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Generalized Mean-Variance Portfolio Selection Model with Regime Switching ⋆

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Abstract: In this paper we deal with a generalized multi-period mean-variance portfolio selection problem with the market parameters subject to Markov random regime switchings. We present necessary and sufficient conditions for obtaining an optimal control policy for this Markovian generalized multi-period mean-variance model, based on a recursive procedure. The analytical solution of our model provides the base for the solution of a great variety of mean-variance formulations.

1. INTRODUCTION

The mean-variance (MV) portfolio selection problem has been investigated since the the Markowitz's seminal work in (Markowitz [1952]). Nowadays, there are a huge literature about this subject, with some extensions, see (Fabozzi et al. [2007]) for an overview of the portfolio selection study development.

Despite of been vastly researched, until recently the market uncertainties were still reproduced like the original models, by a stochastic models in which the key parameters, expected return and volatility, are deterministic. However, there has been an increased interest in the study of financial models in which those key parameters are modulated by a Markov chain, see for instance (Zhang [2000]), (Bauerle and Rieder [2004]), (Çakmak and Özekici [2006]) and (Araujo and Costa [2006]). Such models can better reflect the market environment, since the overall assets usually move according to a major trend given by the state of the underlying economy or by the general mood of the investors.

The generalized multi-period mean-variance problem with Markov regime switching ($PGMV$) can be seen as an unrestricted stochastic control problem in which the objective function is formed by the weighted sum of a linear combination of three elements: the expected wealth, the square of the expected wealth and the expected value of the wealth squared. Several mean-variance models can be derived from the $PGMV$ model, as the traditional formulation in which the objective is to maximize the expected terminal wealth for a given final risk (variance), or the complex one in which the objective function is to maximize the weighted sum of the wealth throughout its investment horizon, with control over maximum wealth lost.

In this paper we consider a multi-period generalized mean-variance model with Markov switching in the key market parameters. Our main result is to derive necessary and sufficient conditions for obtaining an optimal control policy for this multi-period Markovian generalized mean-variance problem, based on a set of interconnected Riccati difference equations and some other recursive equations. It is important to stress that previous papers on this subject obtained only necessary conditions for optimality of the control strategy. To our best knowledge, no sufficient condition had been obtained before. Moreover, when compared with the no jumps case (Zhu et al. [2004]), we provide a more straight full way to compute the optimal control strategy for the multi-period generalized mean-variance problem.

This paper is organized as follows. In section 2 we formulate the model and the problems to be investigated. In Section 3, an optimal control policy for an auxiliary problem as well as the expected value and variance of the investor's wealth are analytically derived. Our main results are in section 4, where we provide necessary and sufficient conditions for the solution of the generalized mean-variance problem. The paper is concluded in section 5 with some final remarks.

2. DEFINITIONS AND THE FINANCIAL MODEL

Throughout the paper we shall denote by $\mathbb{R}^n$ the n-dimensional Euclidean real space and by $\mathbb{R}^{n\times m}$ the Euclidean space of all $n \times m$ real matrices. For a sequence of numbers $a_1, \ldots, a_m$, we shall denote by $diag(a_i)$ the diagonal matrix in $\mathbb{R}^{m\times m}$ formed by the element a_i at the i^{th} diagonal, $i = 1, \ldots, m$. The superscripts $'$ will denote the transpose of a vector or matrix. The variance of a random variable X will be denoted by $Var(X)$.

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We will consider a financial market with $n+1$ risky securities on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. The assets' price will be described by the random vector $\bar{\mathrm{S}}(t) = (\mathrm{S}_0(t), \ldots, \mathrm{S}_n(t))'$ taking values in $\mathbb{R}^{n+1}$ with $t = 0, \ldots, T$. Set $\bar{\mathrm{R}}(t) = (\mathrm{R}_0(t), \ldots, \mathrm{R}_n(t))'$, with $\mathrm{R}_i(t) = \frac{\mathrm{S}_i(t+1)}{\mathrm{S}_i(t)}$. We assume that the random vector $\bar{\mathrm{R}}(t)$ satisfies the following equation:

$$\bar{\mathrm{R}}(t) = [\bar{e} + \bar{\mu}_{\theta(t)}(t)] + \bar{\sigma}_{\theta(t)}(t) W(t), \tag{1}$$

where $\bar{e} = (1, e')'$, with $e \in \mathbb{R}^n$ a vector with $1's$ in all its components. Here $\{\theta(t); t = 0, \ldots, T\}$ is a finite-state discrete-time Markov chain with state space $\mathcal{M} = \{1, \ldots, m\}$, and $\{W(t);\ t = 0, \ldots, T\}$ is a sequence of $(n+1)$-dimensional independent random vectors with zero mean and covariance I (identity matrix). We assume that $\{W(t), \theta(t)\}$ are mutually independent. The set $\mathcal{M}$ represents the possible operations mode of the market. $\mathcal{P}$ is a probability measure such that $\mathcal{P}(\theta(t+1) = j \,|\theta(t) = i) = p_{ij}(t)$, $p_{ij}(t) \geq 0$ and $\sum_{j\in\mathcal{M}} p_{ij}(t) = 1$, for $t = 0, \ldots, T-1$ and $i, j \in \mathcal{M}$. We set for $t = 0, \ldots, T$, $P(t) = [p_{ij}(t)]_{m\times m} \in \mathbb{R}^{m\times m}$, $\pi_i(t) = \mathcal{P}(\theta(t) = i)$, $\pi(t) = (\pi_1(t), \ldots, \pi_m(t))'$. As in (Costa et al. [2005]), for $z = (z_1, \ldots, z_m)' \in \mathbb{R}^m$, we define the operator $\mathcal{E}(z, t) = (\mathcal{E}_1(z, t), \ldots, \mathcal{E}_m(z, t))$ as $\mathcal{E}_i(z, t) = \sum_{j=1}^{m} p_{ij}(t) z_j$, for $i \in \mathcal{M}$. For notational simplicity, we shall omit from now on the variable t in $\mathcal{E}_i(z, t)$. The filtration $\mathcal{F}_t$ is such that the random vectors $\{\bar{\mathrm{S}}(k); k = 0, \ldots, t\}$ and the Markov chain $\{\theta(k); k = 0, \ldots, t\}$ are $\mathcal{F}_t$-measurable.

When the market operation mode is $\theta(t) = i \in \mathcal{M}$, $\bar{\mu}_i(t) \in \mathbb{R}^{n+1}$ represents the vector with the expected returns of the assets, while $\bar{\sigma}_i(t)\bar{\sigma}_i(t)' \in \mathbb{R}^{(n+1)\times(n+1)}$ is the covariance matrix of the returns. It will be convenient to decompose $\bar{\mu}_i(t)$ and $\bar{\sigma}_i(t)$ as $\bar{\mu}_i(t) = (\mu_{i0}(t),\ \mu_i(t))'$, with $\mu_i(t) = (\mu_{i1}(t), \ldots, \mu_{in}(t))' \in \mathbb{R}^n$, and $\bar{\sigma}_i(t) = (\sigma_{i0}(t),\ \sigma_i(t))'$, with $\sigma_{i0}(t) = (\sigma_{i00}(t), \ldots,\ \sigma_{i0n}(t)) \in \mathbb{R}^{1\times n+1}$, and $\sigma_i(t) = [\sigma_{i\ell j}(t)] \in \mathbb{R}^{n\times n+1}$. We shall assume in this paper that $E\left(\bar{\mathrm{R}}(t)\bar{\mathrm{R}}(t)' |\theta(t) = i\right) > 0$, for each $t = 0, \ldots, T-1$ and $i \in \mathcal{M}$.

The set of admissible investment strategies $\mathcal{U} = \{u = (u(0), \ldots, u(T-1))\}$ is such that for each $t = 0, \ldots, T-1$, $u(t) = (u_1(t), \ldots, u_n(t))'$ is a $\mathcal{F}_t$-measurable random vector with finite second moment taking values in $\mathbb{R}^n$. We have that $u(t)$ represents the amount of the wealth allocated among the n securities. Associated to each admissible investment strategy u we have the portfolio's value process $\{V^u(t); t = 0, \ldots, T-1\}$, which represents the investor's wealth at the end of time t. Note that the amount of wealth allocated to the asset $i = 0$ is determined by $V^u(t) - e'u(t)$. For notational simplicity, we shall suppress the superscript ^{u} whenever no confusion may arise. Define $\bar{A}_{\theta(t)}(t) = 1 + \mu_{\theta(t)0}(t)$, $\widetilde{A}_{\theta(t)}(t) = \sigma_{\theta(t)0}(t)$, $\bar{B}_{\theta(t)}(t) = \mu_{\theta(t)}(t) - e\mu_{\theta(t)0}(t)$, and $\widetilde{B}_{\theta(t)}(t) = \sigma_{\theta(t)}(t) - e\sigma_{\theta(t)0}(t)$. Assuming that the initial wealth $V(0) = V_0 > 0$ and that the portfolio is self-financed, the wealth process is represented by:

$$V(t+1) = A_{\theta(t)}(t) V(t) + B_{\theta(t)}(t)' u(t), \tag{2}$$

where $A_{\theta(t)}(t) = \bar{A}_{\theta(t)}(t) + \widetilde{A}_{\theta(t)}(t) W(t)$ and $B_{\theta(t)}(t) = \bar{B}_{\theta(t)}(t) + \widetilde{B}_{\theta(t)}(t) W(t)$.

2.1 The PGMV Model

The generalized multi-period mean-variance problem we shall consider is described as follows. Consider a set of numbers $\alpha(t) > 0$ for $t \in \mathcal{I} := \{\tau_1, \ldots, \tau_{\iota_f}\}$ with $\tau_0 = 0$ and $\tau_{\iota_f} = T$. Consider also a sequence of positive numbers $\rho(t)$, $\nu(t)$, and a sequence of real numbers $\ell(t)$, for $t \in \mathcal{I}$, with $\rho(T) > 0$, $\nu(T) > 0$, and $\ell(T) \neq 0$. We define the following problem:

$$\begin{aligned} PGMV(\rho, \ell, \nu) : \max_{u\in\mathcal{U}} \sum\nolimits_{t\in\mathcal{I}} \Big[&\nu(t) E(V(t))^2 \\ &- \rho(t) E\left(V(t)^2\right) + \ell(t) E(V(t)) \Big]. \end{aligned} \tag{3}$$

It will be convenient to extend the definition of $\alpha(t)$, $\ell(t)$, $\nu(t)$ and $\rho(t)$ for $t \notin \mathcal{I}$ by setting in these cases $\alpha(t) = 0$, $\ell(t) = 0$, $\nu(t) = 0$ and $\rho(t) = 0$. It is also be convenient to set $\tau_0 = 0$, $\ell(0) = 0$, $\nu(0) = 0$, $\rho(0) = 0$ and $\alpha(0) = 0$.

As pointed out in (Li and Ng [2000]), the stochastic problem $PGMV(\rho, \ell, \nu)$ is non separable in the sense of dynamic programming, since it involves a non-linear function of a expectation term $(E(V(t))^2)$. Therefore it cannot be directly solved by dynamic programming. A solution procedure based on a tractable auxiliary problem is proposed in (Zhu et al. [2004]) to seek an optimal dynamic portfolio policy for problem $PGMV(\rho, \ell, \nu)$. We will adopt the same procedure in this paper, considering a similar auxiliary problem. Let $\lambda(t)$, for $t = 1, \ldots, T$, be a set of real numbers, with $\lambda(t) = 0$ for $t \notin \mathcal{I}$. The auxiliary problem is defined as follows:

$$A(\lambda, \rho) : \min_{u\in\mathcal{U}} \sum_{t=1}^{T} E\left\{\rho(t) V(t)^2 - \lambda(t) V(t)\right\}. \tag{4}$$

The solution of problem $A(\lambda, \rho)$ will be presented in Section 3 while an explicit solution of problem $PGMV(\rho, \ell, \nu)$ will be presented in Section 4, based on the necessary and sufficient conditions for optimality that will be derived.

2.2 MV Problems Derived From PGMV

Several mean-variance problems can be derived from the $PGMV$ model, with a convenient definition of the coefficients $\rho(t)$, $\nu(t)$ and $\ell(t)$. We show next one example.

Consider a multi-period MV problem, similar to the one presented in (Zhu et al. [2004]), in which the objective function is formed by the weighted sum of the expected wealth along the time, with restriction over the the probability $(\varrho(t))$ of the wealth falling bellow a minimum value $\psi(t)$. This problem can be formally posed as:

$$PBC(\varrho, \psi) : \max_{u\in\mathcal{U}} \sum_{t\in\mathcal{T}} \alpha(t) E(V(t))$$

$$\text{s.t.: } Var(V(t)) \leq \varrho(t) \left[E(V(t) - \psi(t))\right]^2, \text{ for } t \in \mathcal{T}.$$

By introducing in $PBC(\varrho, \psi)$ the nonnegative Lagrange multipliers $(\omega(t), t \in \mathcal{T})$ a Lagrangian maximization problem $LPBC(\omega, \varrho, \psi)$ formed by attaching the constraints to the objective function can be written as follows:

$$LPBC(\omega,\varrho,\psi): \max_{u\in\mathcal{U}} \sum_{t\in\mathcal{T}} (\alpha(t) - 2\omega(t)\varrho(t)\psi(t))$$
$$\cdot E(V(t)) - \sum_{t\in\mathcal{T}} \omega(t) E\left(V(t)^2\right) + \sum_{t\in\mathcal{T}} \omega(t)(1+\varrho(t))$$
$$\cdot E(V(t))^2 + \sum_{t\in\mathcal{T}} \omega(t)\varrho(t)\psi(t)^2.$$

We can establish a relationship between $LPBC(\omega,\varrho,\psi)$ and (3) by taking for $t\in\mathcal{T}$: $\rho(t) = \omega(t)$, $\nu(t) = \rho(t)(1+\varrho(t))$ and $\ell(t) = \alpha(t) - 2\rho(t)\varrho(t)\psi(t)$.

The problem $PBC(\varrho,\psi)$ can be transformed into the original MV problem, presented in (Markowitz [1952]), just considering for that $\mathcal{T} = \{T\}$ and the right side of the inequality as a constant like σ^2, which represents the desired level of final risk. Other combinations also could be done, giving rise to other MV formulations like the one in (Çakmak and Özekici [2006]).

In Section 4 we present an analytical solution of (3) in terms of ρ, ℓ and ν. With this solution a numerical procedure would be necessary to find the Lagrange multipliers which attend to the respective restrictions, as for $PBC(\varrho,\psi)$. However, there are some situations in which an exact solution can be derived analytically, as the cases in which there is a restriction only on the final time T as the one presented in (Çakmak and Özekici [2006]).

3. OPTIMAL CONTROL POLICY FOR THE AUXILIARY PROBLEM

In this section we obtain an explicit expression for the value function and optimal control policy for the auxiliary problem $A(\lambda,\rho)$ by applying dynamic programming. We also obtain closed expressions for the expected value and variance of the wealth. As in the classical stochastic linear quadratic problem, this optimal control law depends on the solution of a set of recursive coupled Riccati difference equations (see (5) below). Before going to the main result, let us define some intermediate problems. The value function for the auxiliary problem at time $k\in\{0,\ldots,T-1\}$ is defined by: $J(V(k),\theta(k),k) = \min_{u_k\in\mathcal{U}_k} \sum_{t=k}^{T} E\left\{\rho(t)V(t)^2 - \lambda(t)V(t)\middle|\mathcal{F}_k\right\}$, where $\mathcal{U}_k = \{u_k = (u(k),\ldots,u(T-1)); u(t)$ is $\mathcal{F}_t$ measurable for each $t = k,\ldots,T-1\}$. We shall need the following definitions. For each $i\in\mathcal{M}$ and $t = 0,\ldots,T$, set:

$$\phi_i(t) = E\left(B_i(t)B_i(t)'\right) = \bar{B}_i(t)\bar{B}_i(t)' + \widetilde{B}_i(t)\widetilde{B}_i(t)',$$
$$\varphi_i(t)' = E\left(A_i(t)B_i(t)'\right) = \bar{A}_i(t)\bar{B}_i(t)' + \widetilde{A}_i(t)\widetilde{B}_i(t)',$$
$$\beta_i(t) = \bar{B}_i(t)'\phi_i(t)^{-1}\bar{B}_i(t),$$
$$Q_i(t) = E\left(A_i(t)^2\right) - \varphi_i(t)'\phi_i(t)^{-1}\varphi_i(t),$$
$$R_i(t) = \bar{A}_i(t) - \bar{B}_i(t)'\phi_i(t)^{-1}\varphi_i(t),$$

with $Q(t) = diag(Q_i(t))$, $R(t) = diag(R_i(t))$. Notice that from the hypothesis that $E(\bar{\mathrm{R}}(t)\bar{\mathrm{R}}(t)' \mid \theta(t) = i > 0)$, the inverse of $\phi_i(t)$ is well defined and $Q_i(t) > 0$. We compute backwards the m dimensional vectors $K(t) = (K_1(t),\ldots, K_m(t))'$, $Z(t) = (Z_1(t),\ldots,Z_m(t))'$ and $D(t) = (D_1(t),\ldots,D_m(t))'$, with $K_i(t)$, $Z_i(t)$ and $D_i(t)$ as follows: For $t = T-1,\ldots,0$ and $i\in M$:

$$\begin{cases} K_i(t) = \rho(t) + Q_i(t)\mathcal{E}_i[K(t+1)], \\ Z_i(t) = -\lambda(t) + R_i(t)\mathcal{E}_i[Z(t+1)], \\ D_i(t) = -\dfrac{\mathcal{E}_i[Z(t+1)]^2}{4\mathcal{E}_i[K(t+1)]}\beta_i(t) + \mathcal{E}_i[D(t+1)], \end{cases} \tag{5}$$

with $K_i(T) = \rho(T)$, $Z_i(T) = -\lambda(T)$ and $D_i(T) = 0$. It will be convenient to define for $s,t = 0,\ldots,T$ and $s\le t$, $\mathcal{Q}(t,s)\in\mathbb{R}^{m\times m}$, $\mathcal{R}(t,s)\in\mathbb{R}^{m\times m}$ $\mathcal{K}(t,s)\in\mathbb{R}^m$ and $\mathcal{Z}(t,s)\in\mathbb{R}^m$ as follows: $\mathcal{Q}(t,s) = \prod_{k=s}^{t-1}[Q(k)P(k)]$, $\mathcal{K}(t,s) = \mathcal{Q}(t,s)e$, $\mathcal{K}(s) = \mathcal{K}(T,s)$, $\mathcal{R}(t,s) = \prod_{k=s}^{t-1}[R(k)\cdot P(k)]$, $\mathcal{Z}(t,s) = \mathcal{R}(t,s)e$, $\mathcal{Z}(s) = \mathcal{Z}(T,s)$, where $\mathcal{Q}(s,s) = I$ and $\mathcal{R}(s,s) = I$. From (5) we have that $K(t) = \sum_{s=t}^{T}\rho(s)\mathcal{K}(s,t)$ and $Z(t) = -\sum_{s=t}^{T}\lambda(s)\mathcal{Z}(s,t)$. We have the following theorem.

Theorem 1. The optimal control law for problem (4) is given by

$$u(t) = -\phi_{\theta(t)}(t)^{-1}\varphi_{\theta(t)}(t)V(t) - \frac{\mathcal{E}_{\theta(t)}[Z(t+1)]}{2\mathcal{E}_{\theta(t)}[K(t+1)]}\phi_{\theta(t)}(t)^{-1}\bar{B}_{\theta(t)}(t). \tag{6}$$

Furthermore, the value function for the intermediate problem is given by

$$J(V(t),\theta(t),t) = K_{\theta(t)}(t)V(t)^2 + Z_{\theta(t)}(t)V(t) + D_{\theta(t)}(t). \tag{7}$$

Proof. Let us apply induction on t. For $t = T$ we have that $J(V(T),\theta(T),T) = \alpha(T)\rho(T)V(T)^2 - \alpha(T)\lambda(T)V(T) = K_{\theta(T)}(T)V(T)^2 + Z_{\theta(T)}(T)V(T) + D_{\theta(T)}(T)$, in agreement with Theorem 1. Suppose the result holds for $t = k+1$. We show next that the solution also holds for $t = k$. For $\theta(k) = i\in\mathcal{M}$ and $V(k) = v$ we have from the Bellman's principle of optimality that

$$\begin{aligned} J(v,i,k) &= \min_{u(k)} E\{J(V(k+1),\theta(k+1),k+1)|\mathcal{F}_k\} \\ &+ (\rho(k)v^2 - \lambda(k)v) \\ &= \min_{u(k)}\{\mathcal{E}_i[K(k+1)]\left[E\left(A_i(k)^2\right)v^2 + 2\varphi_i(k)'u(k)v\right. \\ &+ \left. u(k)'\phi_i(k)u(k)\right] + \mathcal{E}_i[Z(k+1)]\bar{A}_i(k)v \\ &+ \mathcal{E}_i[Z(k+1)]\bar{B}_i(k)'u(k) + \mathcal{E}_i[D(k+1)]\} \\ &+ (\rho(k)v^2 - \lambda(k)v). \end{aligned} \tag{8}$$

Taking the derivative of (8) over $u(k)$ and making the result equal to zero yields (6). Substituting (6) into (8) yields the value function expressed in (7), providing the desired result. ■

Next we analytically derive expressions for the expected value and variance of the wealth, for $\tau_\kappa\in\mathcal{I}$, under the optimal control law (6). First, let for $k = 0,\ldots,T-1$, $\kappa = 0,\ldots,\iota_f - 1$ and $i\in\mathcal{M}$:

$$\mathrm{B}(k) = P(k)'diag\left(\frac{\pi_i(k)\beta_i(k)}{\mathcal{E}_i(K(k+1))}\right)P(k), \tag{9}$$

$$\mathcal{B}(\tau_\kappa) = \sum_{k=\tau_\kappa}^{\tau_{\kappa+1}-1}\mathcal{R}(\tau_{\kappa+1},k+1)'\mathrm{B}(k)\mathcal{R}(\tau_{\kappa+1},k+1), \tag{10}$$

$$\mathcal{C}(\tau_\kappa) = \sum_{k=\tau_\kappa}^{\tau_{\kappa+1}-1}\mathcal{Q}(\tau_{\kappa+1},k+1)'P(k)'\cdot$$

$$diag\left(\beta_i(k)\left(\frac{\mathcal{E}_i[Z(k+1)]}{\mathcal{E}_i[K(k+1)]}\right)^2\right)\pi(k) \tag{11}$$

$$\begin{cases} q(t) = (q_1(t), ..., q_m(t))', \\ q_i(t) = E(V(t)\,1_{\{\theta(t)=i\}}) \end{cases} \tag{12}$$

$$\begin{cases} g(t) = (g_1(t), ..., g_m(t))', \\ g_i(t) = E(V(t)^2\,1_{\{\theta(t)=i\}}). \end{cases} \tag{13}$$

Theorem 2. Under the optimal control law (6), the expected value and variance the wealth, for each $\tau_\kappa \in \mathcal{I}$ are given by, respectively:

$$E(V(\tau_\kappa)) = q(0)'\,\mathcal{Z}(\tau_\kappa,0) - \frac{1}{2}\sum_{k=0}^{\kappa-1} Z(\tau_{k+1})'\,\mathcal{B}(\tau_k)'\,e \tag{14}$$

and

$$Var(V(\tau_\kappa)) = g(0)'\,\mathcal{K}(\tau_\kappa,0) + \frac{1}{4}\sum_{k=0}^{\kappa-1}\mathcal{C}(k)'\,e - \left(q(\tau_\kappa)'\,e\right)^2. \tag{15}$$

Proof. Using the control law (6) into (2), from Proposition 3.35 in (Costa et al. [2005]) and from (12) it follows that

$$q_j(t+1) = \sum_{i=1}^{m} p_{ij}(t)R_i(t)\,q_i(t) - \frac{1}{2}\sum_{i=1}^{m} p_{ij}(t)\pi_i(t)\,\frac{\mathcal{E}_i[Z(t+1)]}{\mathcal{E}_i[K(t+1)]}\beta_i(t)\,. \tag{16}$$

Now taking square on both sides of (2), using the control law (6) in it, from Proposition 3.35 in (Costa et al. [2005]) and from (13) we have that

$$g_j(t+1) = \sum_{i=1}^{m} p_{ij}(t)Q_i(t)\,g_i(t) + \frac{1}{4}\sum_{i=1}^{m} p_{ij}(t)\pi_i(t)\left(\frac{\mathcal{E}_i[Z(t+1)]}{\mathcal{E}_i[K(t+1)]}\right)^2\beta_i(t)\,. \tag{17}$$

Recalling that $\lambda(t) = 0$ for $t \notin \mathcal{I}$ it follows from (5) that for $\tau_\kappa \le k \le \tau_{\kappa+1} - 1$

$$Z(k+1) = \mathcal{R}(\tau_{\kappa+1}, k+1)Z(\tau_{\kappa+1}) \tag{18}$$

and

$$Z(\tau_\kappa) = -\alpha(\tau_\kappa)\lambda(\tau_\kappa)e + \mathcal{A}_\kappa' Z(\tau_{\kappa+1})\,. \tag{19}$$

It is easy to see that using (10) and (18) in (16), we get:

$$q(\tau_{\kappa+1}) = \mathcal{R}(\tau_{\kappa+1},\tau_\kappa)'q(\tau_\kappa) - \frac{1}{2}\mathcal{B}(\tau_\kappa)\,Z(\tau_{\kappa+1}), \tag{20}$$

and, from (11) and (18) in (17), it yields:

$$g(\tau_{\kappa+1}) = \mathcal{Q}(\tau_{\kappa+1},\tau_\kappa)'\,g(\tau_\kappa) + \frac{1}{4}\mathcal{C}(\tau_\kappa)\,. \tag{21}$$

Recording that $E(V(t)) = q(t)'\,e$ and $Var(V(T)) = g(t)'\,e - E(V(t))^2$, from (20) and (21) we have (14) and (15). ■

4. SOLUTION OF THE GENERALIZED MEAN VARIANCE PROBLEM

We solve in this section the generalized mean-variance problem $PGMV(\rho,\ell,\nu)$. We will represent the set of optimal solutions for problems $A(\lambda,\rho)$ and $PGMV(\rho,\ell,\nu)$ by $\Pi(A(\lambda,\rho))$ and $\Pi(PGMV(\rho,\ell,\nu))$ respectively. In Sub-section 4.1 we present a necessary condition for $u \in PGMV(\rho,\ell,\nu)$, while in Sub-section 4.2 we derive a sufficient condition.

4.1 Necessary condition

We present first a necessary condition for the optimal control law $u \in \Pi(PGMV(\rho,\ell,\nu))$. We have the following result.

Proposition 1. If $u \in \Pi(PGMV(\rho,\ell,\nu))$ then $u \in \Pi(A(\lambda,\rho))$ with, for $t \in \mathcal{I}$,

$$\lambda(t) = \ell(t) + 2\nu(t)E(V^u(t)). \tag{22}$$

Proof. See Theorem 1 in Zhu et al. [2004]. ■

It follows from Proposition 1 that to obtain $u \in \Pi(PGMV(\rho,\ell,\nu))$ we must have $u \in \Pi(A(\lambda,\rho))$ given by (6) with $\lambda(t)$ such that (22) holds. Set for $\kappa = 0,\ldots,\iota_f - 1$, $\mathcal{A}_\kappa = \mathcal{R}(\tau_{\kappa+1},\tau_\kappa)'$. In order to obtain $Z(t)$ such that (22) is satisfied, we evaluate recursively for $\kappa = \iota_f - 1,\ldots,0$ the following matrices $\mathcal{G}_\kappa \in \mathbb{R}^{m\times m}$ and vectors $\mathcal{S}_\kappa \in \mathbb{R}^m$,

$$\begin{cases} \mathcal{G}_\kappa = -2\nu(\tau_\kappa)ee' + \mathcal{A}_\kappa'\left(I + \frac{1}{2}\mathcal{G}_{\kappa+1}\mathcal{B}_\kappa\right)^{-1}\mathcal{G}_{\kappa+1}\mathcal{A}_\kappa, \\ \mathcal{G}_{\iota_f} = -2\nu(T)ee', \end{cases} \tag{23}$$

$$\begin{cases} \mathcal{S}_\kappa = -\ell(\tau_\kappa)e + \mathcal{A}_\kappa'\left(I + \frac{1}{2}\mathcal{G}_{\kappa+1}\mathcal{B}_\kappa\right)^{-1}\mathcal{S}_{\kappa+1}, \\ \mathcal{S}_{\iota_f} = -\ell(T)e, \end{cases} \tag{24}$$

where we are assuming that for each $\kappa = \iota_f - 1,\ldots,0$, the inverse of $\left(I + \frac{1}{2}\mathcal{G}_{\kappa+1}\mathcal{B}_\kappa\right)$ exists. We have the following proposition.

Proposition 2. Suppose that u, as defined in (6), is applied to equation (4). If (22) holds then for each $\kappa = \iota_f,\ldots,1$,

$$Z(\tau_\kappa) = \mathcal{S}_\kappa + \mathcal{G}_\kappa q(\tau_\kappa). \tag{25}$$

Proof. By induction, for $\kappa = \iota_f$ we have that $Z(T) = -\lambda(T)e = -(\ell(T) + 2\nu(T)e'q(T))e$ and the result follows. Suppose it holds for $\kappa + 1$. From (20) and the induction hypothesis, we get that, for $\kappa = 0,\ldots,\iota_f - 1$,

$$q(\tau_{\kappa+1}) = \left(I + \frac{1}{2}\mathcal{B}_\kappa\mathcal{G}_{\kappa+1}\right)^{-1}\mathcal{A}_\kappa' q(\tau_\kappa) - \frac{1}{2}\left(I + \frac{1}{2}\mathcal{B}_\kappa\mathcal{G}_{\kappa+1}\right)^{-1}\mathcal{B}_\kappa\mathcal{S}_{\kappa+1}, \tag{26}$$

with $\bar{q}(0) = q(0)$. From (19),

$$Z(\tau_\kappa) = -\alpha(\tau_\kappa)(\ell(\tau_\kappa) + 2\nu(\tau_\kappa)e'q(\tau_\kappa))e + \mathcal{A}_\kappa'(\mathcal{S}_{\kappa+1} + \mathcal{G}_{\kappa+1}q(\tau_{\kappa+1}))\,. \tag{27}$$

Replacing (26) into (27) and after some algebraic manipulations using the inverse matrix lemma, we obtain (25). ■

Finally we have the following theorem.

Theorem 3. If $u \in \Pi(PGMV(\rho,\ell,\nu))$ then u is as in (6) with $\lambda(\tau_\kappa) = \ell(\tau_\kappa) + 2\nu(\tau_\kappa)e'q(\tau_\kappa)$ in (5), and $q(\tau_\kappa)$ given by (26).

Proof. The proof follows from Propositions 1 and 2. ■

From Theorem 3 we have the following necessary condition algorithm to determine the optimal control strategy for the generalized mean-variance portfolio optimization problem. Step i): Evaluate recursively the matrices $\mathcal{G}_\kappa$ and $\mathcal{S}_\kappa$ as in (23) and (24) for $\kappa = \iota_f, \ldots, 0$. Step ii): Evaluate recursively $q(\tau_\kappa)$ given by (26) for $\kappa = 0, \ldots, \iota_f$. Step iii): Set $\lambda(\tau_\kappa) = \ell(\tau_\kappa) + 2\nu(\tau_\kappa)e'q(\tau_\kappa)$ for $\kappa = 0, \ldots, \iota_f$, 0 otherwise. Step iv): Calculate recursively $K(t)$ and $Z(t)$ as in (5) for $t = T, \ldots, 0$. Step v): The optimal strategy is given by (6).

4.2 Sufficient condition

In this sub-section we establish a sufficient condition for the existence of a solution $u \in \Pi(PGMV(\rho, \ell, \nu))$. We show first that it is enough to search for the optimal control law of (3) in the form

$$u^M(t) = -\phi_{\theta(t)}(t)^{-1}\varphi_{\theta(t)}(t)V(t) - \frac{\mathcal{E}_{\theta(t)}[M(t+1)]}{2\mathcal{E}_{\theta(t)}[K(t+1)]}\phi_{\theta(t)}(t)^{-1}\bar{B}_{\theta(t)}(t). \quad (28)$$

where $M = (M(1), \ldots, M(T))$, $M(t) \in \mathbb{R}^m$. Let us denote by $\mathcal{U}^* \subset \mathcal{U}$ the set of admissible controls u^M written as in (28) for some M. For any $u \in \mathcal{U}$ set

$$\mathcal{J}(u) = \sum_{t\in\mathcal{I}} \rho(t)E\left(V^u(t)^2\right) - \ell(t)E(V^u(t)) - \nu(t)E(V^u(t))^2. \quad (29)$$

Proposition 3. If $\hat{u}^M \in \mathcal{U}^*$ is such that $\mathcal{J}(\hat{u}^M) = \min_{u^M\in\mathcal{U}^*} \mathcal{J}(u^M)$ then:

$$\hat{u}^M \in \Pi(PGMV(\rho, \ell, \nu)).$$

Proof. We will show that for any $u \in \mathcal{U}$ we can find M as above such that $u^M \in \mathcal{U}^*$ and $\mathcal{J}(u^M) \leq \mathcal{J}(u)$. Indeed take $M(t) = Z(t)$, with $Z(t)$ as in (5) and $\lambda(t) = \ell(t) + 2\nu(t)E(V^u(t))$, for $t \in \mathcal{I}$, zero otherwise, and define u^M as in (28). From the fact that $u^M \in \Pi(A(\lambda, \rho))$ we have that

$$\sum_{t\in\mathcal{I}} \left(\rho(t)E(V^{u^M}(t)^2) - \lambda(t)E(V^{u^M}(t))\right) \leq \sum_{t\in\mathcal{I}} \left(\rho(t)E(V^u(t)^2) - \lambda(t)E(V^u(t))\right). \quad (30)$$

Using $\lambda(t)$ as above in (29) and after some manipulations it is easy to see from (30) that

$$\mathcal{J}(u^M) \leq \mathcal{J}(u^M) + \sum_{t\in\mathcal{I}} \nu(t)E\left(V^u(t) - V^{u^M}(t)\right)^2 \leq \mathcal{J}(u),$$

showing the desired result. ■

From Proposition 3 and equations (16), (17) we can rewrite the stochastic problem $PGMV(\rho, \ell, \nu)$ as a deterministic one as follows:

$$\min_M \sum_{t\in\mathcal{I}} \rho(t)e'g^M(t) - \ell(t)e'q^M(t) - \nu(t)(e'q^M(t))^2, \quad (31)$$

with

$$q^M(t+1) = \mathcal{R}(t+1,t)'q^M(t) - h(M,K,t) \quad (32)$$

$$g^M(t+1) = \mathcal{Q}(t+1,t)'g^M(t) + r(M,K,t), \quad (33)$$

with $h(M,K,t) = (h_1(M,K,t), \ldots, h_m(M,K,t)'$ and $r(M,K,t) = (r_1(M,K,t), \ldots, r_m(M,K,t))'$ as to vectors in $\mathbb{R}^m$, with $h_j(M,K,t) = \frac{1}{2}\sum_{i=1}^m p_{ij}(t)\pi_i(t)\frac{\mathcal{E}_i[M(t+1)]}{\mathcal{E}_i[K(t+1)]} \cdot\beta_i(t)$ and $r_j(M,K,t) = \frac{1}{4}\sum_{i=1}^m p_{ij}(t)\pi_i(t)\left(\frac{\mathcal{E}_i[M(t+1)]}{\mathcal{E}_i[K(t+1)]}\right)^2 \cdot\beta_i(t)$, for $i, j \in \mathcal{M}$, and where $M = (M(1), \ldots, M(T))$, $M(t) \in \mathbb{R}^m$ is the control variable. We can apply dynamic programming to solve problem (31). For this we define, for $k = T-1, \ldots, 0$, the intermediate problems

$$\mathcal{V}(g_k, q_k, k) = \min_{M^k} \sum_{t\in\mathcal{I}} \rho(t)e'g^{M^k}(t) - \ell(t)e'q^{M^k}(t) - \nu(t)(e'q^{M^k}(t))^2. \quad (34)$$

where $M^k = (M(k+1), \ldots, M(T))$, $M(t) \in \mathbb{R}^m$, and $g^{M^k}(t)$, $q^{M^k}(t)$ are given by (32), (33) with initial condition $g(k) = g_k$, $q(k) = q_k$. In what follows, recall the definition of $\mathrm{B}(k) \geq 0$ in (9), and that (see, for instance, Saberi et al. [1995], page 12-13)) for any matrix $S \in \mathbb{R}^{m\times m}$, the generalized inverse of S (or Moore-Penrose inverse of S), denoted by $S^\dagger \in \mathbb{R}^{m\times m}$ is such that: i) $SS^\dagger S = S$, ii) $S^\dagger SS^\dagger = S^\dagger$, iii) $(SS^\dagger)' = SS^\dagger$, and iv) $(S^\dagger S)' = S^\dagger S$. Define recursively for $k = T-1, \ldots, 0$ the symmetric $m\times m$ matrices $\Lambda(k)$, $\Psi(k)$, the m-dimensional vectors $L(k)$, and real numbers $\varepsilon(k)$ as follows:

$$\Psi(k) = \mathrm{B}(k) + \mathrm{B}(k)\Lambda(k+1)\mathrm{B}(k), \quad (35)$$

$$\begin{cases} \Lambda(k) = -\nu(k)ee' + \left(\mathcal{R}(k+1,k)\left(I - \Lambda(k+1) \cdot \mathrm{B}(k)\Psi(k)^\dagger\mathrm{B}(k)\right)\right)\Lambda(k+1)\mathcal{R}(k+1,k)', \\ \Lambda(T) = -\nu(T)ee', \end{cases} \quad (36)$$

$$\begin{cases} L(k) = -\ell(k)e + \mathcal{R}(k+1,k)\left(I - \Lambda(k+1) \cdot \mathrm{B}(k)\Psi(k)^\dagger\mathrm{B}(k)\right)L(k+1), \\ L(T) = -\ell(T)e, \end{cases} \quad (37)$$

$$\begin{cases} \varepsilon(k) = -\frac{1}{4}L(k+1)'\mathrm{B}(k)\Psi(k)^\dagger\mathrm{B}(k)L(k+1) + \varepsilon(k+1), \\ \varepsilon(T) = 0. \end{cases} \quad (38)$$

We have the following theorem that establishes a sufficient condition for the existence of a solution $u \in \Pi(PGMV(\rho, \ell, \nu))$. In what follows we denote by $Im(S)$ the range of a matrix $S \in \mathbb{R}^{m\times m}$.

Theorem 4. If for each $k = T-1, \ldots, 0$ we have that $\Psi(k) \geq 0$ and, for any $q \in \mathbb{R}^m$,

$$\mathrm{B}(k)\left(\frac{1}{2}L(k+1) + \Lambda(k+1)\mathcal{R}(k+1,k)'q\right) \in Im(\Psi(k)) \quad (39)$$

then

$$\mathcal{V}(g_k, q_k, k) = K(k)'g_k + L(k)'q_k + q_k'\Lambda(k)q_k + \varepsilon(k) \quad (40)$$

and an optimal solution for problem (31) is given, for $k = 0, \ldots, T-1$, by

$$M^*(k+1) = \Psi(k)^\dagger\mathrm{B}(k)\left(L(k+1) + 2\Lambda(k+1)\mathcal{R}(k+1,k)'q(k)\right). \quad (41)$$

Proof. By induction, we have from (36), (37), (38) that (40) holds for $k = T$. Suppose that (40) holds for $k+1$. From the Bellman's principle of optimality, and noticing that $K(k+1)'r(\mathbf{M}, K, k) = \frac{1}{4}\mathbf{M}'\mathrm{B}(k)\mathbf{M}$, and $L(k+1)'h(\mathbf{M}, K, k) = \frac{1}{2}\mathbf{M}'\mathrm{B}(k)L(k+1)$, we have that

$$\mathcal{V}(g_k, q_k, k) = \alpha(k)\rho(k)e'g_k - \alpha(k)\ell(k)e'q_k - \alpha(k)\nu(k)q_k'ee'q_k + K(k+1)'\mathcal{Q}(k+1,k)'g_k + L(k+1)'\mathcal{R}(k+1,k)'q_k + q_k'\mathcal{R}(k+1,k)\Lambda(k+1) \cdot \mathcal{R}(k+1,k)'q_k + \varepsilon(k+1) + \min_{\mathbf{M}\in\mathbb{R}^m}\left[\frac{1}{4}\mathbf{M}'\left(\mathrm{B}(k) + \mathrm{B}(k)'\Lambda(k+1)\mathrm{B}(k)\right)\mathbf{M} - \mathbf{M}'\mathrm{B}(k)\left(\frac{1}{2}L(k+1) + \Lambda(k+1)\mathcal{R}(k+1,k)'q_k\right)\right]. \tag{42}$$

From Lemma 4.2 in (Rami et al. [2002]) we have that if (39) holds then

$$\mathrm{B}(k)\left(\frac{1}{2}L(k+1) + \Lambda(k+1)\mathcal{R}(k+1,k)'q(k)\right) = \Psi(k)^\dagger\Psi(k)\mathrm{B}(k)\left(\frac{1}{2}L(k+1) + \Lambda(k+1)\mathcal{R}(k+1,k)'q(k)\right)$$

and thus we have that

$$\frac{1}{4}\mathbf{M}'\Psi(k)\mathbf{M} - \mathbf{M}'\mathrm{B}(k)\Big(\frac{1}{2}L(k+1) + \Lambda(k+1)\mathcal{R}(k+1,k)'q_k\Big) = \frac{1}{4}\Big((\mathbf{M} - M^*(k+1))' \Psi(k)(\mathbf{M} - M^*(k+1)) - M^*(k+1)'\Psi(k)M^*(k+1)\Big). \tag{43}$$

If $\Psi(k) \geq 0$ then clearly from (43) the minimum in (42) is reached for $\mathbf{M} = M^*(k+1)$. By making $\mathbf{M} = M^*(k+1)$ in (42) and after some algebraic manipulations we obtain (40), completing the proof. ■

Remark 1. If for each $k = T-1, \ldots, 0$ we have that $\Psi(k) > 0$ then clearly (39) is satisfied, $\Psi(k)^\dagger = \Psi(k)^{-1}$, and the optimal solution (41) is unique. If $I + \Lambda(k+1)\mathrm{B}(k)$ is non-singular then since $\Psi(k) = \mathrm{B}(k)(I + \Lambda(k+1)\mathrm{B}(k))$ it follows that $Im(\Psi(k)) = Im(\mathrm{B}(k))$ and clearly (39) is satisfied.

From Theorem 4 we have the following sufficient condition algorithm to determine the optimal control strategy for the generalized mean-variance portfolio optimization problem. Step i): Evaluate recursively the matrices $\Lambda(k)$ and $\Psi(k)$ and vectors $L(k)$ as in (36)-(37) for $k = T-1, \ldots, 0$. Step ii): Evaluate recursively $q^{M^*}(k)$ given by (32) for the strategy M^* as in (41) for $k = 0, \ldots, T$. Step iii): The optimal strategy is given by (28) with $M = M^*$.

Remark 2. From Theorem 3 the solution obtained from the sufficient condition algorithm must coincide with the solution obtained from the necessary condition algorithm presented in the previous subsection.

Remark 3. We show next that when $\alpha(t) = 0$ for $t = 1, \ldots, T-1$ and $\alpha(T) = 1$, $\ell(T) = 1$, $\rho(T) = \nu(T) = \nu$ (so that $\iota_f = 1$) our results reduce to those in (Çakmak and Özekici [2006]). Indeed, we have that $K(k) = \nu\mathcal{K}(k)$ and $\mathcal{A}_0 = \mathcal{R}(T,0)'$, $\mathcal{B}_0 = \sum_{k=0}^{T-1}\mathcal{R}(T,k+1)'B(k)\mathcal{R}(T,k+1)$, $\mathcal{G}_1 = -2\nu ee'$, $\mathcal{S}_1 = -e$. Thus, $\left(I + \frac{1}{2}\mathcal{B}_0\mathcal{G}_1\right) = (I - \nu\mathcal{B}_0 ee')$ and $e'\left(I - \nu\mathcal{B}_0 ee'\right)^{-1} = \left(I - \nu e'\mathcal{B}_0 eI\right)^{-1}e' = \frac{1}{1-\nu e'\mathcal{B}_0 e}e'$. Defining $a = q(0)'\mathcal{Z}(0), c = g(0)'\mathcal{K}(0)$, $b = \sum_{k=0}^{T-1} h(\mathcal{Z},\mathcal{K},k)'\mathcal{Z}(k+1)$, we get, after some algebraic manipulations, that $\nu e'\mathcal{B}_0 e = 2b$. Clearly the condition for existence of a inverse of the matrix $\left(I + \frac{1}{2}\mathcal{G}_1\mathcal{B}_0\right) = (I - \nu ee'\mathcal{B}_0)$ is that $b \neq \frac{1}{2}$ (indeed, by contradiction, if $b \neq \frac{1}{2}$ and there exists $x \in \mathbb{R}^m$, $x \neq 0$ such that $x'\left(I - \nu ee'\mathcal{B}_0\right) = 0$ then $x'\left(I - \nu ee'\mathcal{B}_0\right)e = x'e(1-2b) = 0$ so that $x'e = 0$ but in this case $0 = x'\left(I - \nu ee'\mathcal{B}_0\right) = x'$ which is a contradiction). From (26) and recalling that $e'q(T) = E(V^u(T))$, $e'\mathcal{A}_0 = \mathcal{Z}(0)'$ we get that $E(V^u(T)) = \frac{1}{1-2b}a + \frac{b}{\nu(1-2b)} = \frac{a\nu+b}{\nu(1-2b)}$. Thus $\lambda(T) = 1 + \nu E(V^u(T)) = \frac{1+2\nu a}{1-2b}$, in agreement with Çakmak and Özekici [2006].

5. CONCLUSION

In this paper we studied a discrete-time generalized mean-variance portfolio selection problem subject to Markovian jumps in the parameters. We analytically derived a closed form expression for an optimal investment strategy and showed that this optimal policy depends upon a set of interconnected Riccati difference equations presented in (5) and other recursive equations. We also provided a necessary condition algorithm and a sufficient condition algorithm for determining this optimal strategy. Finally, we showed that our results coincide with the special case presented in (Çakmak and Özekici [2006]), for the multi-period mean-variance portfolio selection problem subject to Markovian jumps in the parameters when the objective function and constraints consider only the final value of the expected value or the variance of the wealth.

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