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Set membership state estimation for nonlinear systems using contraction theory

Gaétan Videau, Tarek Raïssi, Ali Zolghadri

Université Bordeaux 1, IMS
351 cours de la Libération, 33405 TALENCE *cedex,* FRANCE

Abstract: This paper deals with guaranteed state estimation for nonlinear continuous-time systems. Interval observers are a powerful tool to compute the bounds of the state vector by propagating the uncertainties on the initial state and the parameter vectors and the errors on the measurements. Nevertheless, a widely recognized drawback of interval analyis-based observers is the overestimation due to dependence and wrapping effects. In this paper, contraction theory is used as an alternative in order to reduce the pessimism induced by interval analysis. The proposed interval observer is based on the Luenberger approach where the observation gain is chosen in order to guarantee stability and contraction properties. The methodology is illustrated with simulations on a numerical example.

Keywords: Interval observers, continuous-time systems, nonlinear, bounded errors, contraction.

1. INTRODUCTION

State and parameter estimation problems are usually solved by probabilistic methods, which are relevant when explicit characterizations of the measurement noise and the state perturbations are available. In many practical cases, it is more natural to assume the perturbations belonging to a bounded set but without any stochastic information. In this context, the set of all state and parameter vectors that are consistent with the measured data, a model structure and prior bounds could be characterized.

This paper is devoted to the reliable state estimation of a large class of nonlinear continuous-time systems in a bounded-error context. The underlying systems are described by ordinary differential equations (ODE) where the initial state and a set of parameters are not well-known as it would be the case for many practical applications.

Consider a system described by the following equations:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{p}, t) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{p}, t, \mathbf{w}(t)) \\ \mathbf{x}(t_0) &\in [\mathbf{x}_0] \\ \mathbf{p} &\in [\mathbf{p}] \end{cases} \tag{1}$$

where $\mathbf{f} \in \mathcal{C}^{k-1}(\mathcal{D})$, (the value of k will be discussed later), $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set, n and m are respectively the dimension of the state vector $\mathbf{x}$ and the output vector $\mathbf{y}$. The functions $\mathbf{f}$ and $\mathbf{g}$ are possibly nonlinear. The initial state $\mathbf{x}(t_0)$ is assumed to belong to a prior known box $[\mathbf{x}_0]$. Assume in addition that the measurements $\mathbf{y}_j^m$ are available at the sampling times $t_j \in \{t_1, \dots, t_N\}$.

The measurement noise is assumed additive, bounded but otherwise unknown. The prior domain for the output vector $\mathbf{y}_j$ is the box $[\mathbf{y}_j] = [\mathbf{y}_j^m - \mathbf{e}, \mathbf{y}_j^m + \mathbf{e}]$, where $\mathbf{e}$ represents the set of perturbations sources not taken into account in the model.

The goal is to characterize the set of all the state vectors, solutions of the state equation (ODE), that are consistent with the available measurements and *a priori* errors. In bounded error context, state estimation for nonlinear discrete-time systems has already been studied using set inversion Kieffer et al. [2002], Magnus et al. [2000], Schweppe [1968]. Also, an extension of Kalman filtering to intervals has been introduced in Guanrong et al. [1997], but the domain computed by the latter estimator is not guaranteed to contain the actual value of the state.

To the best of our knowledge, state estimation for nonlinear continuous-time systems in the bounded error context was first introduced in Jaulin [2002] where a state estimator based on the first order enclosure of the state equation has been proposed. Since such enclosures are very pessimistic, it is necessary to partition the state vector into small boxes at each step time in order to control the pessimism. Such a procedure is computationally time-consuming when the dimension of the state vector is high. In Raïssi et al. [2004, 2006], it is shown that pessimism can be controlled without performing state vector partitions by using high order interval Taylor models. Neverthless, the proposed method is efficient only when uncertainty on the state and parameter vectors remains small.

In order to overcome overestimation induced by interval analysis, the authors of Gouzé et al. [2000], Walter and Kieffer [2003], Kieffer and Walter [2006] propose to use the theory of cooperative systems Smith [1995] in order to bracket the whole state of the uncertain system between a lower and an upper deterministic systems, i.e. involving no uncertainty. These two systems permit to compute a minimal outer approximation of the state, i.e. the smallest box containing the actual solution. In the linear case, it is usually possible to use this property even if the model is not cooperative, the observer gain can be selected in order to have cooperativity. In the nonlinear case, it is not always possible to choose an observer gain such that the observer verifies cooperativity.

Based on contraction theory, it will be shown in this paper that an enclosure of the state vector can be computed for a large class of nonlinear systems without using interval analysis even for non cooperative systems.

The paper is structured as follows. In section 2, previous works on guaranteed state estimation are briefly reviewed. Section 3 describes the proposed methodology based on contraction theory, which can be seen as an alternative to the classical Taylor methods. Finally, in section 4, the state estimation algorithms are illustrated on a simulation example.

2. STATE OF THE ART

The estimators proposed in Jaulin [2002], Raïssi et al. [2004], Walter and Kieffer [2003], Kletting et al. [2006] are based on the prediction/correction approach, as in the Kalman filter. The prediction step consists on computing at t_{j+1} the set $\mathcal{X}_{t_{j+1}/t_j}$ of state vectors that are consistent with the state enclosure $\mathcal{X}_{t_j/t_j}$. The correction allows us to contract $\mathcal{X}_{t_{j+1}/t_j}$ by using the information provided by the measurement $\mathbf{y}^m(t_j+1)$.

2.1 Prediction

Prediction allows us to compute $\mathcal{X}_{t_{j+1}/t_j}$ containing the reachable set of the state at t_{j+1} with an initial condition given by $\mathcal{X}_{t_j/t_j}$ at t_j. It consists in solving the equation (2) in a guaranteed way.

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{p}, t) \\ \mathbf{x}(t_0) &\in [\mathbf{x}_0] \\ \mathbf{p} &\in [\mathbf{p}] \end{cases} \tag{2}$$

In this step, uncertainties on initial state and parameters are propagated and the measurements at t_j are not taken into account. In the literature, two methods are proposed to perform prediction. The first one Jaulin [2002], Raïssi et al. [2004], Kletting et al. [2006] is based on a validated integration of the ODE (2) which is performed by using an extension of Taylor expansions to intervals (for more details see Nedialkov [1999] and the references therein). The second is based on cooperative theory Gouzé et al. [2000], Walter and Kieffer [2003], Kieffer and Walter [2006].

Interval Taylor expansions

Consider ODE (2). Using Taylor theorem, one can compute an approximation of the solution of the ODE at t_{j+1} when $\mathbf{x}_j$ and $\mathbf{p}$ are perfectly known.

$$\mathbf{x}_{j+1} = \mathbf{x}_j + \sum_{i=1}^{k-1} h^i \mathbf{f}^{[i]}(\mathbf{x}_j) + h^k \mathbf{f}^{[k]}(\mathbf{x}(t_\xi)) \tag{3}$$

with $t_\xi \in [t_j, t_{j+1}]$, h_j is the step time which is not necesseraly constant and $\mathbf{f}^{[i]}$, $\{i = 1 \dots k\}$, are Taylor coefficients which can be recursively computed by using automatic differentiation Rall [1979]. For this approach, $\mathbf{f} \in \mathcal{C}^{k-1}(\mathbb{R}^n)$ where k is a high number. When the initial state $\mathbf{x}_0$, and hence $\mathbf{x}_j$, are not exactly known, Taylor expansions should be extended to intervals, for more details on interval analysis see Moore [1966], Neumaier [1990], Hansen [2004]. An inclusion function of (3) is then needed. It is easy to prove that if a box $[\xi]$ containing all the state trajectory for $t \in]t_j, t_{j+1}[$ is known, then the interval vector

$$[\mathbf{x}_{j+1}] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} h_j^i \mathbf{f}^{[i]}([\mathbf{x}_j]) + h_j^k \mathbf{f}^{[k]}([\xi]) \tag{4}$$

is guaranteed to contain the state at t_{j+1}. Several methods can be used to compute a set $[\xi]$; the most used is based on Picard-Lindelöf operator and the fixed point theorem Nedialkov [1999]. Actually, the numerical scheme (4) is improved by some matrices factorizations (see for instance Nedialkov [1999] and the references therein). The methods proposed in these references are efficient only when uncertainties on initial state and on parameters are not large. If this is not the case, this interval-based method is not efficient as the numerical evaluation of the solution of the state equation becomes pessimistic Raïssi et al. [2006].

For a particular class of dynamical systems, many authors Walter and Kieffer [2003], Kieffer and Walter [2006], Gouzé et al. [2000] used the monotone systems theory Smith [1995] in order to bracket the whole state flow of the uncertain systems between a lower and an upper deterministic systems (without any uncertainty). These two systems lead to an outer approximation of the state at time t_{j+1} without any numerical conservatism. In the following section, the main properties and definitions of monotone systems are recalled.

Dynamical monotone systems

Definition 1. A dynamical system is called monotone if ordered initial states lead to ordered states at any time t_j with the same order relation.

The monotony property was introduced in 1920^{th} years in the context of ordinary differential equations Müller [1920], for more details see Smith [1995].

Definition 2. A dynamical system is cooperative over a compact set $\mathcal{D}$, if all off-diagonal elements of the jacobian matrix of the state equation are positive, i.e

$$\forall i \neq j, t \geq 0, \mathbf{x} \in \mathcal{D}, \frac{\partial f_i(\mathbf{x}, t)}{\partial x_j} \geq 0 \tag{5}$$

Remark 1. If a dynamical system is cooperative, then it is monotone.

Theorem 3. Smith [1995], Walter and Kieffer [2003], Kieffer and Walter [2006], Gouzé et al. [2000] Consider two cooperative systems

$$\begin{cases} \dot{\overline{\mathbf{x}}} = \overline{\mathbf{f}}(\overline{\mathbf{x}}, t) \\ \dot{\underline{\mathbf{x}}} = \underline{\mathbf{f}}(\underline{\mathbf{x}}, t) \end{cases} \tag{6}$$

which satisfy the conditions

$$\begin{cases} \forall \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}], \forall \mathbf{x} \in \mathbb{D}, \forall t \in [t_0, T] \\ \underline{\mathbf{f}}(\mathbf{x}, t) \leq \mathbf{f}(\mathbf{x}, \mathbf{p}, t) \leq \overline{\mathbf{f}}(\mathbf{x}, t) \end{cases} \tag{7}$$

In addition, if some initial conditions verify

$$\forall \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}], \underline{\mathbf{x}}(t_0) \leq \mathbf{x}(t_0) \leq \overline{\mathbf{x}}(t_0) \tag{8}$$

then the solution of (2) remains in the range

$$\mathcal{X} : \{\underline{\mathbf{x}}(t) \leq \mathbf{x}(t) \leq \overline{\mathbf{x}}(t), \ t_0 \leq t \leq T\} \tag{9}$$

The monotony property can be used to compute minimal outer approximation of the solution of the state ordinary differential equations, i.e. without any numerical conservatism. Nevertheless, it is not always possible to build the bracketing systems when the system under study is not cooperative.

2.2 Correction

In the correction step, a guaranteed enclosure $\mathcal{X}_{j+1/j+1}$ of the state is reconstructed using the measurement $\mathbf{y}_{j+1}^{m}$ and intersected with the set $\mathcal{X}_{j+1/j}$ computed in the prediction. This phase is performed by the inversion of the measurement $\mathbf{y}_{j+1}^{m}$ using the equation

$$\mathbf{x} = \mathbf{g}^{-1}(\mathbf{y}, \mathbf{p}, t) \tag{10}$$

The inversion of the equation (10) is a set inversion problem. In the case of a nonlinear measurement equation, it would not be possible to obtain an explicit solution of (10). In such a case, a guaranteed outer approximation should be computed by application of a contractor which can be, for instance, an extension of Newton method to intervals Neumaier [1990], Hansen [2004], Jaulin et al. [2001]. In this paper, contractors will not be recalled, the interested reader can refer to Hansen [2004] which contains an extensive study of these tools.

3. INTERVAL OBSERVERS BASED ON CONTRACTION ANALYSIS

3.1 Contraction theory

The aim of this approach is to ensure the convergence of the inner and the outer bounds of the state estimate to the actual state trajectory. The following analysis is based on the contraction theory Lohmiller and Slotine [1998]; the proofs of the theorems can be found in Lohmiller and Slotine [1998].

Consider a system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{11}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{f}$ may be nonlinear but continuously differentiable (at least $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^n)$). The differentiation of equation (11) yields the exact differential relation

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t)\delta\mathbf{x} \tag{12}$$

where $\delta\mathbf{x}$ is a virtual displacement between two neighboring trajectories.

If $\lambda_{max}(\mathbf{x}, t)$ is the largest eigenvalue of the symmetric part of the Jacobien $\partial\mathbf{f}/\partial\mathbf{x}$ (i.e. $\frac{1}{2}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T\right)$), then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{x}^T\delta\mathbf{x}) \leq 2\lambda_{max}\delta\mathbf{x}^T\mathbf{x}$$

hence,

$$\| \delta\mathbf{x} \| \leq \| \delta\mathbf{x}_0 \| \, e^{\int_0^t \lambda_{max}(\mathbf{x},t)\mathrm{d}t}. \tag{13}$$

where $\| \, . \, \|$ is a norm on $\mathbb{R}^n$. If $\lambda_{max}(\mathbf{x}, t)$ is uniformly strictly negative, then, from equation (13), any infinitesimal length $\| \delta\mathbf{x} \|$ converges exponentially to zero, and the system is called contracting.

Definition 4. Given the equation (11), a region of the state space is called a contraction region if the Jacobian $\partial\mathbf{f}/\partial\mathbf{x}$ is uniformly negative definite in that region.

By $\partial\mathbf{f}/\partial\mathbf{x}$ uniformly negative we mean that:

$$\exists\beta > 0, \ \forall\mathbf{x}, \ \forall t \geq 0, \ \frac{1}{2}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T\right) \leq -\beta\mathbf{I} < 0 \tag{14}$$

Theorem 5. Given the equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory, which starts in a ball of constant radius centered about a given trajectory and contained at all time in a contraction region, remains in that ball and converges exponentially to this trajectory.

Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region. The proof of this theorem is in Lohmiller and Slotine [1998].

More generally, contraction can be studied with a coordinate transformation

$$\delta\mathbf{z} = \Theta\delta\mathbf{x} \tag{15}$$

where $\Theta(\mathbf{x}, t)$ is a uniformly invertible square matrix.

Using the transformation (15) in (12), we obtain

$$\delta\dot{\mathbf{z}} = \left(\dot{\Theta} + \Theta\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\Theta^{-1}\delta\mathbf{z} \tag{16}$$

Hence, the exponential convergence of $\| \delta\mathbf{z} \|$ to zero is guaranteed if the *generalized Jacobian* matrix

$$\mathbf{F} = \left(\dot{\Theta} + \Theta\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\Theta^{-1} \tag{17}$$

is uniformly negative definite. This implies that all the solutions of the system (11) converge exponentially to a single trajectory independently of the initial state in the contraction domain.

By convention, if the system (11) is contracting, $\mathbf{f}(\mathbf{x}, t)$ is called a contracting function. The absolute value of the largest eigenvalue of the symmetric part of $\mathbf{F}$ is called the system's contraction rate with respect to the uniformly positive definite metric $\mathbf{M} = \Theta^T\Theta$. Note that for a globally contracting autonomuous system, all trajectories converge exponentially to a unique equilibrium point Lohmiller and Slotine [1998].

3.2 Set-membership observers

Consider a nonlinear system described by:

$$\mathcal{S}_1 \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), t, \mathbf{w}(t)) \\ \mathbf{x}(t_0) \in [\mathbf{x}_0] \end{cases} \tag{18}$$

The initial state $\mathbf{x}_0$ is assumed to belong to a prior known ball $\mathcal{B}_0 = \{\hat{\mathbf{x}}_0, R_0\}$.

The proposed observer is based on the classical Luenberger approach, it is given by:

$$\mathcal{O}_1 : \begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), t) + \mathbf{k}(\mathbf{y}(t) - \mathbf{g}(\hat{\mathbf{x}}(t), t)) \\ \hat{\mathbf{y}}(t) = \mathbf{g}(\hat{\mathbf{x}}(t), t) \end{cases} \tag{19}$$

where the matrix gain $\mathbf{k}$ is chosen in order to satisfy the condition:

$$\forall \mathbf{x} \in \mathcal{D}, \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \mathbf{k} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right) < \lambda_{max} I < 0 \tag{20}$$

λ_{max} is the maximum eigenvalue of the symmetric part of $\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \mathbf{k} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right)$. Several studies attempt to compute in a guaranteed way λ_{max} Oishi [2001]. Here, genetic algorithms techniques are used to solve the above problem, for which software packages are available Houck et al. [1995]. They are supposed to give global results, but without any garantee. In a future work, global optimization with interval analysis will be used.

For the sake of simplicity, in this paper a constant gain $\mathbf{k}$ is used. Nevertheless, the authors of Lohmiller and Slotine [1998] have proposed a method based on a nonlinear gain. Nevertheless, the proposed methodology in the latter paper is applicable for academic systems but appears to be very complex to be implemented for high dimensional systems.

The observer (19) permits to compute a punctual estimate at t_j of the center of a ball $\mathcal{B}_j = \{\hat{\mathbf{x}}_j, R_j\}$ which is guaranteed to contain all the state vectors that are consistent with the measurements available up to t_j and with the uncertainty on the initial state. The radius R_j is computed by

$$R_j = R_0 \exp^{\int_{t_0}^{t_j} \lambda_{max} dt} \tag{21}$$

The uncertainty on each state component x_j^i at t_j is obtained by the projection of the ball $\mathcal{B}_j$ on the axe i. Hence, the domain of x_j^i is given by the interval

$$x_j^i \in \left[\hat{x}_j^i - R_j, \hat{x}_j^i + R_j\right] \tag{22}$$

The condition (20) ensures that the uncertainty on the state vector converges to 0 if there is no uncertainty on the state and on the measurement equations. The convergence rate is given by λ_{max}.

Assume now that the measurement error is bounded with a prior known bound $\mathbf{e}$, hence the domain of the output is $[\mathbf{y}^m - \mathbf{e}, \mathbf{y}^m + \mathbf{e}]$. To take into account the uncertainties on the measured data, we propose to use an extended observer which makes it possible to compute the bounds on the state, it is given by:

$$\mathcal{O}_2 : \begin{cases} \dot{\underline{\hat{\mathbf{x}}}}(t) = \mathbf{f}(\underline{\hat{\mathbf{x}}}(t), t) + \mathbf{k}(\mathbf{y}(t) - \mathbf{e} - \mathbf{g}(\underline{\hat{\mathbf{x}}}(t), t)) \\ \dot{\overline{\hat{\mathbf{x}}}}(t) = \mathbf{f}(\overline{\hat{\mathbf{x}}}(t), t) + \mathbf{k}(\mathbf{y}(t) + \mathbf{e} - \mathbf{g}(\overline{\hat{\mathbf{x}}}(t), t)) \\ \underline{\hat{\mathbf{y}}}(t) = \mathbf{g}(\underline{\hat{\mathbf{x}}}(t), t) \\ \overline{\hat{\mathbf{y}}}(t) = \mathbf{g}(\overline{\hat{\mathbf{x}}}(t), t) \end{cases} \tag{23}$$

This estimator generates at the time t_j two balls $\underline{\mathcal{B}}_j = \{\underline{\hat{\mathbf{x}}}_j, \underline{R}_j\}$ and $\overline{\mathcal{B}}_j = \{\overline{\hat{\mathbf{x}}}_j, \overline{R}_j\}$. The domain of the i^{th} component x_j^i of the whole state is given by

$$x_j^i \in \left[\underline{\hat{x}}_j^i - \underline{R}_j, \overline{\hat{x}}_j^i + \overline{R}_j\right] \tag{24}$$

The width of the state enclosure computed by the observer (23) does not converge to 0 but depends on $\mathbf{e}$ and on the observer gain $\mathbf{k}$. The observer (23) is illustrated by the figure 1 for a monodimensional state vector.

3.3 Uncertain systems

Consider a system described by:

$$\mathcal{S}_2 \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}, t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{p}, t, \mathbf{w}(t)) \\ \mathbf{x}(t_0) \in [\mathbf{x}_0] \\ \mathbf{p} \in [\mathbf{P}_0] \end{cases} \tag{25}$$

As in the case of cooperative systems, we propose to bracket the uncertain system described by (25) between two systems involving no parameter uncertainty. Hence, some functions $\mathbf{f}_{low}$ and $\mathbf{f}_{upp}$ satisfying:

$$\mathbf{f}_{low}(\mathbf{x}, t) \leq \mathbf{f}(\mathbf{p}, \mathbf{x}, t) \leq \mathbf{f}_{upp}(\mathbf{x}, t) \tag{26}$$

could be constructed using monotonic properties of the state equation with respect to the parameters. The same procedure is applied to the output function $\mathbf{g}(\mathbf{x}, \mathbf{p})$. The following deterministic lower and upper dynamical systems could be obtained.

$$\mathcal{S}_{low} \begin{cases} \dot{\mathbf{x}}_{low} = \mathbf{f}_{low}(\mathbf{x}_{low}, t) \\ \mathbf{y}_{low} = \mathbf{g}_{low}(\mathbf{x}_{low}, t, \mathbf{w}) \\ \mathbf{x}_{low}(t_0) \in [\mathbf{x}_0] \end{cases} \tag{27}$$

$$\mathcal{S}_{upp} \begin{cases} \dot{\mathbf{x}}_{upp} = \mathbf{f}_{upp}(\mathbf{x}_{upp}, t) \\ \mathbf{y}_{upp} = \mathbf{g}_{upp}(\mathbf{x}_{upp}, t, \mathbf{w}) \\ \mathbf{x}_{upp}(t_0) \in [\mathbf{x}_0] \end{cases} \tag{28}$$

The observer (23) is used for both systems (27) and (28), but only the first equation of (23) is taken into account for (27) and only the second equation could be used for (28). Enclosures of the state of the brackting systems are given by:

$$\mathbf{x}_{low} \in [\underline{\mathbf{x}}_{low}, \overline{\mathbf{x}}_{low}]\,;\ \mathbf{x}_{upp} \in [\underline{\mathbf{x}}_{upp}, \overline{\mathbf{x}}_{upp}] \tag{29}$$

Hence, the state of the whole system described by (25) is given by:

$$\mathbf{x} \in \left[min(\underline{\mathbf{x}}_{low}, \underline{\mathbf{x}}_{upp}), sup(\overline{\mathbf{x}}_{low}, \overline{\mathbf{x}}_{upp})\right] \tag{30}$$

Remark 2. For cooperative systems, initial ordered state leads to ordered state at any time t_j. Thus, for such systems, the state of the whole system is given by:

$$\mathbf{x} \in [\underline{\mathbf{x}}_{low}, \overline{\mathbf{x}}_{upp}] \tag{31}$$

4. NUMERICAL EXAMPLES

Example 1. The observer (23) is illustrated on a hydraulic laboratory system (Fig.2) which is modelled by the equations (32) under the the condition $h_1 > h_3 > h_2$ for all times.

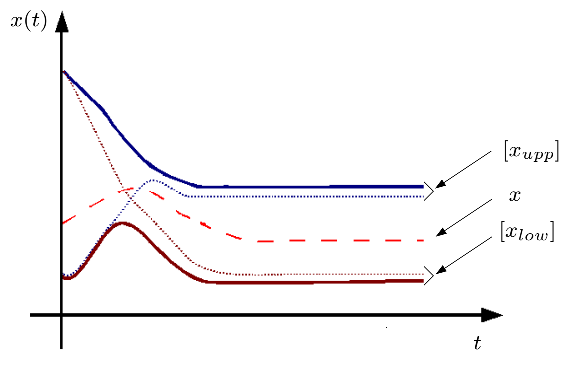

Fig. 1. Bounds of the state with bounded measurement uncertainty

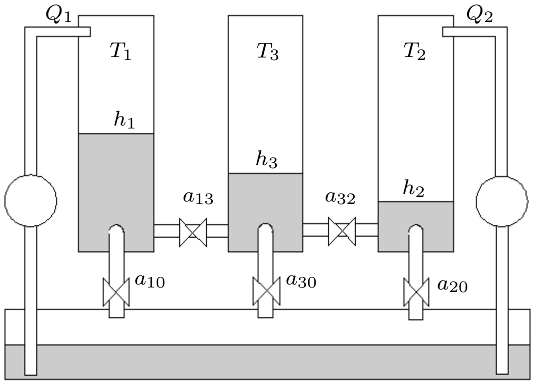

Fig. 2. Three-tanks system

$$\mathcal{S}_3 : \begin{cases} \dot{x}_1(t) &= -\dfrac{a_{13}}{S_c}\sqrt{x_1(t) - x_3(t)} \\ &+ \dfrac{1}{S_c}u_1(t) + v_1(t) \\ \dot{x}_2(t) &= \dfrac{a_{32}}{S_c}\sqrt{x_3(t) - x_2(t)} \\ &- \dfrac{a_{20}}{S_c}\sqrt{x_2(t)} + \dfrac{1}{S_c}u_2(t) + v_2(t) \\ \dot{x}_3(t) &= \dfrac{a_{13}}{S_c}\sqrt{x_1(t) - x_3(t)} \\ &- \dfrac{a_{32}}{S_c}\sqrt{x_3(t) - x_2(t))} + v_3(t) \\ y(t) &= (x_1(t), x_2(t))^T \\ &+ (w_1(t), w_2(t))^T \end{cases} \qquad (32)$$

where $\mathbf{x} = (x_1, x_2, x_3)^T = (h_1, h_2, h_3)^T$ represents the state vector, $\mathbf{u} = (u_1, u_2)^T = (Q_1, Q_2)^T$ is the control vector, $\mathbf{v} = (v_1, v_2, v_3)^T$ represents the state noise, $\mathbf{w} = (w_1, w_2)^T$ is the measurement noise and $\mathbf{p} = (a_{13}, a_{32}, a_{20})^T$ represents the parameter vector. The nominal parameters values are $a_{13} = a_{z13}S_n\sqrt{2g}$, $a_{32} = a_{z32}S_n\sqrt{2g}$ and $a_{20} = a_{z20}S_n\sqrt{2g}$ (where $a_{z13} = 0.6$, $a_{z32} = 0.6$, $a_{z20} = 0.8$, $S_n = 5.10^{-5}[m^2]$, $S = 0.0154[m^2]$, $g = 9.81[n/m^2]$). The nominal value of the parameter vector is $\mathbf{p}_0^T = (13.28, 13.28, 17.72).10^{-5}$.

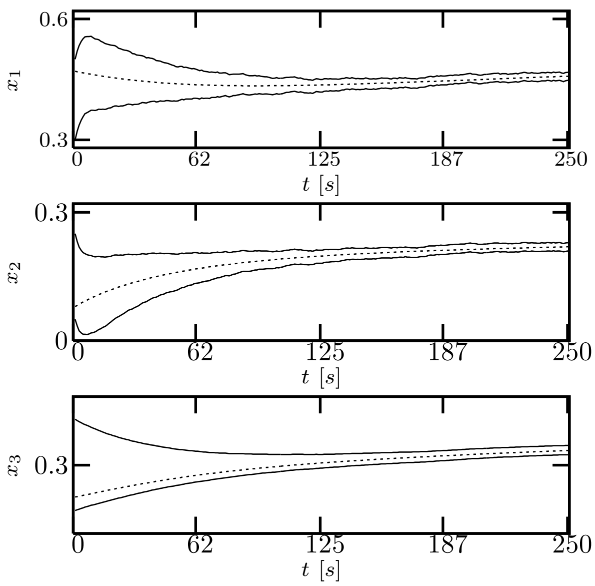

Fig. 3. State Bounds estimated by observer $\mathcal{O}_2$

The observer (23) is used with a gain $\mathbf{k}$ chosen such that the conditions of contraction (20) hold, hence

$$(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \mathbf{k}\frac{\partial \mathbf{g}}{\partial \mathbf{x}}) = \begin{pmatrix} J_{11} - k_{11} & -k_{12} & -J_{11} \\ -k_{21} & J_{22} - k_{22} & J_{23} \\ -J_{11} - k_{31} & J_{23} - k_{32} & J_{11} - J_{23} \end{pmatrix}$$

is uniformly negative definite, where $\mathbf{J} = \{J_{ij}\} = \partial\mathbf{f}/\partial\mathbf{x}$.

In this example, the noise corrupting the measurements is assumed to belong to the interval $[\pm 0.01\,m]$ and the initial state to $([0.3, 0.5], [0.05, 0.25], [0.2, 0.4])^T$. The results of the state estimation are depicted in figure 3. The actual state belongs to the enclosure computed by the proposed observer. In addition, the width of the enclosure decreases which permits to obtain a significant reduction of the uncertainty on the state.

Example 2. Consider the same system described by the equations (32), but assume that the parameter vector $\mathbf{p} = (a_{13}, a_{32}, a_{20})^T$ is not exactly known but belongs to $[\mathbf{p}] = [\mathbf{p}_0 - 20\%, \mathbf{p}_0 + 20\%]$. The results of the simulation based on the equations (25)-(30) are displayed on figure (4). It is shown that the overestimation is well controlled. The width of the domain of the state decreases and converges to the actual values although the parameters are uncertain.

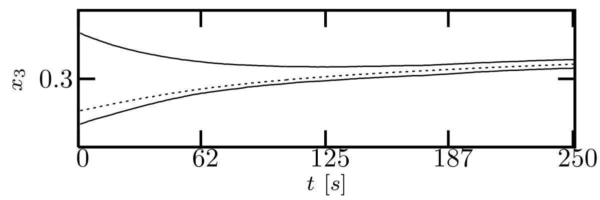

Fig. 4. State estimation for the uncertain nonlinear system (32) using observer $\mathcal{O}_2$

5. CONCLUSION

In this paper, an observer based on contraction theory is proposed. It permits guaranteed state estimation for a large class of nonlinear continuous time systems. This estimator is extended to deal with the cases where some parameters are not exactly known. The presented example clearly shows that the overestimation is well controlled even when the uncertainty on the parameters is large. Further investigations are necessary to extend this methodology to deals with joint parameter and state estimation.

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