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Predictive Control of Hybrid Systems: Stability results for sub-optimal solutions

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Abstract: This article presents a novel model predictive control (MPC) scheme that achieves input-to-state stabilization of constrained discontinuous nonlinear and hybrid systems. Input-to-state stability (ISS) is guaranteed when an optimal solution of the MPC optimization problem is attained. Special attention is paid to the effect that sub-optimal solutions have on ISS of the closed-loop system. This issue is of interest as firstly, the infimum of MPC optimization problems does not have to be attained and secondly, numerical solvers usually provide only sub-optimal solutions. An explicit relation is established between the deviation of the predictive control law from the optimum (called the *optimality margin*) and the resulting deterioration of the ISS property of the closed-loop system (called the *ISS margin*).

Keywords: Hybrid systems, Discontinuous systems, Model predictive control, Sub-optimality, Input-to-state stability.

1. INTRODUCTION

Discrete-time discontinuous nonlinear and hybrid systems form a powerful and general modeling class for the approximation of hybrid and nonlinear phenomena [Branicky et al., 1998], which also includes the class of piecewise affine (PWA) systems [Heemels et al., 2001]. The modeling capability of the latter class of systems has already been shown in several applications, including switched power converters, automotive systems and systems biology, to mention just a few. As a consequence, there is an increasing interest in developing synthesis techniques for robust control of discrete-time hybrid systems. The model predictive control (MPC) methodology [Mayne et al., 2000] has proven to be one of the most successful frameworks for this task, see, for example, [Bemporad and Morari, 1999, Kerrigan and Mayne, 2002, Lazar et al., 2006] and the references therein.

In this paper we are interested in input-to-state stability (ISS) [Jiang and Wang, 2001] as a property to characterize robust stability of hybrid systems in closed-loop with MPC. More precisely, we consider systems that are piecewise continuous, affected by additive disturbances. It is known (for example, see [Lazar, 2006] and the references therein) that for such discontinuous systems most of the results obtained for predictive control of continuous nonlinear systems [Mayne et al., 2000, Limon et al., 2002, Grimm et al., 2007] do not necessarily apply. Only the min-max MPC methodology [Magni et al., 2006, Lazar et al., 2008] could be an alternative, but the prohibitive computational complexity of min-max MPC schemes prevents implementation even for linear systems. As such, computationally feasible input-to-state stabilizing predictive controllers are widely unavailable.

In what follows we propose a tightened constraints MPC scheme for *discontinuous systems* along with conditions for ISS of the resulting closed-loop system, assuming that optimal MPC control sequences are implemented. These results provide advances with respect to the existing works on tightened constraints MPC [Limon et al., 2002, Grimm et al., 2007], where continuity of the system dynamics is assumed, towards discontinuous and hybrid systems. Guaranteeing robust stability and feasibility in the presence of discontinuities is difficult and requires an innovative usage of tightened constraints, which is conceptually different from the approaches in [Limon et al., 2002, Grimm et al., 2007]. Therein tightened constraints are employed for *robust feasibility* only. However, by carefully matching the new tightening approach with the discontinuities in the system dynamics, we achieve *both robust feasibility and ISS* for the optimal case. Another issue that is widely neglected in MPC for hybrid systems is the effect of sub-optimal implementations. In particular, an important result was recently presented in [Spjøtvold et al., 2007], where it was shown that in the case of optimal control of discontinuous PWA systems it is not uncommon that there does not even exist a control law that attains the infimum. Moreover, numerical solvers usually provide only sub-optimal solutions. Thus, for hybrid systems it is necessary to study if and how stability results for optimal predictive control change in the case of sub-optimal implementations, which will be done in this paper.

To cope with MPC control sequences (obtained by solving MPC optimization problems) that are not optimal, but within a margin $\delta \geq 0$ from the optimum, we introduce the notion of ε-ISS as a particular case of the input-to-state practical stability (ISpS) property [Jiang et al., 1996]. Furthermore, we establish an analytic relation between the *optimality margin* δ of the solution of the MPC optimization problem and the *ISS margin* $\varepsilon(\delta)$. Revealing this explicit relation is an important result, as it provides an a priori bound on the evolution of the closed-loop system state and leads to conditions that guarantee ISS even in the presence of unaccounted sub-optimal solutions.

While the ISS results presented in this paper require the use of a specific robust MPC problem formulation (i.e. based on tightened constraints), we also show that nominal asymptotic stability can be guaranteed for sub-optimal MPC of hybrid systems without any modification to the nominal MPC set-up presented in [Mayne et al., 2000]. Note that the classical way to guarantee stability of sub-optimal MPC is to explicitly include an additional stabilization constraint in the original MPC set-up, which enforces that the MPC cost function is a Lyapunov function for any feasible input sequence [Scokaert et al., 1999]. The explicit relation between the optimality margin δ and the ISS margin $\varepsilon(\delta)$ presented in this paper leads to a fundamentally different approach, as we do not change the original MPC set-up.

1.1 Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z} \mid k \geq c_1\}$ and $\{k \in \mathbb{Z} \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}$. For $x \in \mathbb{R}^n$ let $\|x\|$ denote an arbitrary Hölder p-norm and for $Z \in \mathbb{R}^{m\times n}$, let $\|Z\|$ denote the corresponding induced matrix norm. We will use both $(z(0), z(1), \ldots)$ and $\{z(l)\}_{l\in\mathbb{Z}_+}$ with $z(l) \in \mathbb{R}^n$, $l \in \mathbb{Z}_+$, to denote a sequence. For a sequence $\mathbf{z} := \{z(l)\}_{l\in\mathbb{Z}_+}$ let $\|\mathbf{z}\| := \sup\{\|z(l)\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{z}_{[k]} := \{z(l)\}_{l\in\mathbb{Z}_{[0,k]}}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{S}$ the boundary, by $\mathrm{int}(\mathcal{S})$ the interior and by $\mathrm{cl}(\mathcal{S})$ the closure of $\mathcal{S}$. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \sim \mathcal{P} := \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote their Pontryagin difference. For any $\mu > 0$ we define $\mathcal{B}_\mu$ as $\{x \in \mathbb{R}^n \mid \|x\| \leq \mu\}$. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

2. PRELIMINARIES

In this section we introduce the notion of ε-input-to-state stability (ε-ISS) for discrete-time systems of the form:

$$x(k+1) \in G(x(k), w(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{W} \subseteq \mathbb{R}^l$ is an unknown input at discrete-time instant $k \in \mathbb{Z}_+$ and $G : \mathbb{R}^n \times \mathbb{R}^l \to 2^{(\mathbb{R}^n)}$ is an arbitrary nonlinear, possibly discontinuous, set-valued function. For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero input, i.e. $G(0, 0) = \{0\}$. $\mathbb{W} \subseteq \mathbb{R}^l$ is assumed to be a bounded set.

Definition 1. ***RPI*** We call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *robustly positively invariant (RPI)* for system (1) with respect to $\mathbb{W}$ if for all $x \in \mathcal{P}$ and all $w \in \mathbb{W}$ it holds that $G(x, w) \subseteq \mathcal{P}$.

Definition 2. ε***-ISS*** Let $\mathbb{X}$ with $0 \in \mathrm{int}(\mathbb{X})$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. For a given $\varepsilon \in \mathbb{R}_+$, the perturbed system (1) is called ε*-ISS in* $\mathbb{X}$ *for inputs in* $\mathbb{W}$ if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each $x(0) \in \mathbb{X}$ and all $\mathbf{w} = \{w(l)\}_{l\in\mathbb{Z}_+}$ with $w(l) \in \mathbb{W}$ for all $l \in \mathbb{Z}_+$, it holds that all state trajectories of (1) with initial state $x(0)$ and input sequence $\mathbf{w}$ satisfy

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|) + \varepsilon, \quad \forall k \in \mathbb{Z}_{\geq 1}.$$

We call system (1) *ISS in* $\mathbb{X}$ *for inputs in* $\mathbb{W}$ if (1) is 0-ISS in $\mathbb{X}$ for disturbances in $\mathbb{W}$.

Definition 3. ε***-AS*** For a given $\varepsilon \in \mathbb{R}_+$, the 0-input system (1), i.e. $x(k+1) \in G(x(k), 0)$, $k \in \mathbb{Z}_+$, is called ε*-asymptotically stable (ε-AS) in* $\mathbb{X}$ if there exists a $\mathcal{KL}$-function β such that, for each $x(0) \in \mathbb{X}$ it holds that all state trajectories with initial state $x(0)$ satisfy $\|x(k)\| \leq \beta(\|x(0)\|, k) + \varepsilon$, $\forall k \in \mathbb{Z}_{\geq 1}$. We call the 0-input system (1) *AS in* $\mathbb{X}$ if it is 0-AS in $\mathbb{X}$.

The ε-ISS property defined above is a regional version (i.e. for states in $\mathbb{X}$ and disturbances in $\mathbb{W}$) of the ISpS property defined in [Jiang et al., 1996]. However, here ε is not introduced to cope with *persistent disturbances*, but it will be related to sub-optimality of MPC control laws. We refer to ε by the term *ISS (AS) margin*.

Theorem 4. Let d_1, d_2 be non-negative constants, let a, b, c, λ be positive reals with $c \leq b$ and let $\alpha_1(s) := as^\lambda$, $\alpha_2(s) := bs^\lambda$, $\alpha_3(s) := cs^\lambda$ and $\sigma \in \mathcal{K}$. Furthermore, let $\mathbb{X}$ be a RPI set for system (1) with respect to $\mathbb{W}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1, \tag{2a}$$

$$V(x^+) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) + d_2 \tag{2b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $x^+ \in G(x, w)$. Then the system (1) is ε-ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$ with

$$\beta(s, k) := \alpha_1^{-1}(3\rho^k\alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{3\sigma(s)}{1-\rho}\right),$$

$$\varepsilon := \alpha_1^{-1}\left(3\left(d_1 + \frac{d_2}{1-\rho}\right)\right), \quad \rho := 1 - \frac{c}{b} \in [0, 1). \tag{3}$$

If the inequalities (2) hold for $d_1 = d_2 = 0$, the system (1) is ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$.

The proof of Theorem 4 is similar in nature to the proof given in [Lazar, 2006] by replacing the difference *equation* by a difference *inclusion* as in (1) and is omitted here. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem 4 an ε*-ISS function*.

3. MPC SCHEME SET-UP

In this section we present an MPC scheme for discrete-time piecewise continuous (PWC) nonlinear systems

$$x(k+1) = g(x(k), u(k), w(k)) := g_j(x(k), u(k)) + w(k) \quad \text{if } x(k) \in \Omega_j, \quad k \in \mathbb{Z}_+, \tag{4}$$

where each $g_j : \Omega_j \times \mathbb{U} \to \mathbb{R}^n$, $j \in \mathcal{S}$, is a continuous function in x and $\mathcal{S} := \{1, 2, \ldots, s\}$ is a *finite set* of indices. We assume that the state x and the input u are constrained in some sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ that contain the origin in their interior. The collection $\{\Omega_j \subseteq \mathbb{R}^n \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$, with the sets Ω_j not necessarily closed. We also assume that the disturbance w takes values in the set $\mathbb{W} := \mathcal{B}_\mu$ with $\mu \in \mathbb{R}_{>0}$ sufficiently small to be determined.

Consider now the following assumption on system (4).

Assumption 5. For each fixed $j \in \mathcal{S}$, $g_j(\cdot,\cdot)$ satisfies a continuity condition in the first argument in the sense that there exists a $\mathcal{K}$-function η_j such that

$$\|g_j(x,u) - g_j(y,u)\| \leq \eta_j(\|x-y\|),\ \forall x,y \in \Omega_j,\ \forall u \in \mathbb{U},$$

and $\exists j_0 \in \mathcal{S}$ such that $0 \in \operatorname{int}(\Omega_{j_0})$ and $g_{j_0}(0,0) = 0$.

Note that we allow $g(\cdot,\cdot,\cdot)$ to be discontinuous in x over the switching boundaries, which makes discontinuous PWA systems a sub-class of PWC systems of the form (4).

For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $(\phi(1),\ldots,\phi(N))$ denote a state sequence generated by the *unperturbed* system corresponding to (4), i.e.

$$\phi(i+1) := g_j(\phi(i),u(i)) \text{ if } \phi(i) \in \Omega_j,\ i = 0,\ldots,N-1, \tag{5}$$

from initial condition $\phi(0) := x(k)$ and by applying an input sequence $\mathbf{u}_{[N-1]} = (u(0),\ldots,u(N-1)) \in \mathbb{U}^N := \mathbb{U} \times \ldots \times \mathbb{U}$. Let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a set with $0 \in \operatorname{int}(\mathbb{X}_T)$. Define $\eta(s) := \max_{j\in\mathcal{S}} \eta_j(s)$ and let $\eta^{[p]}(s)$ denote the p-times function composition with $\eta^{[0]}(s) := s$ and $\eta^{[k]}(s) = \eta(\eta^{[k-1]}(s))$ for $k \in \mathbb{Z}_{\geq 1}$. As the maximum of a finite number of $\mathcal{K}$-functions is also a $\mathcal{K}$-function, $\eta \in \mathcal{K}$. For any $\mu > 0$ and $i \in \mathbb{Z}_{\geq 1}$, define

$$\mathcal{L}_\mu^i := \left\{ \zeta \in \mathbb{R}^n \,\middle|\, \|\zeta\| \leq \sum_{p=0}^{i-1} \eta^{[p]}(\mu) \right\}.$$

Define the *set of admissible input sequences* for $x \in \mathbb{X}$ as:

$$\mathcal{U}_N(x) := \{\mathbf{u}_{[N-1]} \in \mathbb{U}^N \mid \phi(i) \in \mathbb{X}_i,\ i = 1,\ldots,N-1,\ \phi(0) = x,\ \phi(N) \in \mathbb{X}_T\}, \tag{6}$$

where $\mathbb{X}_i := \cup_{j\in\mathcal{S}}\{\Omega_j \sim \mathcal{L}_\mu^i\} \subset \mathbb{X}, \quad \forall i = 1,\ldots,N-1$.

The purpose of the above set of input sequences will be made clear in Lemma 11. For a given $N \in \mathbb{Z}_{\geq 1}$, notice that $\mu > 0$ has to be sufficiently small so that $0 \in \operatorname{int}(\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1})$. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $F(0) = L(0,0) = 0$ be arbitrary nonlinear mappings.

Problem 6. ***MPC optimization problem*** Let $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given. At time $k \in \mathbb{Z}_+$ let $x(k) \in \mathbb{X}$ be given and *infimize* the cost $J(x(k),\mathbf{u}_{[N-1]}) := F(\phi(N)) + \sum_{i=0}^{N-1} L(\phi(i),u(i))$ over all sequences $\mathbf{u}_{[N-1]}$ in $\mathcal{U}_N(x(k))$.

We call a state $x \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x) \neq \emptyset$. Problem 6 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* for Problem 6. Let $V^*(x) := \inf_{\mathbf{u}_{[N-1]}\in\mathcal{U}_N(x)} J(x,\mathbf{u}_{[N-1]})$. Since $J(\cdot,\cdot)$ is lower bounded by 0, the infimum exists. As such, $V^*(x)$ is well defined for all $x \in \mathbb{X}_f(N)$. However, the infimum is not necessarily attainable, meaning that the infimum is not necessarily a minimum. Indeed, in [Spjøtvold et al., 2007] it was shown that the non-closedness of the regions Ω_j in the partition may lead to optimization problems for which there does not exist a control law that attains the infimum. Hence, only sub-optimal (though arbitrarily close to the optimum) solutions can be found. This phenomenon together with the fact that numerical solvers usually provide sub-optimal solutions only, motivates the need for results that guarantee robustness of predictive control laws in the case of sub-optimal implementations. As such, in this article we will consider the following set of sub-optimal control sequences. For any $x \in \mathbb{X}_f(N)$ and $\delta \geq 0$, we define

$$\Pi_\delta(x) := \{\mathbf{u}_{[N-1]} \in \mathcal{U}_N(x) \mid J(x,\mathbf{u}_{[N-1]}) \leq V^*(x) + \delta\}$$

and $\pi_\delta(x) := \{u(0) \in \mathbb{R}^m \mid \mathbf{u}_{[N-1]} \in \Pi_\delta(x)\}$. We will refer to δ by the term *optimality margin* [1]. For example, an optimality margin δ can be guaranteed a priori by using the sub-optimal mixed integer linear programming (MILP) solver proposed in [Spjøtvold et al., 2007]. Furthermore, quadratic programming (QP) and linear programming (LP) solvers that are employed by existing MIQP and MILP solvers [Holmström, 1999] usually allow the specification of a tolerance with respect to achieving the optimum. Therefore, the tolerance parameter can be used to set a desired optimality margin.

In the next section we will establish $\varepsilon(\delta)$-ISS and ISS results for the sub-optimal MPC closed-loop system corresponding to (4), which is given by the difference inclusion

$$x(k+1) \in \Phi_\delta(x(k),w(k)) := \{g(x(k),u,w(k)) \mid u \in \pi_\delta(x(k))\},\ k \in \mathbb{Z}_+. \tag{7}$$

To simplify the exposition we will make use of the following commonly adopted assumptions in tightened constraints MPC [Limon et al., 2002, Grimm et al., 2007].

Assumption 7. There exist $\mathcal{K}$-functions α_L, $\alpha_F := \tau s^\lambda$, $\alpha_1(s) := as^\lambda$ and $\alpha_2(s) := bs^\lambda$, $\tau, a, b, \lambda \in \mathbb{R}_{>0}$, such that
(i) $L(x,u) \geq \alpha_1(\|x\|)$, $\forall x \in \mathbb{X}$, $\forall u \in \mathbb{U}$;
(ii) $|L(x,u) - L(y,u)| \leq \alpha_L(\|x-y\|)$, $\forall x,y \in \mathbb{X}$, $\forall u \in \mathbb{U}$;
(iii) $|F(x) - F(y)| \leq \alpha_F(\|x-y\|)$, $\forall x,y \in \Omega_{j_0} \cap \mathcal{L}_\mu^{N-1}$;
(iv) $V^*(x) \leq \alpha_2(\|x\|)$, $\forall x \in \mathbb{X}_f(N)$.

Assumption 8. There exist $N \in \mathbb{Z}_{\geq 1}$, $\theta > \theta_1 > 0$, $\mu > 0$ and a mapping $h(\cdot)$ such that
(i) $\alpha_F(\eta^{[N-1]}(\mu)) \leq \theta - \theta_1$;
(ii) $\mathbb{F}_\theta := \{x \in \mathbb{R}^n \mid F(x) \leq \theta\} \subseteq (\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_\mathbb{U}$ and $g_{j_0}(x,h(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$;
(iii) $F(g_{j_0}(x,h(x))) - F(x) + L(x,h(x)) \leq 0$, $\forall x \in \mathbb{F}_\theta$.

Note that the hypotheses in Assumption 7-(i),(ii),(iii) usually hold by definition of $L(\cdot,\cdot)$ and $F(\cdot)$, e.g. when these cost functions are defined via quadratic forms or $1,\infty$-norms. Also, it can be shown that the hypothesis of Assumption 7-(iv) holds, even for discontinuous value functions. For details and for techniques for computing a terminal cost and how to choose N and μ such that the hypotheses in Assumption 8-(i),(ii),(iii) are satisfied we refer the interested reader to [Lazar, 2006].

4. INPUT-TO-STATE STABILITY RESULTS

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ denote a terminal control law and define $\mathbb{X}_\mathbb{U} := \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$.

Theorem 9. Let $\delta \in \mathbb{R}_{>0}$ be given, suppose that Assumption 5, Assumption 7 and Assumption 8 hold for the nonlinear hybrid system (4) and Problem 6, and set $\mathbb{X}_T = \mathbb{F}_{\theta_1}$. Then:

(i) If Problem 6 is feasible at time $k \in \mathbb{Z}_+$ for state $x(k) \in \mathbb{X}$, then Problem 6 is feasible at time $k+1$ for any state $x(k+1) \in \Phi_\delta(x(k),w(k))$ and all $w(k) \in \mathcal{B}_\mu$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$;

(ii) The closed-loop system $x(k+1) \in \Phi_\delta(x(k),w(k))$ is $\varepsilon(\delta)$-ISS in $\mathbb{X}_f(N)$ for inputs in $\mathcal{B}_\mu$ with ISS margin $\varepsilon(\delta) := \left(\frac{3b}{a^2}\delta\right)^{\frac{1}{\lambda}}$.

[1] Note that $\delta = 0$ and $\Pi_\delta(x) \neq \emptyset$ corresponds to the situation when the global optimum is attained in Problem 6.

To prove Theorem 9 we will make use of the following technical lemmas (see the appendix for the proofs).

Lemma 10. Let $x \in \Omega_j \sim \mathcal{L}_\mu^{i+1}$ for some $j \in \mathcal{S}, i \in \mathbb{Z}_+$, and let $y \in \mathbb{R}^n$. If $\|y - x\| \leq \eta^{[i]}(\mu)$, then $y \in \Omega_j \sim \mathcal{L}_\mu^i$.

Lemma 11. Let $(\phi(1), \ldots, \phi(N))$ be a state sequence of the unperturbed system (5), obtained from initial state $\phi(0) := x(k) \in \mathbb{X}$ and by applying an input sequence $\mathbf{u}_{[N-1]} = (u(0), \ldots, u(N-1)) \in \mathcal{U}_N(x(k))$. Let $(j_1, \ldots, j_{N-1}) \in \mathcal{S}^{N-1}$ be the corresponding mode sequence in the sense that $\phi(i) \in \Omega_{j_i} \sim \mathcal{L}_\mu^i \subset \Omega_{j_i}$, $i = 1, \ldots, N-1$. Let $(\bar{\phi}(1), \ldots, \bar{\phi}(N))$ be also a state sequence of the unperturbed system (5), obtained from the initial state $\bar{\phi}(0) := x(k+1) = \phi(1) + w(k)$ for some $w(k) \in \mathcal{B}_\mu$ and by applying the shifted input sequence $\bar{\mathbf{u}}_{[N-1]} := (u(1), \ldots, u(N-1), h(\bar{\phi}(N-1)))$.

Then, it holds that

$$(\bar{\phi}(i), \phi(i+1)) \in \Omega_{j_{i+1}} \times \Omega_{j_{i+1}} \quad \text{for} \quad i = 0, \ldots, N-2, \tag{8a}$$

$$\|\bar{\phi}(i) - \phi(i+1)\| \leq \eta^{[i]}(\|w(k)\|) \quad \text{for} \quad i = 0, \ldots, N-1. \tag{8b}$$

Proof. *(Proof of Theorem 9)*
(i) We will show that $\bar{\mathbf{u}}_{[N-1]}$, as defined in Lemma 11, is a feasible sequence of inputs at time $k+1$. Let $(j_1, \ldots, j_{N-1}) \in \mathcal{S}^{N-1}$ be such that $\phi(i) \in \Omega_{j_i} \sim \mathcal{L}_\mu^i \subset \Omega_{j_i}$, $i = 1, \ldots, N-1$. Then, due to property (8b) and $\phi(i+1) \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^{i+1}$, it follows from Lemma 10 that $\bar{\phi}(i) \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^i \subset \mathbb{X}_i$ for $i = 1, \ldots, N-2$. From

$$\|\bar{\phi}(N-1) - \phi(N)\| \leq \eta^{[N-1]}(\|w(k)\|) \leq \eta^{[N-1]}(\mu)$$

and Assumption 7-(iii) it follows that

$$F(\bar{\phi}(N-1)) - F(\phi(N)) \leq \alpha_F(\eta^{[N-1]}(\mu)),$$

which implies $F(\bar{\phi}(N-1)) \leq \theta_1 + \alpha_F(\eta^{[N-1]}(\mu)) \leq \theta$ due to $\phi(N) \in \mathbb{X}_T = \mathbb{F}_{\theta_1}$ and $\alpha_F(\eta^{[N-1]}(\mu)) \leq \theta - \theta_1$. Hence $\bar{\phi}(N-1) \in \mathbb{F}_\theta \subset \mathbb{X}_\mathbb{U} \cap (\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1}) \subset \mathbb{X}_\mathbb{U} \cap \mathbb{X}_{N-1}$, so that $h(\bar{\phi}(N-1)) \in \mathbb{U}$ and $\bar{\phi}(N) \in \mathbb{F}_{\theta_1} = \mathbb{X}_T$. Thus, the sequence $\bar{\mathbf{u}}_{[N-1]}$ is feasible at time $k+1$, which proves the first part of (i). Moreover, since $g_{j_0}(x, h(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$ and $\mathbb{F}_{\theta_1} \subset \mathbb{F}_\theta$ it follows that $\mathbb{F}_{\theta_1}$ is a positively invariant set for system $x(k+1) = g_{j_0}(x, h(x(k))), k \in \mathbb{Z}_+$. Then, since $\mathbb{F}_{\theta_1} \subset \mathbb{F}_\theta \subseteq (\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_\mathbb{U} \subset \mathbb{X}_i \cap \mathbb{X}_\mathbb{U}$ for all $i = 1, \ldots, N-1$ and $\mathbb{X}_T = \mathbb{F}_{\theta_1}$, the sequence $(h(\phi(0)), \ldots, h(\phi(N-1)))$ is feasible for Problem 6 for all $\phi(0) := x(k) \in \mathbb{F}_{\theta_1}$, $k \in \mathbb{Z}_+$. Therefore, $\mathbb{X}_T = \mathbb{F}_{\theta_1} \subseteq \mathbb{X}_f(N)$, which concludes the proof of (i).

(ii) The result of part (i) implies that $\mathbb{X}_f(N)$ is an RPI set for the closed-loop system $x(k+1) \in \Phi_\delta(x(k), w(k))$, $k \in \mathbb{Z}_+$. Moreover, $0 \in \text{int}(\mathbb{X}_T)$ implies that $0 \in \text{int}(\mathbb{X}_f(N))$. We will now prove that $V^*(\cdot)$ is an ε-ISS function for the closed-loop system (7). Since for any $x \in \mathbb{X}$ and $\mathbf{u}_{[N-1]} \in \mathcal{U}_N(x)$ it holds that $J(x, \mathbf{u}_{[N-1]}) \geq L(x, u(0))$, from Assumption 7-(i) it follows that $V^*(x) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}_f(N)$, with $\alpha_1(s) = as^\lambda$. Furthermore, by Assumption 7-(iv), for all $x \in \mathbb{X}_f(N)$ we have that $V^*(x) \leq \alpha_2(\|x\|)$, $\alpha_2(s) = bs^\lambda$. Hence, $V^*(\cdot)$ satisfies inequality (2a) with $d_1 = 0$ for all $x \in \mathbb{X}_f(N)$.

Next, we prove that $V^*(\cdot)$ satisfies inequality (2b). Let $x(k+1) \in \Phi_\delta(x(k), w(k))$ for some arbitrary $w(k) \in \mathcal{B}_\mu$. Furthermore, for any $\mathbf{u}_{[N-1]} \in \mathcal{U}_N(x(k))$ let $\bar{\mathbf{u}}_{[N-1]}$ be defined as in Lemma 11. Notice that $J(x(k), \bar{\mathbf{u}}_{[N-1]}) \leq V^*(x(k)) + \delta$ implies $-V^*(x(k)) \leq -J(x(k), \bar{\mathbf{u}}_{[N-1]}) + \delta$. Using Assumption 8-(iii), i.e.

$$F(g_{j_0}(x, h(x))) - F(x) + L(x, h(x)) \leq 0, \ \forall x \in \mathbb{F}_\theta,$$

property (8a), Assumption 7-(ii),(iii) and $\bar{\phi}(N-1) \in \mathbb{X}_T$, it follows that:

$$\begin{aligned}
&V^*(x(k+1)) - V^*(x(k)) \\
&\leq J(x(k+1), \bar{\mathbf{u}}_{[N-1]}) - J(x(k), \mathbf{u}_{[N-1]}) + \delta \\
&= -L(\phi(0), u(0)) + F(\bar{\phi}(N)) + \delta \\
&\quad + \left[-F(\bar{\phi}(N-1)) + F(\bar{\phi}(N-1))\right] \\
&\quad - F(\phi(N)) + L(\bar{\phi}(N-1), h(\bar{\phi}(N-1))) \\
&\quad + \sum_{i=0}^{N-2} \left[L(\bar{\phi}(i), u(i+1)) - L(\phi(i+1), u(i+1))\right] \\
&\leq -L(\phi(0), u(0)) + F(\bar{\phi}(N)) - F(\bar{\phi}(N-1)) \\
&\quad + L(\bar{\phi}(N-1), h(\bar{\phi}(N-1))) \\
&\quad + \alpha_F(\eta^{[N-1]}(\|w(k)\|)) + \sum_{i=0}^{N-2} \alpha_L(\eta^{[i]}(\|w(k)\|)) + \delta \\
&\leq -\alpha_3(\|x(k)\|) + \sigma(\|w(k)\|) + \delta,
\end{aligned}$$

with $\sigma(s) := \alpha_F(\eta^{[N-1]}(s)) + \sum_{i=0}^{N-2} \alpha_L(\eta^{[i]}(s))$ and $\alpha_3(s) := \alpha_1(s) = as^\lambda$. Notice that $\sigma \in \mathcal{K}$ due to $\alpha_F, \alpha_L, \eta \in \mathcal{K}$. The statement then follows from Theorem 4. Moreover, from (3) it follows that the ε-ISS property of Definition 2 holds with $\varepsilon(\delta) = \left(\frac{3b}{a^2}\delta\right)^{\frac{1}{\lambda}}$. □

Theorem 9 enables the proper selection of an optimality margin δ in the numerical solver by choosing a desirable ISS margin $\varepsilon(\delta)$ and finding the corresponding value of δ. Also, Theorem 9 recovers as a particular case the following result for the *optimal* case published in [Lazar et al., 2005], where only PWA systems were considered.

Corollary 12. Suppose that the hypothesis of Theorem 9 is satisfied and the global optimum is attained in Problem 6 for all $k \in \mathbb{Z}_+$. Then, the closed-loop system $x(k+1) \in \Phi_0(x(k), w(k))$ is ISS in $\mathbb{X}_f(N)$ for inputs in $\mathcal{B}_\mu$.

Remark 13. The result of Corollary 12 recovers the result in [Limon et al., 2002] as the following particular case: $\mathbb{X} = \Omega_{j_0}$, $\mathcal{S} = \{j_0\}$ and $g_{j_0}(\cdot, \cdot)$ is Lipschitz continuous in $\mathbb{X}$. In this case, the set of admissible input sequences $\mathcal{U}_N(x)$ only plays a role in guaranteeing recursive feasibility of Problem 6, while ISS can be established directly from Lipschitz continuity of the dynamics, see [Limon et al., 2002] for details. See also [Grimm et al., 2007] where the Lipschitz continuity of system dynamics is relaxed to basic continuity. Corollary 12 also relaxes the Lipschitz continuity requirement to a kind of uniform continuity in the region Ω_{j_0} and furthermore, *allows for discontinuous nonlinear dynamics*, while the assumptions on the MPC cost, prediction horizon and disturbance bound $\mu > 0$ are not stronger than the ones employed in [Limon et al., 2002].

Next, we present a modification to the set of δ sub-optimal MPC controllers that will enable to guarantee ISS of the closed-loop system a priori, even for non-zero optimality margins. For any $x \in \mathbb{X}_f(N)$ and $\delta \geq 0$ let

$$\overline{\Pi}_\delta(x) := \{\mathbf{u}_{[N-1]} \in \mathcal{U}_N(x) \mid J(x, \mathbf{u}_{[N-1]}) \leq V^*(x) + \delta\|x\|^\lambda\}$$

and $\overline{\pi}_\delta(x) := \{u(0) \in \mathbb{R}^m \mid \mathbf{u}_{[N-1]} \in \overline{\Pi}_\delta(x)\}$. The MPC closed-loop system corresponding to (4) is now given by

$$x(k+1) \in \overline{\Phi}_\delta(x(k), w(k)) := \{g(x(k), u, w(k)) \mid u \in \overline{\pi}_\delta(x(k))\},\ k \in \mathbb{Z}_+.$$

Note that for the above set of δ sub-optimal MPC control actions it holds that $\overline{\pi}_\delta(0) \equiv \overline{\pi}_0(0)$ for all $\delta > 0$. Hence, compared to the *absolute* δ sub-optimal MPC control laws, now δ is a relative optimality margin that varies with the size of the state norm. The closer the state gets to the origin, the better the approximation of the optimal MPC control law has to be. This is a realistic assumption, as there exists a sufficiently small neighborhood of the origin where all constraints in Problem 6 become inactive and there is no more switching in the predicted trajectory and as such the numerical problem to be solved becomes significantly simpler.

Theorem 14. Suppose that the hypotheses of Theorem 9 are satisfied with the $\mathcal{K}$-function $\alpha_1(s) := as^\lambda$, $a, \lambda \in \mathbb{R}_{>0}$, as introduced in Assumption 7. Let $\delta \in \mathbb{R}_{>0}$ be given such that $0 < \delta < a$. Then:

(i) If Problem 6 is feasible at time $k \in \mathbb{Z}_+$ for state $x(k) \in \mathbb{X}$, then Problem 6 is feasible at time $k+1$ for any state $x(k+1) \in \overline{\Phi}_\delta(x(k), w(k))$ and all $w(k) \in \mathcal{B}_\mu$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$;

(ii) The closed-loop system $x(k+1) \in \overline{\Phi}_\delta(x(k), w(k))$ is ISS in $\mathbb{X}_f(N)$ for inputs in $\mathcal{B}_\mu$.

Proof. The proof of Theorem 14 readily follows by applying the reasoning used in the proof of Theorem 9. The modified set of sub-optimal control laws $\overline{\pi}_\delta(x)$ makes a difference only in the proof of statement (ii), where $J(x(k), \bar{\mathbf{u}}_{[N-1]}) \leq V^*(x(k)) + \delta\|x(k)\|^\lambda$ implies $-V^*(x(k)) \leq -J(x(k), \bar{\mathbf{u}}_{[N-1]}) + \delta\|x(k)\|^\lambda$ and thus,

$$\begin{aligned} V^*(x(k+1)) - V^*(x(k)) \leq &\ \ldots \\ &\leq -\alpha_1(\|\phi(0)\|) + \sigma(\|w(k)\|) + \delta\|x(k)\|^\lambda \\ &= -\alpha_3(\|x(k)\|) + \sigma(\|w(k)\|), \end{aligned}$$

with $\sigma(s) := \alpha_F(\eta^{[N-1]}(s)) + \sum_{i=0}^{N-2} \alpha_L(\eta^{[i]}(s))$ and $\alpha_3(s) := (a-\delta)s^\lambda$. Note that $\alpha_3 \in \mathcal{K}$ as $a - \delta > 0$. □

Remark 15. In the particular case when system (4) is PWA, $\mathbb{X}$, $\mathbb{U}$, Ω_j, $j \in \mathcal{S}$ are polyhedral sets and the MPC cost function is defined using $1, \infty$-norms, Problem 6 can be formulated as a MILP problem, which is standard in hybrid MPC [Bemporad and Morari, 1999]. For methods to compute a terminal cost and control law $h(\cdot)$ that satisfy Assumption 7, Assumption 8 and for illustrative examples we refer the interested reader to [Lazar et al., 2005, 2006].

5. ASYMPTOTIC STABILITY RESULTS

Sufficient conditions for asymptotic stability of discrete-time PWA systems in closed-loop with MPC controllers were presented in [Lazar et al., 2006], under the standing assumption of global optimality for the MPC control law. As already mentioned in the introduction, it is important to analyze if and how the stability results of [Lazar et al., 2006] change in the case of sub-optimal implementations. For this purpose, we employ the ε-asymptotic stability property introduced in Section 2 and we consider the more general class of PWC nonlinear systems, i.e.

$$x(k+1) = \xi(x(k), u(k)) := g_j(x(k), u(k)) \ \text{ if } \ x(k) \in \Omega_j, \tag{9}$$

where the notation is similar to the one in Section 3. We still assume that each $g_j(\cdot,\cdot)$, $j \in \mathcal{S}$, satisfies a continuity condition as was defined in Assumption 5. However, we do not require anymore that the origin lies in the interior of one of the regions Ω_j in the state-space partition. The MPC problem set-up remains the same as the one described by Problem 6, with the only difference that the *set of admissible input sequences* for an initial condition $x \in \mathbb{X}$ is now defined as:

$$\mathcal{U}_N(x) := \{\mathbf{u}_{[N-1]} \in \mathbb{U}^N \mid \phi(i) \in \mathbb{X},\ i = 1, \ldots, N-1, \ \phi(0) = x,\ \phi(N) \in \mathbb{X}_T\}. \tag{10}$$

All the definitions introduced in Section 3 and Section 4 remain the same (e.g., $\mathbb{X}_f(N)$, $V^*(\cdot)$, $\Pi_\delta(\cdot)$, $\pi_\delta(\cdot)$, $\overline{\Pi}_\delta(\cdot)$, $\overline{\pi}_\delta(\cdot)$, etc.) with the observation that set of admissible input sequences defined in (6) is replaced everywhere with the set defined in (10). We will use

$$\Xi_\delta(x(k)) := \{\xi(x(k), u) \mid u \in \pi_\delta(x(k))\}$$

and

$$\overline{\Xi}_\delta(x(k)) := \{\xi(x(k), u) \mid u \in \overline{\pi}_\delta(x(k))\}.$$

Theorem 16. Let $\delta \in \mathbb{R}_{>0}$ be given and suppose that Assumption 7 holds for system (9) and Problem 6. Take $N \in \mathbb{Z}_{\geq 1}$, $\mathbb{X}_T$ with $0 \in \text{int}(\mathbb{X}_T)$ as a positively invariant set for system (9) in closed-loop with $u(k) = h(x(k))$, $k \in \mathbb{Z}_+$. Furthermore, suppose $F(\xi(x, h(x))) - F(x) + L(x, h(x)) \leq 0$ for all $x \in \mathbb{X}_T$.

(i) If Problem 6 is feasible at time $k \in \mathbb{Z}_+$ for state $x(k) \in \mathbb{X}$, then Problem 6 is feasible at time $k+1$ for any state $x(k+1) \in \Xi_\delta(x(k))$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$;

(ii) The closed-loop system $x(k+1) \in \Xi_\delta(x(k))$ is ε-AS in $\mathbb{X}_f(N)$ with $\varepsilon(\delta) := \left(\frac{2b}{a^2}\delta\right)^{\frac{1}{\lambda}}$;

(iii) Suppose that $\delta \in \mathbb{R}_{>0}$ satisfies $0 < \delta < a$, where $a \in \mathbb{R}_{>0}$ is the gain of the $\mathcal{K}$-function $\alpha_1(s) := as^\lambda$, introduced in Assumption 7. Then, the closed-loop system $x(k+1) \in \overline{\Xi}_\delta(x(k))$ is *AS in* $\mathbb{X}_f(N)$.

The proof of the above theorem can be obtained *mutatis mutandis* by combining the reasoning used in the proof of Theorem III.2 in [Lazar et al., 2006] (see also [Lazar, 2006]), and Theorem 4 for the case when $\sigma(s) \equiv 0$.

Remark 17. The result of Theorem 16, statement (ii), establishes that δ sub-optimal nonsmooth MPC is $\varepsilon(\delta)$-AS without requiring any additional assumption, other than the ones needed for AS of optimal smooth MPC [Mayne et al., 2000]. Furthermore, the result of Theorem 16, statement (iii), introduces a slightly stronger condition, under which even AS can be guaranteed a priori for a specific class of sub-optimal predictive control laws. In contrast with the results in [Scokaert et al., 1999] this is achieved without introducing additional stabilization constraints in the original MPC problem set-up.

6. CONCLUSION

In this paper we have considered hybrid systems in closed-loop with predictive control laws. We presented conditions for ε-ISS and ε-AS of the resulting closed-loop systems. These conditions do not require continuity of the system dynamics nor optimality of the predictive control law. The latter is especially important as firstly, the infimum in an MPC optimization problem does not have to be attained and secondly, numerical solvers usually provide only sub-optimal solutions. An explicit relation was established between the deviation of the MPC control action from the optimum (the so-called optimality margin δ) and the resulting deterioration of the ISS (AS) property of the closed-loop system in terms of the so-called ISS (AS) margin $\varepsilon(\delta)$. The link between the optimality margin of the MPC control action and the ISS (AS) margin of the closed-loop system was further exploited to derive stronger conditions that yield sub-optimal MPC controllers with an ISS (AS) guarantee, without adding additional constraints to the MPC optimization problem.

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Appendix A. PROOF OF LEMMA 10

Consider $y \in \mathbb{R}^n$ with $\|y - x\| \leq \eta^{[i]}(\mu)$. Let $\zeta \in \mathcal{L}_\mu^i$ and define $z := y - x + \zeta$. Then it holds that

$$\|z\| \leq \|y - x\| + \|\zeta\| \leq \eta^{[i]}(\mu) + \sum_{p=0}^{i-1} \eta^{[p]}(\mu) = \sum_{p=0}^{i} \eta^{[p]}(\mu)$$

and thus, $z \in \mathcal{L}_\mu^{i+1}$. Together with $x \in \Omega_j \sim \mathcal{L}_\mu^{i+1}$ this yields $x + z \in \Omega_j$. Hence, $y + \zeta = x + z \in \Omega_j$. Since $\zeta \in \mathcal{L}_\mu^i$ was arbitrary, we have $y \in \Omega_j \sim \mathcal{L}_\mu^i$. □

Appendix B. PROOF OF LEMMA 11

Property (8a) obviously holds for $i = 0$, since $\bar{\phi}(0) = \phi(1) + w(k)$, $w(k) \in \mathcal{B}_\mu = \mathcal{L}_\mu^1$ and $\phi(1) \in \Omega_{j_1} \sim \mathcal{L}_\mu^1$. Property (8b) holds for $i = 0$ as $\|\bar{\phi}(0) - \phi(1)\| = \|w(k)\| = \eta^{[0]}(\|w(k)\|)$.

We proceed by induction. Suppose that both (8a) and (8b) hold for $0 \leq i - 1 < N - 2$. Then, since $\phi(i-1) \in \Omega_{j_i}$ and $\|\bar{\phi}(i-1) - \phi(i)\| \leq \eta^{[i-1]}(\|w(k)\|)$, it follows that:

$$\begin{aligned}\|\bar{\phi}(i) - \phi(i+1)\| &= \|g_{j_i}(\bar{\phi}(i-1), u(i)) - g_{j_i}(\phi(i), u(i))\| \\ &\leq \eta_{j_i}(\|\bar{\phi}(i-1) - \phi(i)\|) \leq \eta(\|\bar{\phi}(i-1) - \phi(i)\|) \\ &\leq \eta(\eta^{[i-1]}(\|w(k)\|)) = \eta^{[i]}(\|w(k)\|), \qquad \text{(B.1)}\end{aligned}$$

and thus, (8b) holds for i. Next, as $\eta^{[i]}(\|w(k)\|) \leq \eta^{[i]}(\mu) \leq \sum_{p=0}^{i} \eta^{[p]}(\mu)$, it follows that $\bar{\phi}(i) - \phi(i+1) \in \mathcal{L}_\mu^{i+1}$. Then, since $\phi(i+1) \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^{i+1}$, we have that

$$\phi(i+1) + (\bar{\phi}(i) - \phi(i+1)) = \bar{\phi}(i) \in \Omega_{j_{i+1}}.$$

Hence, (8a) holds for i. Therefore, we have proven that (8a) holds for $i = 0, \ldots, N-2$ and (8b) holds for $i = 0, \ldots, N-2$. Finally, (8a) and (8b) with index $i = N-2$ imply (8b) with index $i = N-1$ via the reasoning used in (B.1). □