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ON INPUT-TO-OUTPUT STABILITY OF SWITCHED NONLINEAR SYSTEMS

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Abstract: The problem of input-to-output stability for switched nonlinear systems is considered.We investigate methods based both, on constant and average dwell-times. In particular, we show that constant dwell-time should depend on initial conditions and disturbances amplitudes to ensure global stability properties for a generic nonlinear switched system. *IFAC Copyright 2008.*
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1 Introduction

It is a well known fact that a switched system is not necessarily (Lyapunov) stable provided that the individual systems (without switches) are stable; conversely, a set of unstable systems may be rendered stable via proper switching –*cf.* (Liberzon 2003). For instance, commonly used assumptions include that switching is sufficiently slow –that the switching signal has the so-called *dwell-time*. Indeed, it may be shown that under a sufficiently large *constant* dwell-time stability (in various senses) of the individual subsystems implies a similar property for the switched system –*cf.* (Morse 1995, Hespanha *et al.* 2003, Hespanha and Morse 1999, Liberzon 1999, Liberzon 2003, Vu *et al.* 2005, Xie *et al.* 2001). The dwell-time assumption was weakened in (Hespanha and Morse 1999, Vu *et al.* 2005) by introducing the *average dwell-time* assumption; it is supposed that over a sufficiently large interval, a dwell-time spaced switch occurs yet, locally (in time) it is possible to switch from one subsystem to another at any desired rate. See also the recent article (Hespanha and Teel 2005) on input-output stability. Further, *state dependent dwell-time* was introduced in (Persis *et al.* 2003); this consists in making the length of time intervals between switches a function of the current system state, which in certain cases, leads to a more efficient switching rule.

The contribution of this paper is twofold: firstly, we consider uniform output stability –*cf.* (Fradkov *et al.* 1999, Rumyantsev and Oziraner 1987, Sontag and Wang 1999, Sontag and Wang 2001, Vorotnikov 1998) as opposed to classical Lyapunov stability; secondly, we relax the assumption on constant dwell-time to ensure global stability for the switched system, based on a uniform (in the initial conditions) stability property imposed on each subsystem. That is, we show that, in general, dwell-time shall depend on *initial conditions*. In the same context, we analyse output stability for systems with input disturbances and establish that dwell-time depends on the "amplitude" of such disturbances. Finally, we relax such conditions for exponentially stable systems.

2 Preliminaries

Consider a family of systems

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}, \mathbf{d}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \quad q \in I, \tag{1}$$

where $\mathbf{x} \in R^n$, $\mathbf{d} \in R^m$ is a disturbance, $\mathbf{y} \in R^p$ is the output of interest and q is an index, element of $I \subset \mathbb{Z}_{>0}$. We assume that $\mathbf{f}_q : R^{n+l} \to R^n$, $\mathbf{h} : R^n \to R^p$ are continuous and locally Lipschitz and that $\mathbf{d} : R_+ \to R^m$ is Lebesgue measurable and locally essentially bounded:

$$||\mathbf{d}||_{[0,t)} = ess \sup \{ |\mathbf{d}(s)|, \ s \in [0, t) \} .$$

We denote by M_{R^m} the set of globally essentially bounded functions, *i.e.*, that satisfy $||\mathbf{d}|| := \|\mathbf{d}\|_{[0,+\infty)} < \infty$.

A continuous function $\sigma : R_+ \to R_+$ is of class K if it is strictly increasing and $\sigma(0) = 0$; it is of class K_∞ if it is also radially unbounded; a continuous function $\beta : R_+ \times R_+ \to R_+$ is of class KL, if $\beta(\cdot, t)$ it is of class K for each t and $\beta(s, \cdot)$ is strictly decreasing to zero for each s.

Let $i : R_+ \to I$ be piecewise constant and right-continuous, the family of systems (1) defines the following switched system

$$\dot{\mathbf{x}} = \mathbf{f}_{i(t)}(\mathbf{x}, \mathbf{d}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}) . \tag{2}$$

Following (Morse 1995, Hespanha and Morse 1999, Vu *et al.* 2005, Persis *et al.* 2003) we recall the definition of dwell-time.

Definition 1 The switching signal $i(t)$ is said to have *average dwell-time* $0 < \tau_D < +\infty$ *if between switches and for any* $t_2 \geq t_1 \geq 0$*we have, for any integer* $1 \leq N_0 < +\infty$,

$$N_{[t_1,t_2)} \leq N_0 + \frac{t_2 - t_1}{\tau_D},$$

where $N_{[t_1,t_2)}$ *is the number of switches. If the interval between any two switches is not less than* τ_D*the switching signal* i *has the dwell-time property and* $N_0 = 1$*. The switching signal* $i(t)$ *is said to have state-dependent dwell-time* $\tau_D : R^n \to R_+$ *if, for any* $\mathbf{x} \in R^n$ *and some* $\delta \in K$*, the estimate* $0 < \tau_D(\mathbf{x}) \leq \delta(1 + |\mathbf{x}|)$ *holds.*

The system (2) where i has average dwell-time, state-dependent dwell-time or constant dwell-time, has a finite number of switches on any finite-time interval and its solution is continuous and defined, at least, locally.

The switched system, for a switching signal $i(t)$, is called forward complete if, for all initial conditions $\mathbf{x}_0 \in R^n$ and all inputs $\mathbf{d} \in M_{R^m}$, the solutions $\mathbf{x}(t, \mathbf{x}_0, \mathbf{d})$ of the switched system (2) are defined for all $t \geq 0$. We denote the output trajectories by $\mathbf{y}(t, \mathbf{x}_0, \mathbf{d}) = \mathbf{h}(\mathbf{x}(t, \mathbf{x}_0, \mathbf{d}))$ and, on occasions, we use the short-hand notation $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{x}_0, \boldsymbol{d})$, $\boldsymbol{y}(t) = \boldsymbol{y}(t, \boldsymbol{x}_0, \boldsymbol{d})$ and $\boldsymbol{y}_0 := \boldsymbol{y}(0, \boldsymbol{x}_0, 0))$. Furthermore, we recall the following two definitions from (Sontag and Wang 1999, Sontag and Wang 2001).

Definition 2 We say that for some fixed $q \in I$ *the forward complete system (1) is state independent IOS (SIIOS) with respect to the output* $\mathbf{y}$ *and the input* $\mathbf{d}$ *if, for all* $\mathbf{x}_0 \in R^n$ *and*

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$\mathbf{d} \in M_{R^m}$, *there exist functions* $\beta'_q \in KL, \gamma'_q \in K$ *such that* , for all $t \geq 0$,

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \ \leq \beta'_q(|\mathbf{y}_0)|, t) + \gamma'_q(||\mathbf{d}||_{[0,t)}) .$$

We say that the switched forward complete system (2) with $i : R_+ \to I$ *is SIIOS with respect to the output* $\mathbf{y}$ *and the input* $\mathbf{d}$ *if, for all* $\mathbf{x}_0 \in R^n$ *and* $\mathbf{d} \in M_{R^m}$, *there exist functions* $\beta' \in KL, \gamma' \in K$ *such that,* for all $t \geq 0$,

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \ \leq \beta'(|\mathbf{y}_0|, t) + \gamma'(||\mathbf{d}||_{[0,t)}) .$$

The systems are exponentially SIIOS if $\beta'_q(s, r) = a\, s\, e^{-b\, r}$ *or* $\beta'(s, r) = a\, s\, e^{-b\, r}$ *for some* $a > 0$, $b > 0$.

Definition 3 We say that a switched forward-complete system (2) with $i : R_+ \to I$ *is IOS with respect to the output* $\mathbf{y}$ *and the input* $\mathbf{d}$ *if for all* $\mathbf{x}_0 \in R^n$ *and* $\mathbf{d} \in M_{R^m}$ *there exist functions* $\beta \in KL, \gamma \in K$ *such that* , for all $t \geq 0$,

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \ \leq \beta(|\mathbf{x}_0|, 0) + \gamma(||\mathbf{d}||_{[0,t)}) .$$

The difference between IOS and SIIOS consists in the dependence on initial conditions: for SIIOS system if initial amplitude of variable $\mathbf{y}$ is small, then the overall amplitude of $\mathbf{y}$ during transients is also "small". For an IOS system the output $\mathbf{y}(t)$, even with small initial values may have large transient values if $\mathbf{x}$ is "large" in norm. In contrast to this, the asymptotic behaviour of IOS and SIIOS systems' trajectories is similar: in either case, and in the absence of disturbances, the trajectories converge to the manifold $\{y = 0\}$. Consequently, in the particular case when $\mathbf{y} = \mathbf{x}$ both properties boil down to the well known input-to-state stability (ISS). Other closely connected input-output stability properties for nonlinear dynamical systems can be found, *e.g.*, in (Sontag and Wang 2001).

Assumption 1 For each fixed $q \in I$ *system from (1) is forward complete and SIIOS with respect to output* $\mathbf{y}$ *and input* $\mathbf{d}$ *for functions* $\beta_q \in KL$, $\gamma \in K$.

Note that from Definition 2 it holds that $\beta_q(s, 0) \geq s$, $s \geq 0$.

3 Main Results

3.1 Uniform decrease for nonlinear stable systems

In (Xie *et al.* 2001) it was shown, under suitable smoothness assumptions on the switching systems, that if all systems in family (1) are ISS then the switched system (2) is also ISS, provided that switching signal has average dwell-time $\tau_d > 0$ and there exists a constant $0 < \mu < 1$ s.t. for all $i, j \in I$, and all $s \geq 0$,

$$\beta_i(2\beta_j(2s, \tau_D), \tau_D) \leq \mu s. \tag{3}$$

This implies that systems in $\mathcal{I}$ are exponentially stable:

Lemma 1 The class $\mathcal{KL}$ functions β_i, for $i \in \mathcal{I}$, satisfy the inequality (3) for all $s \geq 0$ and for some constants $\tau_d > 0$ and μ (s.t. $0 < \mu < 1$) if and only if there exists λ (with $0 < \lambda < 1$) such that $\beta_i(s, t) \leq \lambda s$ for all $i \in \mathcal{I}$. □

Proof. *Sufficiency.* Let $\lambda = \sqrt{\mu}/2$. Assume that $\beta_i(s, t) \leq \lambda s$ for all $i \in \mathcal{I}$ and all $s \geq 0$. Then,

$$\beta_i(2\beta_j(2s, \tau_D), \tau_D) \ \leq \ 2\lambda\beta_j(2s, \tau_D) \ \leq \ 4\lambda^2 s =: \mu s\, .$$

Necessity. We show that inequality (3) implies that, defining $\lambda := \max\{\mu, 1/2\}$, we have, for any $i \in \mathcal{I}$ and all $s \geq 0$,

$$\beta_i(s, \tau_d) \leq \lambda s\, . \tag{4}$$

We proceed by *reductio ad absurdum.* Assume that (4) does not hold *i.e.* there exist $i \in \mathcal{I}$ and $s_* \in R_{\geq 0}$ such that $\beta_i(s_*, \tau_d) > \lambda s_*$. Then, noting that $\beta(\cdot, \tau_d) \in \mathcal{K}_\infty$ we obtain

$$\begin{aligned} \beta_i(2\beta_i(2s_*, \tau_d), \tau_d) \ &> \ \beta_i(2\beta_i(s_*, \tau_d), \tau_d) \\ &> \ \beta_i(2\lambda s_*, \tau_d) \\ &\geq \ \beta(s_*, \tau_d) \ > \ \lambda s_* \ \geq \ \mu s_*, \end{aligned}$$

where in the third inequality we used the fact that $2\lambda \geq 1$. From (3) we have $\beta_i(2\beta_i(2s_*, \tau_d), \tau_d)\mu s_*$. ■

Proposition 1 Let Assumption 1 and Ineq. (4) hold. Then, each system in (1) admits an exponential estimate for all $\mathbf{x}_0 \in R^n$, $\mathbf{d} \in M_{R^m}$, $t \geq 0$ *and* $q \in I$, *i.e.,*

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \leq \lambda^{-1}\beta_q(|\mathbf{y}_0)|, 0)e^{\ \ln(\lambda)t/\tau_D} + \Lambda\gamma(||\mathbf{d}||_{[0,t)}) \tag{5}$$

$$\sum_{k=0}^{+\infty} \lambda^k = \Lambda < +\infty. \tag{6}$$

Proof. Let $q \in I$ and $t \geq \tau_D$ be arbitrary; define n as the largest integer smaller than t/τ_D, denoted $n = \lfloor t/\tau_D \rfloor$. Then, using Assumption 1 we obtain, for any $t \geq 0$,

$$|\mathbf{y}(t)| \leq \beta_q(|\mathbf{y}_0|, t) + \gamma(||\mathbf{d}||_{[0,t)})\, .$$

Then,

$$\begin{aligned} |\mathbf{y}(t)| &\leq \beta_q(|\mathbf{y}(\{n-1\}\tau_D)|, t - \{n-1\}\tau_D) + \gamma(||\mathbf{d}||_{[\{n-1\}\tau_D, t)}) \\ &\leq \lambda|\mathbf{y}(\{n-1\}\tau_D)| + \gamma(||\mathbf{d}||_{[0,t)}) \end{aligned}$$

and, for any $k \in \{1, \ldots, n\}$,

$$\begin{aligned} |\mathbf{y}(\{n-k\}\tau_D)| &\leq \beta_q(|\mathbf{y}(\{n-k\}\tau_D)|, \tau_D) + \gamma(||\mathbf{d}||_{[0,t)}) \\ &\leq \lambda|\mathbf{y}(\{n-k\}\tau_D)| + \gamma(||\mathbf{d}||_{[0,t)}) \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathbf{y}(t)| &\leq \lambda[\, \beta_q(|\mathbf{y}(\{n-2\}\tau_D)|, \tau_D) + \gamma(||\mathbf{d}||_{[0,t)})\,] + \gamma(||\mathbf{d}||_{[0,t)}) \\ &\leq \lambda^2|\mathbf{y}(\{n-2\}\tau_D)| + (1+\lambda)\gamma(||\mathbf{d}||_{[0,t)}) \\ &\leq \lambda^2[\, \beta_q(|\mathbf{y}(\{n-3\}\tau_D)|, \tau_D) + \gamma(||\mathbf{d}||_{[0,t)})\,] \\ &\qquad + (1+\lambda)\gamma(||\mathbf{d}||_{[0,t)}) \\ &\leq \lambda^3|\mathbf{y}(\{n-3\}\tau_D)| + (1+\lambda+\lambda^2)\gamma(||\mathbf{d}||_{[0,t)})\, . \end{aligned}$$

By induction we obtain

$$\begin{aligned} |\mathbf{y}(t)| &\leq \lambda^n|\mathbf{y}_0| + \sum_{k=1}^{n-1} \lambda^k\gamma(||\mathbf{d}||_{[0,t)}) + \gamma(||\mathbf{d}||_{[0,t)}) \\ &\leq \lambda^n|\mathbf{y}_0| + \Lambda\gamma(||\mathbf{d}||_{[0,t)}) \\ &\leq \lambda^{-1}|\mathbf{y}_0|e^{\ \ln(\lambda)t/\tau_D} + \Lambda\gamma(||\mathbf{d}||_{[0,t)})\, . \end{aligned} \tag{7}$$

For $t \in [\, 0, \tau_D)$ it holds that

$$|\mathbf{y}(t)| \ \leq \beta_q(|\mathbf{y}_0|, 0) + \gamma(||\mathbf{d}||_{[0,t)})\, . \tag{8}$$

The result follows by combining the bounds (7) and (8). ■

Remark 1 According to Lemma 8 from (Sontag 1998) (see also Lemma A1 from (Sontag and Wang 1999) and Lemma A2 from (Sontag and Wang 2001)) there always exists a function $\sigma_q \in K$ such that $\beta_q(s,r) \le \sigma_q(s)\,\sigma_q(e^{-r})$). If $\beta_q(s,r) = \sigma_q(s)\,\sigma_q(e^{-r})$ for some $\sigma_q \in K$, $s \ge 0$, $q \in I$ and β_q satisfies (4) *i.e.*, $\sigma_q(s) \le \lambda\, s/\sigma_q(e^{-\tau_D})$, $s \ge 0$ then, under the conditions of Proposition 1 each system in the family (1) is *exponentially* SIIOS for all $\mathbf{x}_0 \in R^n$, $\mathbf{d} \in M_{R^m}$ and $q \in I$:

$$|\mathbf{y}(t)| \le \ |\mathbf{y}_0|/\sigma_q(e^{-\tau_D})\, e^{\,\ln(\lambda)\, t/\tau_D} + \Lambda\,\gamma(||\mathbf{d}||_{[0,t)})\,.$$

Such conclusion fails for a general functions $\beta_q \in KL$ *e.g.*,

$$\beta_q(s,r) = \begin{cases} s/(1+r) \ \textit{if}\ s \le 1\,; \\ s^2/(1+s\,r) \ \textit{if}\ s > 1\,, \end{cases} \quad \beta_q(s,0) = \begin{cases} s \ \textit{if}\ s \le 1\,; \\ s^2\, \textit{if}\ s > 1\,, \end{cases}$$

belongs to class KL and has dwell-time $\tau_D = \lambda^{-1} - 1$, but $\beta_q(s,0)$ does not possess linear grow rate condition. □

In Theorem 1 of (Xie *et al.* 2001) the authors of the latter implicitly impose exponential stability for each system in family (1). An attempt to overcome the requirement on exponential stability was made in (Persis *et al.* 2003), where the authors introduced state-dependent dwell-time $\tau_D : R^n \to R_+$. A key property to ensure its existence is local Lipschitz continuity of functions β_j with respect to the first argument, near zero. For $\beta_j(s,r) = \sigma(s)\,\sigma(e^{-t})$, $\sigma \in K$, as considered in (Persis *et al.* 2003), this condition may be formulated as

$$\lim_{s\to 0} \frac{\sigma(s)}{s} < +\infty.$$

Now, let us recast this hypothesis for family (1):

Assumption 2 For all $r \ge 0$ and all $j \in I$ let

$$\lim_{s\to 0} \frac{\beta_j(s,r)}{s} \le c < +\infty \tag{9}$$

As it follows from the next propositions such requirement is equivalent to exponential stability of the system for any compact set of initial conditions. For example, the system $\dot{x} = -x^2$, whose solution is $x(t) = x_0/(1 + x_0\,t)$ with initial state $x_0 \in R_+$, is non-exponentially asymptotically stable. In this case, $\beta(s,r) = s/(1+sr)$ and the system does not satisfy (9) for $r = +\infty$ and there exists no τ_D such that the system's trajectories satisfy, locally, an exponential bound.

Proposition 2 Let Assumption 2 hold then, for any $R > 0$ there exists $\tau_R > 0$ such that (4) holds locally for any $0 < \lambda < 1$, $0 \le s \le R$, $q \in I$ and $\beta_q(s,\tau_R) \le \lambda\, s$.

Proof. For $\tau \ge 0$ consider the function

$$b(\tau) = \sup_{0\le s\le R} \frac{\beta_q(s,\tau)}{s},$$

which is well-defined and bounded (due to continuity of β_q the supremum exists and is finite for any $\varepsilon \le s \le R$, $\varepsilon > 0$, while the existence of the supremum for $\varepsilon \to 0$ is guaranteed by (9)). It follows that

$$0 \le b(\tau) \le B, \quad B = \sup_{0\le s\le R} \frac{\beta_q(s,0)}{s},$$

since $\beta_q(s,\tau) \le \beta_q(s,0)$ for any $\tau \ge 0$ (note also that $\beta_q(s,0) \le B\,s$ for $0 \le s \le R$). The function $b(\tau)$ is decreasing; to see this, take $\tau_2 > \tau_1 \ge 0$ then, $\beta_q(s,\tau_2) < \beta_q(s,\tau_1)$ and

$$\begin{aligned} b(\tau_2) - b(\tau_1) &= \sup_{0\le s\le R} \frac{\beta_q(s,\tau_2)}{s} - \sup_{0\le s\le R} \frac{\beta_q(s,\tau_1)}{s} \\ &\le \sup_{0\le s\le R} \frac{\beta_q(s,\tau_2) - \beta_q(s,\tau_1)}{s} \le 0\,. \end{aligned}$$

Assume now that function $b(\cdot)$ does not "decrease" to zero then, for all $\tau \ge 0$ there exists $\delta > 0$ such that $b(\tau) \ge \delta$ and

$$\begin{aligned} \delta \le \lim_{\tau\to+\infty} b(\tau) &= \lim_{\tau\to+\infty} \sup_{0\le s\le R} \frac{\beta_q(s,\tau)}{s} \\ &= \sup_{0\le s\le R} \lim_{\tau\to+\infty} \frac{\beta_q(s,\tau)}{s} = 0 \le B, \end{aligned}$$

which is a contradiction, so $\delta = 0$. Thus, for any $0 < \lambda < 1$ there exists $\tau_R > 0$ such that $b(\tau_R) \le \lambda$. ■

Proposition 3 For each system in (1), let Assumptions 1 and 2 hold. Then, for any $R \ge 0$ and $\mathrm{b}_{\mathbf{d}} \ge 0$ there exist $0 < \lambda < 1$, $\tau_R = \tau_R(R, \mathrm{b}_{\mathbf{d}}, \lambda) > 0$ such that, for all $\mathbf{x}_0 \in X_R$, $X_R = \{\, \mathbf{x} \in R^n : |\mathbf{h}(\mathbf{x})| \le R\}$, $\|\mathbf{d}\| \le \mathrm{b}_{\mathbf{d}}$ and all $t \ge 0$,

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{d})| \ \le \lambda^{-2}\,|\mathbf{y}_0| e^{\,\ln(\lambda)\, t/\tau_R} + \Lambda\,\gamma(||\mathbf{d}||_{[0,t)})\,.$$

Proof. By Assumption 1 for any $q \in I$, $R \ge 0$, $\mathbf{x}_0 \in R^n$ such that $|\mathbf{y}_0| \le R$ and any $\|\mathbf{d}\| \le \mathrm{b}_{\mathbf{d}}$ we have

$$|\mathbf{y}(t)| \ \le \tilde{R}, \quad \tilde{R} = \beta_q(R,0) + \gamma(\mathrm{b}_{\mathbf{d}})\,, \quad t \ge 0\,.$$

As it was proved in Proposition 2, for such $\tilde{R}$ there exists $\tau_R > 0$ such that relation (4) holds for any solution with $|\mathbf{y}_0| \le R$ and $\|\mathbf{d}\| \le \mathrm{b}_{\mathbf{d}}$. Repeating the arguments of Proposition 1 we obtain, for $t \ge \tau_R$, the estimate

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{d})| \ \le \lambda^{-1}|\mathbf{y}_0|\, e^{\,\ln(\lambda)\, t/\tau_R} + \Lambda\,\gamma(||\mathbf{d}||_{[0,t)}),$$

and, considering that $\beta_q(s,0) \le \tilde{B}\,s$ for $0 \le s \le \tilde{R}$, where $\tilde{B} = \sup_{0\le s\le\tilde{R}} \frac{\beta_q(s,0)}{s}$) we obtain

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{d})| \ \le \tilde{B}|\mathbf{y}_0)| + \gamma(||\mathbf{d}||_{[0,t)}) \qquad \forall\, t \le \tau_R\,.$$

Taking $\lambda^{-1} \ge \tilde{B}$ and combining the last two estimates we obtain the desired result. ■

According to the proof of Proposition 3, to calculate $\lambda = \lambda(R)$ and the local dwell-time constant $\tau_R = \tau_R(R, \mathbf{b}_d, \lambda)$ for a system $q \in I$, one can solve the equations:

$$\sup_{0\le s\le\beta_q(R,0)+\gamma(\mathbf{b}_d)} \frac{\beta_q(s,0)}{s} = \lambda^{-1},$$

$$\sup_{0\le s\le\beta_q(R,0)+\gamma(\mathbf{b}_d)} \frac{\beta_q(s,\tau_R)}{s} = \lambda.$$

3.2 *SIIOS stability of nonlinear switched systems with constant dwell-time*

The results contained in (Xie *et al.* 2001) and (Persis *et al.* 2003) for switched systems of the form (2) follow from Proposition 3.

Lemma 2 *Let Assumptions 1 and 2 hold. Then, for any constants R and $\mathbf{b}_d > 0$, any $0 < \lambda < 1$ there exists dwell-time $\tau_R = \tau_R(R, \mathbf{b}_d, \lambda) > 0$, $B = B(R, \mathbf{b}_d, \lambda) > 0$ such that the switched system (2) is forward complete and, for all $\mathbf{d} \in M_{R^m}$ such that $\|\mathbf{d}\| \le \mathbf{b}_d$, $\mathbf{x}_0 \in X_R$, $X_R = \{\mathbf{x} \in R^n : |\mathbf{h}(\mathbf{x})| \le R\}$, $R \in R_+$, the following bound holds:*

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \le \lambda^{-1} B |\mathbf{y}_0| e^{\,ln(\lambda) t/\tau_R} + \Lambda^2 \gamma(||\mathbf{d}||_{[0,t)}),$$

for all $t \ge 0$ and Λ is defined in (6). □

Proof. The system (2) with dwell-time switching signal are locally continuous on some interval $[0, T)$; indeed, $T = +\infty$ since for each t and for some fixed $q \in I$ the solutions of (2) match those of the family (1), which is forward complete by assumption. Let $R \ge 0$ be arbitrary, then as in proof of Proposition 2 for any $0 < \lambda < 1$ there exists $\tau_R^q = \tau_R^q(R, \mathbf{b}_d, \lambda) > 0$ such that

$$\sup_{0 \le s \le \bar\beta(R + \Lambda^2 \gamma(\mathbf{b}_d)) + \gamma(\mathbf{b}_d)} \frac{\beta_q(s, \tau_R^q)}{s} = \lambda,$$

$$B_q = \sup_{0 \le s \le \bar\beta(R + \Lambda^2 \gamma(\mathbf{b}_d)) + \gamma(\mathbf{b}_d)} \frac{\beta_q(s, 0)}{s} \ge 1,$$

$$\bar\beta(s) = \max_{q \in I} \{\beta_q(s, 0)\}.$$

Then, for any system from (1), any $\mathbf{x}_0 \in X_R$, $\mathbf{d} \in M_{R^m}$ such that $\|\mathbf{d}\| \le \mathbf{b}_d$ and all $t \ge 0$,

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \le \lambda^{-1} B_q |\mathbf{y}_0| e^{\,\ln(\lambda) t/\tau_R^q} + \Lambda\gamma(||\mathbf{d}||_{[0,t)})\,.$$

Let $B = \max_{q \in I}\{B_q\}$ and let $\tau_R = \ln(\lambda^2/B) \max_{q \in I}\{\tau_R^q\} / \ln(\lambda)$ be the dwell-time constant then, for any $q \in I$ and all $t \ge 0$

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \le \lambda^{-1} B |\mathbf{y}_0| e^{\,\ln(\lambda^2/B) t/\tau_R} + \Lambda\gamma(||\mathbf{d}||_{[0,t)})\,.$$

Let us now partition the interval $[0, T)$ into a concatenation of subintervals whose limits are given by $i(t)$. For simplicity let as assume that there exists $k \ge 0$ such that $T = t_{k+1}$), $[0, T) = \bigcup_{j=0}^{k} [t_j, t_{j+1})$, $i(t) = i(t_j)$ for $t \in [t_j, t_{j+1})$ and $t_0 = 0$. By the dwell-time property, for all intervals $[t_j, t_{j+1})$, $0 \le j \le k$ we have $t_{l+1} - t_l \ge \tau_R$. Now, by assumption we have

$$\begin{aligned}|\mathbf{y}(t_{j+1})| &\le \lambda^{-1} |\mathbf{y}(t_j)| e^{\,\ln(\lambda^2/B)[t_{j+1} - t_j]/\tau_R} + \Lambda\gamma(||\mathbf{d}||_{[t_j, t_{j+1})}) \\ &\le \lambda |\mathbf{y}(t_j)| + \Lambda\gamma(||\mathbf{d}||_{[0,T)})\,.\end{aligned}$$

Further substituting the latter inequality for $\mathbf{y}(t_{j+1})$ in the estimate for the next interval $[t_{j+1}, t_{j+2})$ we obtain

$$\begin{aligned}|\mathbf{y}(t_{j+2})| &\le \lambda^{-1}[\lambda|\mathbf{y}(t_j)| + \Lambda\gamma(||\mathbf{d}||_{[t_j, t_{j+1})})] \\ &\quad \times e^{\,\ln(\lambda^2/B)[t_{j+2} - t_{j+1}]/\tau_R} + \Lambda\gamma(||\mathbf{d}||_{[t_{j+1}, t_{j+2})})) \\ &\le \lambda^2 |\mathbf{y}(t_j)| + (1 + \lambda)\Lambda\gamma(||\mathbf{d}||_{[0,T)}).\end{aligned}$$

Repeating recursively these calculations, for any arbitrary $k > 0$, we obtain

$$\begin{aligned}|\mathbf{y}(t_{j+k})| &\le \lambda^k |\mathbf{y}(t_j)| + \Lambda \sum_{r=0}^{k} \lambda^r \gamma(||\mathbf{d}||_{[0,T)}) \\ &\le \lambda^k |\mathbf{y}(t_j)| + \Lambda^2 \gamma(||\mathbf{d}||_{[0,T)}) \\ &\le \lambda^{-1} |\mathbf{y}(t_j)| e^{\,\ln(\lambda) t/\tau_R} + \Lambda^2 \gamma(||\mathbf{d}||_{[0,T)})\,.\end{aligned}$$

We now show that $|\mathbf{y}(t)| \le \bar\beta(R + \Lambda^2 \gamma(\mathbf{b}_d)) + \gamma(\mathbf{b}_d)$. The property holds for $j = 0$ by definition ($\mathbf{x}_0 \in X_R$, $\Lambda \ge 1$). In view of Assumption 1 it holds for all $t \in [t_j, t_{j+1})$ with $j \ge 1$

$$\begin{aligned}|\mathbf{y}(t)| &\le \bar\beta(|\mathbf{y}(t_j)|) + \gamma(||\mathbf{d}||_{[t_j, t_{j+1})}) \\ &\le \bar\beta(|\mathbf{y}(0)| + \Lambda^2 \gamma(||\mathbf{d}||)) + \gamma(||\mathbf{d}||),\end{aligned}$$

Additionally, on each interval it holds that for all $t \in [t_j, t_{j+1})$ with $j = 0, 1, 2, \ldots,$,

$$|\mathbf{y}(t)| \le \lambda^{-1} B |\mathbf{y}(t_j)| + \Lambda\,\gamma(||\mathbf{d}||_{[t_j, t_{j+1})}).$$

Combining the last estimates the result follows. ■

3.3 SIIOS stability of nonlinear switched systems with average dwell-time

We consider now, switching systems driven by $i(t)$ with average dwell-time $0 < \tau_D < +\infty$ and $1 < N_0 < +\infty$. We seek to relax the exponential stability requirement. The main technical difficulty strives in the complexity of computing the lengths of different time intervals between switches; *e.g.*, one may have arbitrarily small delays between switches. To avoid this, we wish to establish conditions under which intervals with a minimal "useful" length exists. Before presenting the main result of this section we need to introduce some preliminary statements on the maximal and minimal length of intervals between switches.

In the sequel we denote such value by $\tau_D\, N_0^{-1}$.

Claim 1 *On any time interval $[t_s, t_e)$ with $t_e - 4\,\tau_D \ge t_s \ge 0$ there exists at least one time interval between switches, such that its length is larger than $\tau_D\, N_0^{-1}$, $N_0 > 1$.*

Proof. The total number of switches over $[t_s, t_e)$ is bounded by

$$N_{[t_s, t_e)} \le N_0 + \frac{t_e - t_s}{\tau_D}.$$

Assume that the claim does not hold *i.e.*, intervals between switches have length smaller than $\tau_D\, N_0^{-1}$. It follows that the maximum total length of time intervals between switches is limited by

$$\left(N_0 + \frac{t_e - t_s}{\tau_D} + 1 \right) \frac{\tau_D}{N_0} = \frac{(N_0 + 1)\,\tau_D + t_e - t_s}{N_0}$$

while the residual length of time over the interval $[t_s, t_e)$ can be estimated to be

$$\begin{aligned}t_e - t_s - &\frac{(N_0 + 1)\,\tau_D + t_e - t_s}{N_0} = \\ &\frac{(N_0 - 1)\,(t_e - t_s) - (N_0 + 1)\,\tau_D}{N_0} \ \ge\ (3\,N_0 - 5)\frac{\tau_D}{N_0}.\end{aligned}$$

Then, for any whole number $N_0 > 1$ we obtain that the residual length is bigger than zero which is a contradiction hence, there exists an interval with length bigger than $\tau_D\, N_0^{-1}$. ■

Now, the maximum number of intervals over which the signal $i(t)$ is continuous (constant) on an interval $[t_s, t_e)$ is denoted by $I_{[t_s, t_e)}$ and bounded by

$$I_{[t_s, t_e)} \le N_0 + \frac{t_e - t_s}{\tau_D} + 1,$$

while the maximum number of intervals with length $\tau_D\, N_0^{-1}$, denoted by $I_{[\,t_s,t_e)}^{\tau_D\, N_0^{-1}}$, admits the upper-bound

$$I_{[\,t_s,t_e)}^{\tau_D\, N_0^{-1}} \leq N_0 \frac{t_e - t_s}{\tau_D} + 1\,.$$

Note that for $N_0 > 1$ and $(t_e - t_s)\,\tau_D^{-1} \geq 3$ we have $I_{[\,t_s,t_e)}^{\tau_D\, N_0^{-1}} > I_{[\,t_s,t_e)}$.
the number of subintervals with length

Next, we may make a statement on the number of subintervals with length $\tau_D\, N_0^{-1}$ within intervals of length bigger than $\tau_D\, N_0^{-1}$.

Claim 2 Consider any interval $[\,t_s, t_e)$ *with* $t_e - 4\,\tau_D \geq t_s \geq 0$. *Between any two switchings within* $[\,t_s, t_e)$*, spaced at least* $\tau_D\, N_0^{-1}$ *units of time, there exist at least* $\kappa_{[\,t_s,t_e)}$ *subintervals of length* $\tau_D\, N_0^{-1}$ *with* $N_0 > 1$*, where*

$$\kappa_{[\,t_s,t_e)} \geq \frac{(N_0 - 1)\,(t_e - t_s)}{\tau_D} - N_0.$$

Proof. Assume that there is $1 \leq n \leq N_0 + (t_e - t_s)\,\tau_D^{-1} + 1$ intervals between switches with length bigger than $\tau_D\, N_0^{-1}$. Then, the maximum total length of time intervals between switches with length less than $\tau_D\, N_0^{-1}$ is limited by

$$\left(N_0 + \frac{t_e - t_s}{\tau_D} + 1 - n\right)\frac{\tau_D}{N_0} = \frac{(N_0 + 1 - n)\,\tau_D + t_e - t_s}{N_0}$$

and the residual over $[\,t_s, t_e)$ can be estimated from

$$\begin{aligned} t_e - t_s - \frac{(N_0 + 1 - n)\,\tau_D + t_e - t_s}{N_0} \\ = \frac{(N_0 - 1)\,(t_e - t_s) - (N_0 + 1 - n)\,\tau_D}{N_0}. \end{aligned}$$

The number κ_n of subintervals with length $\tau_D\, N_0^{-1}$ in n intervals may be estimated to satisfy

$$\begin{aligned} \kappa_n &\geq \frac{(N_0 - 1)(t_e - t_s) - (N_0 + 1 - n)\tau_D}{N_0}\frac{N_0}{\tau_D} - n + 1 \\ &= \frac{(N_0 - 1)(t_e - t_s) - (N_0 + 1 - n)\tau_D}{\tau_D} - n + 1 \geq n \end{aligned}$$

where $n - 1$ was subtracted from the last expression to account for th effect that to compute the the whole number of subintervals with length bigger than $\tau_D\, N_0^{-1}$ we must divide the length of residual intervals modulo n. Thus,

$$\begin{aligned} \kappa_{[\,t_s,t_e)} &= \min_{1 \leq n \leq N_0 + (t_e - t_s)\,\tau_D^{-1} + 1} \{\,\kappa_n : \kappa_n \geq n\,\} \\ &= \frac{(N_0 - 1)\,(t_e - t_s)}{\tau_D} - N_0 \end{aligned}$$

and the minimum of the above expression is reached for $n = 1$. ■

Next, we consider the relation between the number $\kappa_{[\,t_s,t_e)}$ of subintervals with length bigger than $\tau_D\, N_0^{-1}$ and the total possible number of intervals $I_{[\,t_s,t_e)}$:

$$\begin{aligned} \alpha_{[t_s,t_e)} &= \frac{I_{[t_s,t_e)}}{\kappa_{[t_s,t_e)}} \\ &= \left(N_0 + \frac{t_e - t_s}{\tau_D} + 1\right)\left(\frac{(N_0 - 1)(t_e - t_s)}{\tau_D} - N_0\right)^{-1} \\ &= \frac{(N_0 + 1)\tau_D + (t_e - t_s)}{(N_0 - 1)(t_e - t_s) - N_0\tau_D}\,. \end{aligned}$$

For $t_e - t_s \geq q\,\tau_D$ with $q \geq 3$ and $N_0 > 1$ the denominator of the above expression is always positive and for $q \geq 4$

$$\frac{1}{N_0 - 1} \leq \alpha_{[\,t_s,t_e)} \leq \frac{N_0 + 5}{3\,N_0 - 4}.$$

The following lemma on SIIOS of switched systems is reminiscent of Theorem III.1 from (Vu *et al.* 2005), which was formulated in terms of "exponential" Lyapunov functions.

Lemma 3 Let Assumptions 1 and 2 hold. Then, for any constants $R \in R_+$*,* $\mathbf{b}_d > 0$*, and* $0 < \lambda < 1$ *there exists a dwell-time* $\tau_R = \tau_R(R, \mathbf{b}_d, \lambda) > 0$ *and a number* $B = B(R, \mathbf{b}_d, \lambda) > 0$ *such that the switched system (2) is forward complete and, for all* $t \geq 0$, $\mathbf{d} \in M_{R^m}$ such that $\|\mathbf{d}\| \leq \mathbf{b}_d$, $\mathbf{x}_0 \in X_R$, *with* $X_R = \{\,\mathbf{x} \in R^n : |\mathbf{h}(\mathbf{x})| \leq R\,\}$ we have

$$\begin{aligned} |\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \leq (\lambda^{-1}B)^{N_0+4}[\,\lambda^{-1}|\mathbf{y}_0|e^{0.25\,\ln(\lambda)t/\tau_R} \\ + (N_0 + 4)\,\Lambda^3\gamma(||\mathbf{d}||_{[\,0,T)})\,], \end{aligned}$$

where Λ is defined in (6). □

Proof . The solutions of System (2) are continuous and defined, at least locally, on $[\,0, T)$. Actually, $T = +\infty$ since for each fixed t the solutions of system (2) equal those of a system from family (1) for some fixed $q = i(t) \in I$, which is forward complete by assumption. Now, for any $0 < \lambda < 1$ there exists $+\infty > \tau_R^q = \tau_R^q(R, \mathbf{b}_d, \lambda) > 0$ such that

$$\sup_{0 \leq s \leq \bar{\beta}(R + \Lambda^2\,\gamma(||\mathbf{d}||)) + \gamma(||\mathbf{d}||)} \frac{\beta_q(s, \tau_R^q)}{s} = \lambda,$$

$$B_q = \sup_{0 \leq s \leq \bar{\beta}(R + \Lambda^2\,\gamma(||\mathbf{d}||)) + \gamma(||\mathbf{d}||)} \frac{\beta_q(s, 0)}{s} \geq 1,$$

$$\bar{\beta}(s) = \max_{q \in I}\{\,\beta_q(s, 0)\,\}$$

then, for any $\mathbf{x}_0 \in X_R$, $\mathbf{d} \in M_{R^m}$ and for all $t \geq 0$ we have

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \leq \lambda^{-1}B_q|\mathbf{y}_0|e^{\,\ln(\lambda)t/\tau_R^q} + \Lambda\gamma(||\mathbf{d}||_{[0,t)})\,.$$

Next, let $B = \max_{q \in I}\{\,B_q\,\}$, $\tau_R = \ln(\lambda^2/B)\max_{q \in I}\{\,\tau_R^q\,\}/\ln(\lambda)$ be the dwell-time constant; then, for any $q \in I$ and for all $t \geq 0$,

$$|\mathbf{y}(t, \mathbf{x}_0, \mathbf{d})| \leq \lambda^{-1}B|\mathbf{y}_0|e^{\,\ln(\lambda^2/B)t/\tau_R} + \Lambda\gamma(||\mathbf{d}||_{[0,t)})\,.$$

This estimate is valid while $|\mathbf{y}(t)| \leq \bar{\beta}(R + \Lambda^2\,\gamma(||\mathbf{d}||)) + \gamma(||\mathbf{d}||)$, $t \geq 0$. Consider the interval $[\,0, T)$ which can be divided into subintervals of length $4\,\tau_R$ *i.e.*:

$$[\,0, T) = \bigcup_{r=0}^{K}\,[\,4\,\tau_R\, r, 4\,\tau_R\,(r + 1)) + [\,4\,\tau_R\,(K + 1), T)\,.$$

For each $0 \leq r \leq K$ the interval $[\,4\,\tau_R\, r, 4\,\tau_R\,(r + 1))$ can be constructed as a concatenation of intervals where the signal $i(t)$ takes the same value *i.e.*, $[\,4\,\tau_R\, r, 4\,\tau_R\,(r + 1)) = \bigcup_{j=0}^{k}\,[\,t_j, t_{j+1})$, $i(t) = i(t_j)$ for $t \in [\,t_j, t_{j+1})$; $t_0 = 4\,\tau_R\, r$, $t_{k+1} = 4\,\tau_R\,(r + 1)$.

By the average dwell-time property and Claim 1 there exists at least one interval $[\,t_l, t_{l+1})$ with $0 \leq l \leq k$ such that $t_{l+1} - t_l \geq \tau_R\, N_0^{-1}$ while for all other $0 \leq j \leq k$ with $j \neq l$ it possibly holds that $t_{j+1} - t_j < \tau_R N_0^{-1}$. On each interval and for each SIIOS system from (1) we have

$$|\mathbf{y}(t)| \leq \lambda^{-1} B |\mathbf{y}(t_j)| e^{\ln(\lambda^2/B)(t-t_j)/\tau_R} + \Lambda\gamma(||\mathbf{d}||_{[t_j,t)}),$$

for all $t \in [t_j, t_{j+1})$ and $0 \leq j \leq k$. Furthermore, for $j \neq l$ we have

$$|\mathbf{y}(t_{j+1})| \ \leq \lambda^{-1} B |\mathbf{y}(t_j)| + \Lambda\,\gamma(||\mathbf{d}||_{[\,t_j,t_{j+1})}),$$

while, for $j = l$,

$$|\mathbf{y}(t_{l+1})| \ \leq \beta_{i(t_l)}(|\mathbf{y}(t_l)|, t_{l+1} - t_l) + \gamma(||\mathbf{d}||_{[\,t_l,t_{l+1})})\,.$$

Invoking Claim 2 we obtain

$$k + 1 \leq N_{[4\,\tau_D\,r, 4\,\tau_D\,(r+1))} \leq N_0 + 4,$$

$$\kappa_{[4\,\tau_D\,r, 4\,\tau_D\,(r+1))} \geq 2, \quad t_{l+1} - t_l \geq \tau_d\, N_0^{-1}\,.$$

Now, without loosing generality, we can assume that the first $k-1$ intervals have length less than $\tau_R N_0^{-1}$ while the lth (with $l = k$), has length larger than $\tau_R N_0^{-1}$. Then,

$$\begin{aligned}
|\mathbf{y}(t_k)| &\leq \lambda^{-1} B |\mathbf{y}(t_{k-1})| + \Lambda\gamma(||\mathbf{d}||_{[t_{k-1},t_k)}) \\
&\leq \lambda^{-1} B [\lambda^{-1} B |\mathbf{y}(t_{k-2})| \\
&\quad + \Lambda\gamma(||\mathbf{d}||_{[t_{k-2},t_{k-1})})] + \Lambda\gamma(||\mathbf{d}||_{[t_{k-1},t_k)}) \\
&\quad \vdots \\
&\leq (\lambda^{-1} B)^k |\mathbf{y}(t_0)| \\
&\quad + \Lambda \sum_{r=0}^{k-1} (\lambda^{-1} B)^r \gamma(||\mathbf{d}||_{[t_{k-r-1},t_{k-r})})\,.
\end{aligned}$$

For $\mathbf{y}(t_{l+1}) = \mathbf{y}(t_{k+1})$ we have

$$\begin{aligned}
|\mathbf{y}(t_{k+1})| &\leq \lambda(\lambda^{-1} B)^{-N_0-3}\big((\lambda^{-1} B)^k |\mathbf{y}(t_0)| \\
&\qquad + \Lambda \sum_{r=0}^{k-1} (\lambda^{-1} B)^r \gamma(||\mathbf{d}||_{[t_{k-r-1},t_{k-r})})\big) \\
&\quad + \Lambda\gamma(||\mathbf{d}||_{[t_k,t_{k+1})}) \leq \lambda |\mathbf{y}(t_0)| \\
&\qquad + \Lambda \sum_{r=0}^{k} (\lambda B^{-1})^{k-r} \gamma(||\mathbf{d}||_{[t_{k-r},t_{k-r+1})}) \\
&\quad \leq \lambda |\mathbf{y}(t_0)| + \Lambda^2 \gamma(||\mathbf{d}||_{[t_0,t_{k+1})}).
\end{aligned}$$

For all $t \in [t_0, t_{k+1})$ it holds that

$$|\mathbf{y}(t)| \leq (\lambda^{-1} B)^{N_0+4} [|\mathbf{y}(t_0)| + \Lambda\{N_0 + 4\}\gamma(||\mathbf{d}||_{[t_0,t_{k+1})})]\,.$$

Such estimates hold for all intervals $0 \leq r \leq K$:

$$|\mathbf{y}(4\tau_R(r+1))| \leq \lambda |\mathbf{y}(4\tau_R r)| + \Lambda^2 \gamma(||\mathbf{d}||_{[4\tau_R r, 4\tau_R(r+1))}), \quad (10)$$

$$\begin{aligned}
|\mathbf{y}(t)| \leq (\lambda^{-1} B)^{N_0+4} [|\mathbf{y}(4\tau_R r)| + \Lambda\{N_0 + 4\} \\
\times \gamma(||\mathbf{d}||_{[4\tau_R r, 4\tau_R(r+1))})],
\end{aligned}$$

for all $t \in [4\tau_D r, 4\tau_D(r + 1))$. Using the estimate (10), for all $0 \leq r \leq K$, we obtain

$$\begin{aligned}
|\mathbf{y}(4\tau_D(K+1))| &\leq \lambda(\lambda \cdots \{\lambda|\mathbf{y}_0| + \Lambda^2\gamma(||\mathbf{d}||_{[0,4\tau_R)})\} \cdots \\
&\quad + \Lambda^2 \gamma(||\mathbf{d}||_{[4\tau_R(r-1),4\tau_R r)})) \\
&\quad + \Lambda^2 \gamma(||\mathbf{d}||_{[4\tau_R r, 4\tau_R(r+1))}) \\
&\leq \lambda^K |\mathbf{y}_0| + \Lambda^3 \gamma(||\mathbf{d}||_{[0,T)}) \\
&\leq \lambda^{-1} |\mathbf{y}_0| e^{0.25\,\ln(\lambda)t/\tau_R} + \Lambda^3 \gamma(||\mathbf{d}||_{[0,T)})
\end{aligned}$$

Then, the following estimate holds for all $t \geq 0$:

$$\begin{aligned}
|\mathbf{y}(t)| \leq (\lambda^{-1} B)^{N_0+4} [\lambda^{-1} |\mathbf{y}_0| e^{0.25\,\ln(\lambda)t/\tau_R} \\
+ (N_0 + 4)\Lambda^3 \gamma(||\mathbf{d}||_{[0,T)})].
\end{aligned}$$

Finally, let us verify that $|\mathbf{y}(t)| \leq \bar{\beta}(R + \Lambda^2\gamma(||\mathbf{d}||)) + \gamma(||\mathbf{d}||)$ any $t \geq 0$. This holds true for $t \in [0, 4\tau_R)$ by construction and, from (10), it also holds for all other $t \in [4\tau_R, T)$. ■

Results of 2 and 3 establish that, taking average dwell time and sufficiently large constant dwell-time, we may ensure SI-IOS for switched system (2) assuming that the systems from (1) are, independently, locally exponentially stable.

4 Conclusion

The problem of output stability of switched nonlinear systems has been considered. Two solutions are proposed: based on constant and average dwell-time. We have shown that, for fairly generic nonlinear switched systems, to ensure a *global* stability property the dwell-time constant must depend on the initial conditions as well as disturbances "infinity norms".

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