Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Elementwise decoupling and convergence of the Riccati equation in the SG-algorithm [1]

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Abstract: It is shown that the difference Riccati equation of the Stenlund-Gustafsson (SG) algorithm for estimation of linear regression models can be solved elementwise. Convergence estimates for the elements of the solution to the Riccati equation are provided, directly relating convergence rate to the signal-to-noise ratio in the regression model. It is demonstrated that the elements of the solution lying in the direction of excitation exponentially converge to a stationary solution while the other elements experience bounded excursions around their current values.

1. INTRODUCTION

One of the recent contributions to the plethora of covariance anti-windup algorithms for the Kalman filter is a specialization of the Riccati equation suggested in Stenlund and Gustafsson [2002], further referred to as Stenlund-Gustafsson (SG) algorithm. A similar idea is presented somewhat later in Cao and Schwartz [2004].

Being suggested as an *ad hoc* solution, no convergence analysis has been conducted for the SG algorithm but it appeared to work well in simulations. An important feature of the algorithm that has already been demonstrated in the seminal publication is that the difference Riccati equation can be written for this particular case in the form of a linear (Sylvester) matrix equation. This fact has facilitated further results on the dynamics of the SG algorithm and is proven for a general case in Evestedt and Medvedev [2006b].

In Medvedev [2004], a formal proof for non-divergence of the Riccati equation for the SG algorithm was given using theory of converging matrix products. It was also shown in the same paper that the linear mapping corresponding to the Riccati (Sylvester) equation is not a contraction in most common norms but a paracontraction in some unspecified norm. The latter discovery posed the question why and under what conditions the SG algorithm converges.

The remarkable robustness of the SG algorithm against lack of excitation called for its use in a number of engineering applications such as active vibration control Olsson [2005], acoustic echo cancelation Evestedt et al. [2005] and change detection Evestedt and Medvedev [2006a].

In this paper, convergence properties of the SG algorithm are studied via an elementwise decomposition of the Riccati equation, clearly revealing the underlying convergence mechanism and its close relationship to the excitation condition of recursive parameter estimation, Ljung and Gunnarsson [1990]. After some preliminaries introducing necessary notions and notation, recursive relationships for elementwise solution of the Riccati equation are derived. Further, exponential convergence of the Riccati equation in the direction of excitation is proved followed by convergence rate bounds. It is also shown that each element outside of the current excitation direction remains within a bounded interval whose range is defined by excitation and the parameters of the algorithm. Obtained convergence results are illustrated by simulation.

[1] This work has been carried out with financial support by Swedish Research Council.

2. PRELIMINARIES

Consider the following difference Riccati equation typical to parameter estimation of regression models

$$P(t) = P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{r(t) + \varphi^T(t)P(t-1)\varphi(t)} + Q(t) \quad (1)$$

where $P(\cdot) \in R^{n\times n}$, $Q(\cdot) \in R^{n\times n}, Q(\cdot) = Q^T(\cdot), Q(\cdot) \geq 0$, $P(0) = P^T(0), P(0) \geq 0$, $\varphi(\cdot) \in R^n$ is a regressor vector, $r(t)$ is a positive scalar, and $t \in \{1, 2, \ldots, \infty\}$.

In Stenlund and Gustafsson [2002], $Q(t)$ is treated as an "input signal" that can be designed to provide desirable behavior of $P(t)$, for instance convergence to a certain point or manifold.

A special case of the difference Riccati equation arises when the following form of the free matrix term

$$Q(t) = \frac{P_d\varphi(t)\varphi^T(t)P_d}{r(t) + \varphi^T(t)P_d\varphi(t)} \quad (2)$$

is introduced for a given constant $P_d \in R^{n\times n}, P_d > 0$. Evidently, the matrix P_d is then a stationary point of (1). Notice that the classical stability proof for the Kalman filter assuming $Q(t) > 0$ provided e.g. in Jazwinski [1970] does not apply to the SG algorithm since rank $Q(t) = 1$, according to (2).

The persistence of excitation condition plays a significant role in the dynamic behavior of (1). The condition is that there exists such $c \in R^+$ and integer N that for all k

$$cI \leq \sum_{t=k}^{k+N} \varphi(t)\varphi(t)^T \quad (3)$$

When condition (3) is not satisfied, some eigenvalues of $P(\cdot)$ grow linearly in time. This phenomenon is usually referred to as (covariance) windup. The SG algorithm has been developed in Stenlund and Gustafsson [2002] specifically in order to deal with the windup problem in a systematic manner and the Riccati equation (1,2) is proved to be non-diverging under lack of excitation in Medvedev [2004].

Though the idea of the above method also works in a multivariable setting, only the scalar case, i.e. when φ is a vector, is treated here for the sake of brevity.

2.1 Alternative equation form

As shown in Stenlund and Gustafsson [2002] for non-singular $P(t)$ and in Evestedt and Medvedev [2006b] for a general case, the difference $E(t) = P(t) - P_d$ obeys the recursion

$$E(t+1) = A_t^{-1}(P(t))E(t)A_t^{-T}(P_d) \tag{4}$$

where $A_t(X) = I + r^{-1}(t)X\varphi(t)\varphi^T(t)$.

To distinguish between intensity and direction of excitation, introduce the following notation

$$\rho_t = \frac{\varphi^T(t)\varphi(t)}{r(t)},\ U_t = \frac{\varphi(t)\varphi^T(t)}{\varphi^T(t)\varphi(t)}$$

where U_t is a unitary projection matrix specifying the direction of excitation. Now, ρ_t characterizes the instantaneous intensity of excitation and can be loosely called signal-to-noise ratio since $\varphi^T(t)\varphi(t)$ is the squared Euclidian norm of the regression vector and $r(t)$ usually stands for the variance of measurement noise in the underlying regression model. Denote

$$\rho_* = \inf_t \rho_t$$

As shown in Evestedt and Medvedev [2006b], for the special choice of the matrix $Q(t)$ given by (2), Riccati equation (1) can generally be rewritten in the form of a (linear) discrete Sylvester difference equation

$$\begin{aligned} P(t+1) &= A_t^{-1}(P(t))P(t)A_t^{-T}(P_d) \\ &\quad - A_t^{-1}(P(t))P_dA_t^{-T}(P_d) + P_d \end{aligned} \tag{5}$$

A complete parameterization of all stationary solutions of (5), including non-symmetric ones, is provided in Evestedt and Medvedev [2006b]. However, being initiated with a symmetric matrix, the solution $P(\cdot)$ stays symmetric, which fact clearly follows from (1), (2). Therefore, $E(t)$ in (4) is symmetric whenever $P(0)$ is symmetric.

In vectorized form, (4) becomes (see *e. g.* Horn and Johnson [1991], p. 254)

$$\begin{aligned} e(t+1) &= M\left(P_d, P(t)\right)e(t) \\ M\left(P_d, P(t)\right) &= A_t^{-1}(P_d) \otimes A_t^{-1}(P(t)) \end{aligned} \tag{6}$$

where $\otimes$ denotes Kronecker (tensor) product and $e(\cdot) = \text{vec}\ E(\cdot)$.

2.2 Lyapunov transformation

It is practical to bring equation (4) to a more compact and structure-revealing form, as suggested in Medvedev [2004].

Denote the normalized eigenvector of U_t corresponding to the eigenvalue equal to one as ξ_1^t. Eigenvalue-eigenvector decomposition of U_t immediately yields $U_t = \xi_1^t {\xi_1^t}^T$. Thus, the direction of excitation at time t can be characterized by the vector ξ_1^t. The excitation condition written for the regressor

$$\sum_{i=t}^{t+k} \varphi_i\varphi_i^T > 0,\ k \geq n \tag{7}$$

can be equivalently stated as

$$\text{rank}\ \left[\ \xi_1^t\ \ldots\ \xi_1^{t+k}\ \right] = n$$

i. e. the vectors $\xi_1^i,\ i = t, \ldots, t+k$ span the whole R^n. For the singular case $\rho_t = 0$, it is assumed that $\xi_1^t = 0$.

Let the sequence $\{U_t\}$ be persistently exciting on each interval of n consecutive steps i.e.

$$\det T(\tau) \neq 0,\ T(\tau) = \left[\ \xi_1^\tau\ \ldots\ \xi_1^{\tau+n}\ \right]$$

Note that the exact order of ξ_1^t in T is not important. When future values of $\{U_t\}$ are not available, one can in a similar manner use past values of the regressor vector and propagate T one step ahead.

The matrix T is constant on each interval $t = \tau, \ldots, \tau+n-1$ and is a Lyapunov transformation, Rugh [1996]. Indeed, T is bounded since it is composed of normalized vectors and $|\det\ T| > 0$ by construction. Therefore, it preserves stability properties of a dynamic system when used as a state vector transformation.

Essential properties of T are that it includes all possible directions of excitation and it is always nonsingular. Thus, the columns of the transformation matrix are denoted as

$$T(t) = \left[\ \xi_1^1(t)\ \ldots\ \xi_1^n(t)\ \right]$$

with time variable dropped when appropriate to save space.

Introduce a new state matrix $Z(t) = T^T(t)E(t)T(t)$. Since $E(t)$ is a symmetric matrix, the transformed state $Z(t)$ is also symmetric. It can be concluded, see Medvedev [2004] that $T^{\otimes^2}$ is a Lyapunov transformation for the vectorized sequence $e(t)$ and therefore $T^T(t)E(t)T(t)$ is such for the matrix sequence $E(t)$.

After the Lyapunov transformation, (4) reads

$$Z(t+1) = \bar{A}_t^{-1}(P(t))Z(t)\bar{A}_t^{-T}(P_d) \tag{8}$$

where

$$\bar{A}_t(\cdot) = T^T A_t(\cdot)T^{-T}$$

It is essential to realize that $T(t)$ is a time-varying transformation and changes after each n steps of the algorithm. This implies that the state elements of $Z(t)$ evolve not only due to recursion (8) but as well as a function of $T(t)$.

For some $X \in R^{n\times n}, X \geq 0$, consider two vectors

$$d_i^T(X) = \left[\ {\xi_1^1}^T X\xi_1^i\ \ldots\ {\xi_1^n}^T X\xi_1^i\ \right]$$

and

$$D_i^T(X) = \left[\ D_i^1(X)\ \ldots\ D_i^n(X)\ \right]$$

which are related to each other as to

$$D_i(X) = \frac{\rho_i d_i(X)}{1 + \rho_i {\xi_1^i}^T X\xi_1^i} \tag{9}$$

The inverse of $\bar{A}_i(X)$ is calculated in Medvedev [2004] to be

$$\bar{A}_i^{-1}(X) = I - \left[\ 0_{n\times(i-1)}\ D_i(X)\ 0_{n\times(n-i)}\ \right]$$

Then $\bar{A}_i(X)$ is a sum of a unit matrix and a matrix whose non-zero elements are collected in just one column. The position of the column is defined by the current excitation direction ξ_1^i.

The structure of $\bar{A}_i(X)$ brought about by the Lyapunov transformation makes it possible to distinguish between the elements of Z in the direction of excitation and those outside of it. Formally, the elements $z_{kl}, (k = i) \vee (l = i)$ are in the excitation direction and all other (e.g. $z_{kl}, k = 1, \ldots, n; l = 1, \ldots, n; k \neq i, l \neq i$) are outside.

For $X \geq 0$, let $\lambda_i(X) \leq \lambda_{i+1}(X), i = 1, \ldots, n-1$ be the eigenvalues of the matrix.

Proposition 1. (Medvedev [2004]). If $X \geq 0, X \neq 0$ then

$$\max_{\xi_1^k,\xi_1^i} |D_i^k(X)| = \frac{\rho_i \lambda_n(X)}{1+\rho_i\lambda_n(X)} < 1 \tag{10}$$

Proposition 2. Let $P(t)$ be a solution to (5), then

$$D_i^k\left(P(t)\right) = \frac{\rho_i\left({\xi_1^k}^T P_d \xi_1^i + z_{ik}(t)\right)}{1+\rho_i\left({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t)\right)}$$

Proof. From the definitions of Z and E, it follows

$$P(t) = P_d + E(t) = P_d + T^{-T}(t)Z(t)T^{-1}(t)$$

Then

$$D_i^k\left(P(t)\right) = \frac{\rho_i\left({\xi_1^k}^T P_d \xi_1^i + {\xi_1^k}^T T^{-T}(t)Z(t)T^{-1}(t)\xi_1^i\right)}{1+\rho_i\left({\xi_1^i}^T P_d \xi_1^i + {\xi_1^i}^T T^{-T}(t)Z(t)T^{-1}(t)\xi_1^i\right)} \tag{11}$$

Now, taking into account the identity

$${\xi_1^k}^T T^{-T}(t) = [\,0 \,\ldots\, \underset{k}{1} \; 0 \,\ldots]$$

leads to

$${\xi_1^k}^T T^{-T}(t)Z(t)T^{-1}(t)\xi_1^i = z_{ik}(t)$$

The above equality together with (11) yields the sought result.

3. ELEMENTWISE SOLUTION

In Medvedev [2004], it was demonstrated how the original recursive scheme (4) can be at each step decomposed into a set of fourth-order linear autonomous systems of triangular structure. Furthermore, at each t, the dynamics of each element z_{kl} of the matrix Z is shown to be defined by four numbers, namely $D_i^l(P_d), D_i^i(P_d), D_i^k(P), D_i^i(P)$.

Proposition 3. (Medvedev [2004]). Let $\xi_1^t = \xi_1^i$. Then the dynamics of the elements of $z(t) = \{z_{kl}, k = 1,\ldots,n; l = 1,\ldots,n\}$ are governed by

$$\begin{aligned}\bar{z}_{kl}(t+1) = &\,\mathcal{E}\left(D_i^k(P_d), D_i^i(P_d)\right) \\ &\otimes \mathcal{E}\left(D_i^l(P(t)), D_i^i(P(t))\right)\bar{z}_{kl}(t)\end{aligned} \tag{12}$$

where

$$\mathcal{E}(a,b) = \begin{bmatrix} 1 & -a \\ 0 & 1-b\end{bmatrix}$$

and

$$\bar{z}_{kl}^T = [\, z_{kl} \;\; z_{ki} \;\; z_{il} \;\; z_{ii}\,]$$

Notice that for symmetric solutions, $\bar{z}_{kl}$ and $\bar{z}_{lk}$ are equivalent up to permutation.

The result above shows that the updates of the elements of Z outside of the current excitation direction are completely defined by the values of the elements in the excitation direction.

Provided (5) is initialized at a symmetric matrix $P(0)$, the solution of it is obviously symmetric. Thus, a further simplification of (12) can be obtained for such practically important cases.

Another complication inflicted by the structure of $\mathcal{E}(\cdot,\cdot)$ is that it has an eigenvalue equal to one which hinders convergence analysis. Decomposition (12) is valid for any element of Z no matter where it is situated in the matrix with respect to the current excitation. It can be intuitively expected that the elements of Z in the direction of excitation converge and all other elements do not. In order to get an insight into the underlying convergence mechanism, an elementwise form of (12) is required.

Proposition 4. For symmetric solutions of (4), i.e. $P(t) = P^T(t)$, the elements of $z(t) = \{z_{kl}, k = 1,\ldots,n; l = 1,\ldots,n\}$ are governed by

$$\begin{aligned}z_{kl}(t+1) = z_{kl}(t) &+ \frac{D_i^k(P)D_i^l(P)\left(1-D_i^i(P_d)\right)}{D_i^i(P_d)-D_i^i(P)} z_{ii}(t) \\ &- \frac{D_i^k(P_d)D_i^l(P_d)\left(1-D_i^i(P)\right)}{D_i^i(P_d)-D_i^i(P)} z_{ii}(t)\end{aligned} \tag{13}$$

which expression reads for $k = l = i$ as

$$z_{ii}(t+1) = (1-D_i^i(P_d))(1-D_i^i(P))z_{ii}(t) \tag{14}$$

Proof. Writing (12) elementwise gives

$$\begin{aligned}z_{kl}(t+1) &= z_{kl}(t) - D_i^k(P)z_{ki}(t) - D_i^l(P_d)z_{il}(t) \\ &\quad + D_i^k(P)D_i^l(P_d)z_{ii}(t) \qquad (15)\\ z_{ki}(t+1) &= (1-D_i^i(P))z_{ki}(t) - D_i^l(P_d)(1-D_i^i(P))z_{ii}(t) \\ z_{il}(t+1) &= (1-D_i^i(P_d))z_{il}(t) - D_i^k(P)(1-D_i^i(P_d))z_{ii}(t) \\ z_{ii}(t+1) &= (1-D_i^i(P_d))(1-D_i^i(P))z_{ii}(t)\end{aligned}$$

For the case of symmetric P, Z is also symmetric since

$$Z(t) = T^T(t)(P(t)-P_d)T(t)$$

and $P_d = P_d^T$. Defining elementwise equations similar to (15) but for z_{lk}, z_{ik}, z_{li} and taking into account that Z is symmetric leads to the following identities valid for all t

$$\begin{aligned}\left(D_i^l(P)-D_i^l(P_d)\right)z_{il} &= \left(D_i^k(P)-D_i^k(P_d)\right)z_{ki} \\ &\quad + \left(D_i^k(P_d)D_i^l(P) - D_i^k(P)D_i^l(P_d)\right)z_{ii} \\ \left(D_i^i(P)-D_i^i(P_d)\right)z_{ki} &= \left(D_i^l(P_d)(1-D_i^i(P)) \right.\\ &\quad \left. + D_i^l(P)(1-D_i^i(P_d))\right)z_{ii} \\ \left(D_i^i(P)-D_i^i(P_d)\right)z_{il} &= \left(D_i^k(P_d)(1-D_i^i(P)) \right.\\ &\quad \left. + D_i^k(P)(1-D_i^i(P_d))\right)z_{ii}\end{aligned}$$

Now z_{ki} and z_{il} in the first equation of (15) can be expressed in terms of z_{ii} using the last two identities above, thus producing (13).

Notice that (14) is immediately obtained from (13) by letting $k = l = i$.

Proposition 4 provides a general form for elementwise evaluation of $Z(t)$. However, (13) still includes the functions $D_i^j(P), j = \{k,l\}$ which, in their turn, depend on elements of $Z(t)$.

Corollary 1. Equation (13) can be equivalently written as

$$z_{kl}(t+1) = z_{kl}(t) + \rho_i\left(\frac{{\xi_1^l}^T P_d \xi_1^i {\xi_1^k}^T P_d \xi_1^i}{1+\rho_i {\xi_1^i}^T P_d \xi_1^i} - \frac{({\xi_1^l}^T P_d\xi_1^i + z_{il})({\xi_1^k}^T P_d \xi_1^i + z_{ik})}{1+\rho_i({\xi_1^i}^T P_d\xi_1^i + z_{ii}(t))}\right) \tag{16}$$

or, for the special case of $l = i$

$$z_{ki}(t+1) = \frac{(1+\rho_i {\xi_1^i}^T P_d \xi_1^i) z_{ki}(t) - \rho_i {\xi_1^k}^T P_d \xi_1^i z_{ii}(t)}{\left(1+\rho_i {\xi_1^i}^T P_d \xi_1^i\right)\left(1+\rho_i ({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t))\right)} \quad (17)$$

which expression reads for $k = l = i$ as

$$z_{ii}(t+1) = \frac{z_{ii}(t)}{\left(1+\rho_i {\xi_1^i}^T P_d \xi_1^i\right)\left(1+\rho_i ({\xi_1^i}^T P_d \xi_1^i + z_{ii})\right)} \quad (18)$$

Proof. The proof is straightforward by substituting in (13) the following equalities

$$1 - D_i^i(P) = \frac{1}{1+\rho_i\left({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t)\right)} \quad (19)$$

$$1 - D_i^i(P_d) = \frac{1}{1+\rho_i {\xi_1^i}^T P_d \xi_1^i}$$

$$D_i^i(P) - D_i^i(P_d) = \frac{\rho_i z_{ii}}{(1+\rho_i {\xi_1^i}^T P_d \xi_1^i)\left(1+\rho_i\left({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t)\right)\right)} \quad (20)$$

Exactly as before, (17) and (18) are just the specializations of (16) for $l = i$ and $k = l = i$.

4. CONVERGENCE

From experimental studies in Evestedt and Medvedev [2006a], Evestedt et al. [2005], Olsson [2005], it can be concluded that the solution of Riccati equation (1,2) converges to P_d whenever excitation condition (3) is satisfied. However, no formal proof of this fact can be found in the literature. On the negative side, it is known from Medvedev [2004] that the mapping $M(\cdot, \cdot)$ in (6) is not a contraction. This does not mean that (6) (or, equivalently (4)) cannot converge but significantly complicates the choice of tools for convergence analysis.

Regarding stationary behavior of the SG algorithm under lack of excitation studied in Evestedt and Medvedev [2006b], it is shown that solutions of (4) converge to P_d in the elements that lie in the persistent directions of excitation. However, the mechanism of convergence has so far not been revealed.

The result of Corollary 1 explains why and how the solutions of (4) converge. Indeed, (18) implies very fast exponential convergence of the diagonal elements of Z in the excited directions. Therefore, the following approximation is justified

$$z_{ii}(t+1) \approx \frac{z_{ii}(t)}{\left(1+\rho_i {\xi_1^i}^T P_d \xi_1^i\right)^2}$$

Clearly, the rate of convergence increases for higher signal-to-noise ratios. Now, it is safe to assume that z_{ii} is small which yields the following approximation for the off-diagonal elements of Z in the directions of excitation, cf. (17)

$$z_{ki}(t+1) \approx \frac{z_{ki}(t)}{1+\rho_i {\xi_1^i}^T P_d \xi_1^i}$$

The elements $z_{ki}, k \neq i$ also exhibit exponential convergence but at a lower rate than for the diagonal elements. Thus, the assumption that $z_{ki}, k = 1, \ldots, n$ are small is as well justified. This turns (16) into

$$z_{kl}(t+1) \approx z_{kl}(t), \; k \neq i, l \neq i$$

Recapitulating for the simplifications above, it can be concluded that all elements of Z in the directions of excitation converge exponentially to zero while all other elements remain approximately constant. The argument is not a formal proof but applies locally in a neighborhood of the stationary solution of (8).

At this point, it is instructive to recall that the Lyapunov transformation $T(t)$ used to obtain (8) changes each n steps therefore changing the values of generally all elements of Z. After such transformation change, any z_{kl} can either increase or decrease. This however does not influence the type of convergence due to the properties of Lyapunov transformations, Rugh [1996].

It is seldom one has much control over noise and excitation conditions in an identification experiment which makes the choice of P_d quite important in applications. From (17) and (18), it becomes clear in what way P_d should be selected to obtain a higher convergence rate for (1, 2). Since $\lambda_1(P_d) \leq \|P_d \xi_1^i\|_2 \leq \lambda_n(P_d)$, P_d has to have eigenvectors corresponding to large $\lambda_i(P_d)$ collinear to the frequent excitation directions.

In the sequel of the section, formal convergence proofs for solutions of (8) are provided. Let's first turn to the convergence in the current direction of excitation.

Proposition 5. The mapping

$$\begin{bmatrix} z_{ki}(t+1) \\ z_{ii}(t+1) \end{bmatrix} = F_{ki} \begin{bmatrix} z_{ki}(t) \\ z_{ii}(t) \end{bmatrix} \quad (21)$$

where

$$F_{ki} = \frac{\mathcal{E}(\rho_i {\xi_1^k}^T P_d \xi_1^i, \rho_i {\xi_1^i}^T P_d \xi_1^i) + \rho_i {\xi_1^i}^T P_d \xi_1^i I}{(1+\rho_i {\xi_1^i}^T P_d \xi_1^i)\left(1+\rho_i\left({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t)\right)\right)}$$

is a contraction for all $k = 1, \ldots, n; \; k \neq i$.

Proof. For a matrix X of the following structure

$$X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

it applies that

$$\lambda_{12}(X^T X) = \frac{a^2 + b^2 + c^2 \pm \sqrt{((a-c)^2 + b^2)((a+c)^2 + b^2)}}{2}$$

Now, a direct evaluation gives

$$\lambda_2(F_{ki}^T F_{ki}) = \frac{0.5}{(1+\rho_i {\xi_1^i}^T P_d \xi_1^i)^2 \left(1+\rho_i\left({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t)\right)\right)^2} \Bigg((1+\rho_i {\xi_1^i}^T P_d \xi_1^i)^2 + \rho_i^2 ({\xi_1^k}^T P_d \xi_1^i)^2 + 1 + \rho_i \sqrt{\left(({\xi_1^i}^T P_d \xi_1^i)^2 + ({\xi_1^k}^T P_d \xi_1^i)^2\right) f_{ki}} \Bigg) \quad (22)$$

where

$$f_{ki} = (2 + \rho_i {\xi_1^i}^T P_d \xi_1^i)^2 + \rho_i^2 ({\xi_1^k}^T P_d \xi_1^i)^2$$

Noticing that

$$({\xi_1^k}^T P_d \xi_1^i)^2 \leq \lambda_n^2(P_d)$$

and that $\lambda_2(F_{ki}^T F_{ki})$ is monotonically decreasing in ρ_i and ${\xi_1^i}^T P_d \xi_1^i$ with

$$\lim_{\rho_i \to 0} \lambda_2(F_{ki}^T F_{ki}) < 1$$
$$\lim_{\lambda_1(P_d) \to 0} \lambda_2(F_{ki}^T F_{ki}) < 1$$

completes the proof.

Proposition 5 proves that all the elements of $Z(t)$ in the current excitation direction exponentially converge.

Now let's show that the increase in the elements outside of the current excitation direction is bounded at each step from above and below.

Proposition 6. For each element $z_{kl}(t), k \neq i, l \neq i$ of $Z(t)$ in (8), the following inequalities apply

$$|z_{kl}(t+1) - z_{kl}(t)| < z_{ii}(t) + \frac{2}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) \quad (23)$$

Proof. Consider the second term in the right-hand side of (16). Recalling that

$$|D_i^k(P)| = \frac{\rho_i |({\xi_1^k}^T P_d \xi_1^i + z_{ik})|}{1 + \rho_i({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t))} < 1$$

the following inequalities follow

$$\begin{aligned}
&-|{\xi_1^l}^T P_d \xi_1^i| - |{\xi_1^l}^T P_d \xi_1^i + z_{il}| \\
&< \rho_i \left(\frac{{\xi_1^l}^T P_d \xi_1^i {\xi_1^k}^T P_d \xi_1^i}{1 + \rho_i {\xi_1^i}^T P_d \xi_1^i} - \frac{({\xi_1^l}^T P_d \xi_1^i + z_{il})({\xi_1^k}^T P_d \xi_1^i + z_{ik})}{1 + \rho_i({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t))} \right) \\
&< |{\xi_1^l}^T P_d \xi_1^i| + |{\xi_1^l}^T P_d \xi_1^i + z_{il}|
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\frac{1}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) - \frac{1}{\rho_i}\left(1 + \rho_i({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t))\right) \\
&< z_{kl}(t+1) - z_{kl}(t) \\
&< \frac{1}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) + \frac{1}{\rho_i}\left(1 + \rho_i({\xi_1^i}^T P_d \xi_1^i + z_{ii}(t))\right)
\end{aligned}$$

which immediately gives (23).

Notice also that the lower bound is always negative and the upper bound is always positive despite indefinite sign of z_{ii}. Indeed, due to Proposition 1, $D_i^i(\cdot)$ is bounded $0 < D_i^i(P(t)) < 1$. Thus, taking into account (19)

$$\begin{aligned}
&\frac{1}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) < z_{ii}(t) + \frac{2}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) \quad (24) \\
&-z_{ii}(t) - \frac{2}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) < -\frac{1}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i)
\end{aligned}$$

5. BOUNDS

Bounds (23) capture a phenomenon observed in simulation of (8), namely that the excursions in the elements of $Z(t)$ outside of the current excitation direction decrease with time. This is explained by exponential convergence of z_{ii}. Another property of (8) is that the excursions lessen with higher excitation intensity. This can be as well derived from (23).

Constant and therefore less sharp bounds can be easily obtained around the stationary solution, i.e. assuming that z_{ii} is small

$$\begin{aligned}
-\frac{2}{\rho_*}(1 + \rho_* \lambda_n(P_d)) &< z_{kl}(t+1) - z_{kl}(t) \\
&< \frac{2}{\rho_*}(1 + \rho_* \lambda_n(P_d))
\end{aligned}$$

For bounds valid even near the initial conditions on (8), more insight into evolutions of z_{ii} is needed.

As mentioned before, $z_{ii}(t)$ can take both positive and negative values. Inequality (24) implies that the negative values of $z_{ii}(t)$ are for any t bounded from below.

$$z_{ii}(t) > -\frac{1}{\rho_i}(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i) \geq -\frac{1}{\rho_i}(1 + \rho_i \lambda_n(P_d))$$

This means that the range of negative z_{ii} is basically defined by P_d for large signal-to-noise ratios and by the ratio itself when it is poor.

Furthermore, by comparing (20) and (18), one obtains

$$z_{ii}(t+1) = \frac{1}{\rho_i}\left(D_i^i(P(t)) - D_i^i(P_d)\right)$$

Since both $D_i^i(P(t))$ and $D_i^i(P_d)$ are positive and less than one

$$|D_i^i(P(t)) - D_i^i(P_d)| < 1$$

and, for all t

$$z_{ii}(t+1) < \frac{1}{\rho_i} \quad (25)$$

The inequality above shows that, for good excitation conditions, z_{ii} should take a small value already after the first step of the algorithm in the excitation direction in question. Taking into account the exponential convergence due to (18), all z_{ii} are quite small for $t > n$, provided the excitation is good and the noise variance r in the regression model is low.

The convergence of (1,2) is fast for high values of ρ_t and a properly chosen P_d. One can also conclude that increasing the convergence rate through P_d also increases the range of variation for z_{ii}.

The result of Proposition 5 give rise to the following inequality

$$\left\| \begin{array}{c} z_{ki}(t+1) \\ z_{ii}(t+1) \end{array} \right\|_2^2 \leq \lambda_2(F_{ki}^T F_{ki}) \left\| \begin{array}{c} z_{ki}(t) \\ z_{ii}(t) \end{array} \right\|_2^2, \ k \neq i \quad (26)$$

Taking into account (18) yields

$$\begin{aligned}
z_{ki}^2(t+1) &\leq \lambda_2(F_{ki}^T F_{ki}) z_{ki}^2(t) + \Bigg(\lambda_2(F_{ki}^T F_{ki}) \\
&- \frac{1}{\left(1 + \rho_i {\xi_1^i}^T P_d \xi_1^i\right)^2 \left(1 + \rho_i({\xi_1^i}^T P_d \xi_1^i + z_{ii})\right)^2} \Bigg) z_{ii}^2(t) \\
&< \lambda_2(F_{ki}^T F_{ki}) \left(z_{ki}^2(t) + z_{ii}^2(t)\right)
\end{aligned}$$

It can be easily checked that the factor by z_{ii}^2 in the first inequality is non-negative and strictly less than one.

6. SIMULATION RESULTS

The bounds on the elements of $Z(t)$ are illustrated using a simulation example with $n = 3$. Regressor vectors are chosen to vary in length but be periodic in the direction of excitation which gives a constant transformation matrix T.

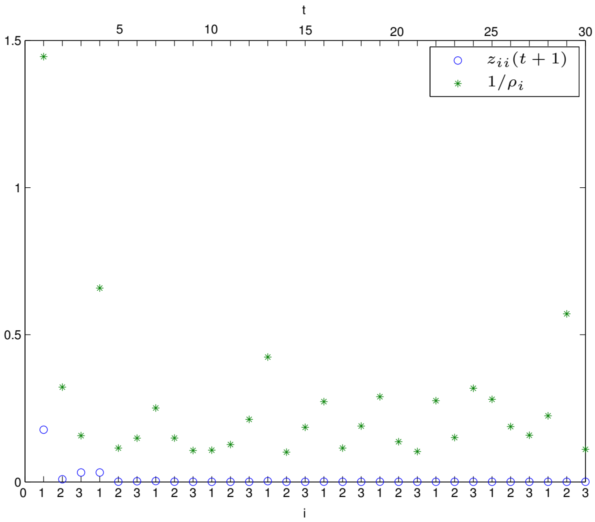

Fig. 1. Bound (25) on the diagonal element in the direction of excitation.

Letting $r(t) = 1$ yields $\rho_t = \varphi^T(t)\varphi(t)$. In this simulation, $\rho_i(\tau)$ is taken as a random sequence of numbers between 0 and 10. This choice is only for the purpose of illustration since it is not what typically occurs when the Riccati equation is used in a parameter estimation algorithm. In the latter case, the regressor vectors include input and output signal values and possess certain structure.

Furthermore, the matrices $P(0)$ and P_d are selected as random positive definite matrices to make the example as general as possible. The Riccati equation in the SG-algorithm was implemented elementwise according to Corollary 1 and the evaluated elements were utilized to validate (26) and (25).

In Fig. 1, the actual values of z_{ii} are given and compared with the upper bound provided by (25). As anticipated, the bound is useful during initial iteration steps of the Riccati equation but becomes conservative further on.

The contraction property proved in Proposition 5 is illustrated in Fig. 2. The three subplots correspond to three elements of $Z(t)$ in the current excitation direction outside of the main diagonal.

7. CONCLUSION

It is shown that the Riccati equation of the SG algorithm can be solved elementwise. The Riccati equation in question has a free term that is of rank 1 and its convergence properties are therefore not guaranteed by the standard Kalman filter theory. The expressions for the elementwise solution reveal the underlying convergence mechanism and are instrumental in the derivation of the upper and lower bounds on the elements of the Riccati equation solution.

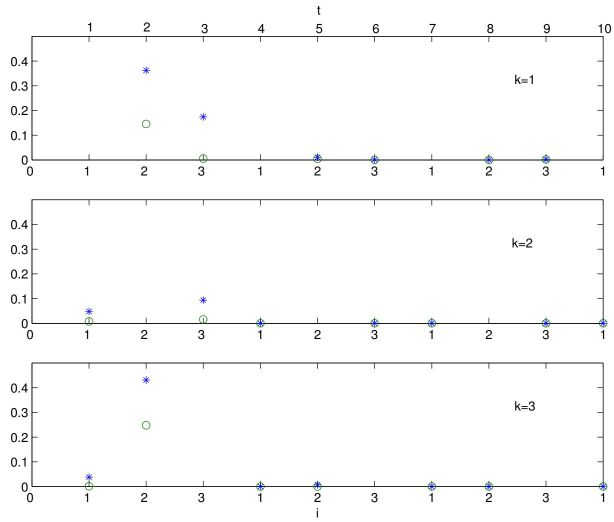

Fig. 2. The stepwise contraction property of the SG-algorithm in the excitation direction. The plots show the right-hand side (∗) and the left-hand side (o) of (26). The instances when $z_{ki} \equiv z_{ii}$ are omitted, see Fig. 1 instead.

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