II}(\dot{y}) = \operatorname{sgn}(\dot{y}) \cdot (\dot{y})^2$. The DF formula for the second nonlinearity is:

$$N_{II}(a) = \frac{2}{\pi a} \int_0^{\pi} (a \sin\theta)^2 \sin\theta d\theta = \frac{2a}{3\pi}.$$

Also, the DF of the relay function is given by formula (4). Therefore, the DF of the first controller is

$$N_1(a) = \frac{4c_1}{\pi a} + j\frac{4c_2}{\pi a},$$

and of the second controller is

$$N_2(a) = \frac{4c_1}{\pi a} + j\frac{2ac_2}{3\pi}.$$

Select $c_1 = 10$, $c_2 = 2$ and compute and plot the negative reciprocal functions of $N_1(a)$ and $N_2(a)$ (Fig. 5).

Indeed, the system with controller having the DF $N_1(a)$ reveals a finite time convergence, while the controller with DF $N_2(a)$ shows asymptotic convergence (Fig. 6). A zoomed image shows the process after 4.5s, which reveals the type of convergence.

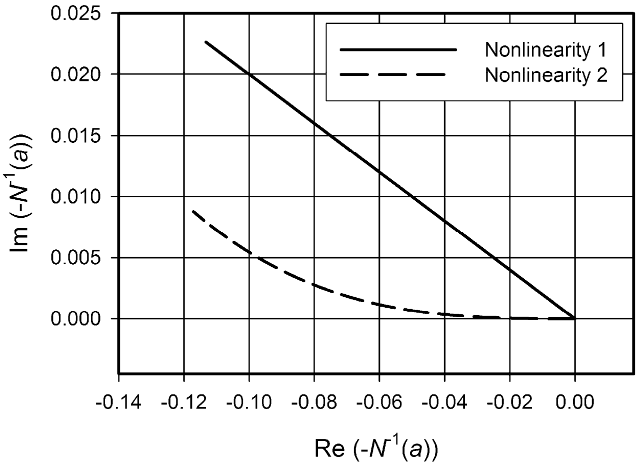


Fig. 5. Negative reciprocal of $N_1(a)$ and $N_2(a)$

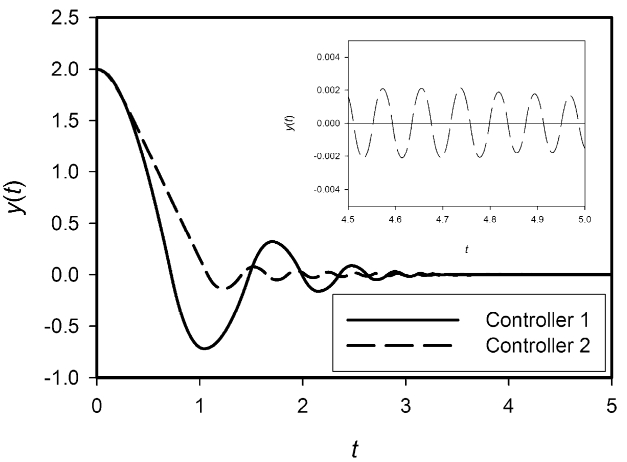


Fig. 6. Transients in the system with controllers 1 and 2; zoomed picture for time 4.5s-5.0s

## 5. CONCLUSIONS

Therefore, a frequency-domain method of analysis of transient oscillatory processes has been developed. The proposed method involves modification of the describing function methodology of analysis and consideration of the instantaneous values of the frequency, amplitude and decay of the oscillations. The time evolution of those variables can be assessed via integration of the instantaneous values.

The proposed method of analysis leads to a simple criterion of the existence of a finite-time or asymptotic conversion, which involves just one characteristic – the *phase deficit*. The proposed methodology can be extended to higher-order linear plants and can be used for analysis of all available SOSM algorithms, as well as for the development of new ones.

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