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SWITCHED MUTUAL–MASTER-SLAVE SYNCHRONISATION: APPLICATION TO MECHANICAL SYSTEMS

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Abstract: We show that the problems of mutual, master-slave synchronisation are equivalent (up to a transformation) to a classical output tracking control for certain nonlinear time-varying systems. Therefore, solving any of these problems solves the others however, in the presence of disturbances such an over-simplification of the tracking/synchronisation problem may lead to loss of performance and increase of control efforts. Then, we propose a supervisor controller that ensures asymptotic tracking while keeping synchronisation errors "small". Further, in the presence of disturbances, we establish input-output-to-state stability. In particular, we address the simultaneous tracking and synchronisation problems for mechanical systems. *IFAC Copyright 2008.*

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1 INTRODUCTION

Synchronisation has been a centre of attention in vibration mechanics (Blekhman 1988), beginning with (Pecora and Carroll 1990), in chaotic systems; physics in control (Pogromsky *et al.* 2002) and in control theory (Nijmeijer 2001), (G.Sugar and Kumar 2002). Recent applications in *controlled* synchronisation include synchronisation of *networks* of nonlinear oscillators –*cf.* (Blekman *et al.* 1997), (Pogromsky *et al.* 2002). There are two basic synchronisation schemes: *master-slave* and *mutual.* The first consists in making one or more *slave* systems to follow, dynamically, a leading system called the master; in the second case, several systems are required to synchronise their dynamics without any particular hierarchy. From a control theory viewpoint, master-slave synchronisation may be re-casted in an observer design problem –*cf.* (Nijmeijer and Mareels 1997). Other aspects classical in control theory, such as parameter uncertainty, robustness, optimality, etc. arise naturally –*cf.* (Blekman *et al.* 1997).

In some cases, like for example in cooperative coordination of mobile robots, satellites or robot manipulators –*cf.* (Bondhus *et al.* 2005, Luh 1983, G.Sugar and Kumar 2002, Sun 2003) the controlled synchronisation problem includes two subtasks: tracking of a desired trajectory that is common to all robots and, second, the synchronisation of robots behaviour relative to each other. Strictly speaking only the second problem is about synchronisation. In this paper we consider the problems of tracking and master-slave synchronisation, *simultaneously*; that is, we address the control problem of making a master system follow a reference desired trajectory and to make a set of slave systems synchronise with the master. We show that, actually, in certain cases these problems are equivalent themselves and to mutual synchronisation, up to an invertible mapping. Such state transformation introduces a *gain* relation that may be significant when the systems are affected by external disturbances or in presence of neglected dynamics. The proposed control approach is novel in the sense that it is based on a supervisor which switches between a tracking and a synchronisation controller, depending on the respective errors. We show that for mechanical systems this results in a significant performance improvement with respect to classic solutions to synchronisation problems.

The rest of the paper is organised as follows. In the following section we introduce the problem statement and describe in mathematical detail the problems of tracking, master-slave and mutual synchronisation; in Section 3 we propose a switching strategy for controlled synchronisation and trajectory tracking of robot manipulators and conclude with some remarks in Section 4.

Notation. We use $R_+ := [0, \infty)$. A continuous function $\sigma : R_+ \to R_+$ is of class K if it is strictly increasing and $\sigma(0) = 0$; additionally it belongs to class K_∞ if it is also radially unbounded; a continuous function $\beta : R_+ \times R_+ \to R_+$ is of class KL, if $\beta(\cdot, s)$ is of class K for each s and $\beta(r, \cdot)$ is strictly decreasing to zero for each r.

2 Problem statement

Consider $N > 1$ nonlinear dynamical systems

$$\dot{x}_i = f_i(x_i, u_i, d_i), \quad y_i = h_i(x_i), \quad i \in [1, N], \tag{1}$$

where $x_i \in R^{n_i}$ are state vectors of the subsystems in (1); $u_i \in R^{m_i}$ are control inputs; $d_i \in R^{l_i}$ are disturbances, $y_i \in R^p$ are the outputs to be synchronised, functions $f_i : R^{n_i+m_i+l_i} \to R^{n_i}$ and $h_i : R^{n_i} \to R^p$ are continuous and locally Lipschitz $\forall\, i \in [0, N]$; $x = [x_1^T \dots x_N^T]^T \in R^n$, $n = \sum_i n_i$; $d = [d_1^T ... d_N^T]^T \in R^l$, $l = \sum_i l_i$. The Euclidean norm is denoted by $|\cdot|$, and $\|\cdot\|_{[t_0,t]}$ denotes the $L_\infty^{m_i}$ norm. Control inputs are denoted by $u_i : R_+ \to R^{m_i}$ and d_i are supposed to be Lebesgue measurable and locally essentially bounded functions *i.e.*,

$$\| d_i \|_{[t_0,T]} = ess \sup \{ | d_i(t) |, \ t \in [t_0, T] \}.$$

We denote the set of all such functions, with the property $||d||_{[0,+\infty)} =: ||d|| < +\infty$ by M_{R^m}. Let $x_i(t, t_0, x_i^0, u_i, d_i)$ for all $i \in [0, N]$ denote the solutions of the system (1) with initial conditions $t_0 \in R_+$, $x_i^0 \in R^{n_i}$ generated by inputs $u_i \in M_{R^{m_i}}$ and $d_i \in M_{R^{l_i}}$ then, the outputs are functions $y_i(t, t_0, x_i^0, u_i, d_i) = h_i(x_i(t, t_0, x_i^0, u_i, d_i))$. On occasions we may write $x_i(t)$ and $y_i(t)$ if all other arguments are clear from the context. The solutions are assumed to exist on finite intervals $[0, T)$ and, if $T = +\infty$ for every initial pair $t_0 \in R_+$, x_0^i the system is said to be forward complete.

Consider the following output regulation problems for system (1) with the given common output reference trajectory $y_d : R \to R^p$:

1. Independent tracking problem. To design a controller $U_i : R_+ \times R^{n_i} \to R^{m_i}$, $i \in [1, N]$ such that the closed-loop system is forward complete for all $t_0 \in R_+$, $x_i^0 \in R^{n_i}$ and Input to Output State Stable (IOSS) –*cf.* (Sontag and Wang 1999) with respect to the input disturbances $d_i \in M_{R^{l_i}}$ and the output *tracking errors* $e_i(t, t_0, x_i^0, d_i) = y_i(t, t_0, x_i^0, U_i, d_i) - y_d(t)$, $i \in [1, N]$ $(e = (e_1^T \ldots e_N^T)^T)$ that is, for some $\beta_i \in KL$, $\gamma_i \in K$, all $t_0 \in R_+$ and $i \in [1, N]$,

$$|e_i(t, t_0, x_i^0, d_i)| \le \beta_i(|x_i^0|, t - t_0) + \gamma_i(||d_i||_{[t_0,\infty)}) .$$

2. Master-slave synchronisation problem. Let, for implicitly, the first subsystem in (1) be the master system. It is required to design control laws $U_i : R_+ \times R^n \to R^{m_i}$, $i \in [1, N]$ such that: the closed-loop system is forward complete; the master system is trajectory tracking controlled and for all $t_0 \in R_+$, $x^0 \in R^n$ and $d \in M_{R^l}$ the following estimates for the *master-slave synchronisation errors* $\varepsilon_1 = -e_1$, $\varepsilon_i = y_i - y_1$, $i \in [2, N]$, $\varepsilon = [\varepsilon_1^T \ldots \varepsilon_N^T]^T$ for some $\beta_i' \in KL$, $\gamma_i' \in K$, all $t \ge t_0 \ge 0$, $i \in [2, N]$:

$$|\varepsilon_i(t, x^0, d)| \le \beta_i'(|x^0|, t - t_0) + \gamma_i'(||d||), .$$

3. Mutual synchronisation problem. To design control laws $U_i : R_+ \times R^n \to R^{m_i}$, $i \in [1, N]$ such that the closed-loop system is forward complete and the *mutual synchronisation errors*

$$\zeta_i = y_i - y_d + \sum_{j=1}^{N, i\neq j} k_{i,j} \left[y_i - y_j \right] ,$$

$(\zeta = [\zeta_1^T \ldots \zeta_N^T]^T))$ where $i \in [0, N]$, $k_{i,j} > 0$, $k_{i,i} = 0)$, satisfy the following estimates for all $t_0 \in R_+$, $x^0 \in R^n$, $d \in M_{R^l}$, $t \ge t_0$:

$$|\zeta_i(t, x^0, d)| \le \beta_i''(|x^0|, t - t_0) + \gamma_i''(||d||),$$

where the functions $\beta_i'' \in KL$, $\gamma_i'' \in K$ may depend on parameters $k_{i,j}$.

In the absence of disturbances, the problems above consist in finding controllers such that the output tracking (resp. synchronisation) errors converge to zero uniformly with respect to the size of initial state errors. In other words, it is required that the system be IOS –*cf.* (Sontag and Wang 1999). Synchronisation problems, specifically for mechanical and chaotic systems has been extensively studied in the literature of control systems and applied physics (e.g., synchronisation of chaotic systems) –*cf.* references above; however, it is important to remark that, in the absence of disturbances, all three problems are *equivalent* up to a transformation which introduces a "gain" relation between the errors. More precisely, one has the following:

$$\varepsilon = \Upsilon e, \quad \zeta = \Psi e, \tag{2}$$

where

$$\Upsilon = \begin{bmatrix} -I_p & 0 & \cdots & 0 \\ -I_p & I_p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -I_p & 0 & \cdots & I_p \end{bmatrix},$$

$$\Psi = \begin{bmatrix} \left(1 + \sum\limits_{j=2}^{N} k_{1,j}\right) I_p & -k_{1,2} I_p & \cdots & -k_{1,N} I_p \\ -k_{2,1} I_p & \left(1 + \sum\limits_{j=1}^{N, j\neq 2} k_{2,j}\right) I_p & \cdots & -k_{2,N} I_p \\ \vdots & \vdots & \ddots & \vdots \\ -k_{N,1} I_p & -k_{N,2} I_p & \cdots & \left(1 + \sum\limits_{j=1}^{N-1} k_{N,j}\right) I_p \end{bmatrix}$$

where I_p is identity matrix of dimension $p \times p$. Both matrices, Υ and Ψ are invertible. *Thus, even in the generic case the synchronisation errors are linearly dependent and proportional to each other and to tracking errors.* The importance of this claim cannot be overestimated: on one hand, it means that up to some extent solving an output tracking control problem is equivalent to solving the master-slave or the mutual synchronisation problems; on the other hand, the term *proportional* hides that the gain relation (introduced by the matrix norms of Υ and Ψ) between one tracking error and synchronisation errors may be quite significant, if regarded from a practical viewpoint; in particular, in problems related to mechanical systems –*cf.* following section.

The above-described scenario motivates the following problem, that we solve in this paper: to ensure acceptable output tracking control (in the IOS sense above) while keeping synchronisation errors under a meaningful specified bound, given *a priori*. It is clear that such problem, is in general unsolvable via pure tracking or pure synchronisation control as it has been proposed in the literature so far (to the best of our knowledge).

3 Switched synchronisation of mechanical systems

Consider a set of N mechanical systems indexed by $i \in [1, N]$ *i.e.*,

$$M(q_i)\, \ddot{q}_i = -C(q_i, \dot{q}_i)\, \dot{q}_i - g(q_i) + u_i + d_i, \tag{3}$$

where $q_i \in R^\nu$, $\dot{q}_i \in R^\nu$ are generalised coordinates and their velocities; functions M, C, g are continuous; M is positive definite and bounded *i.e.*, $0 < M_{\min} \le |M(q_i)| \le M_{\max} < +\infty$; for all $q_i \in R^\nu$ and all $i \le N$; the Coriolis matrix is constructed using the Christoffel symbols of the second kind, hence $\dot{M}(q_i) = C(q_i, \dot{q}_i) + C(q_i, \dot{q}_i)^\top$ and $|C(q_i, \dot{q}_i)| \le k|\dot{q}_i|$, $k > 0$ for all $q_i, \dot{q}_i \in R^\nu$. We assume that $q_i(t)$ and their velocities $\dot{q}_i(t)$ are measurable. As it is customary, it is also assumed that the *given* desired trajectory $t \mapsto q_d$, $t \ge t_0 \ge 0$ is twice continuously differentiable and its time derivatives $\dot{q}_d(t)$, $\ddot{q}_d(t)$ are bounded for all $\ge t_0 \ge 0$.

Tracking errors are defined as

$$e_i = [q_i \; \dot{q}_i]^T - [q_d \; \dot{q}_d]^T, \quad i \in [1, N]$$

while synchronisation errors are defined as

$$\delta_j(t) = [q_{j+1} \; \dot{q}_{j+1}]^T - [q_1 \; \dot{q}_1]^T, \quad j \in [1, N-1]$$

[1]This is considered for simplicity but other synchronisation errors may also be considered under minor obvious modifications.

that is, we consider master-slave synchronisation[1]. The switched synchronisation control problem consists in the following: let $\Delta > 0$ be a *given* synchronisation error tolerance; find a controller such that, for all $t \geq t_0 \geq 0$, the closed-loop system error trajectories satisfy the bounds:

$$|e(t)| \leq a|e(t_0)|e^{-c\,(t-t_0)} + b||d||;$$
$$|\delta(t)| \leq \kappa \max\{\,\Delta, |\delta(t_0)|\,\} + r\,||d||$$

for some positive constants a, c, b, κ, r.

The control approach consists in *switching* between controllers that achieve pure tracking or pure synchronisation in the absence of disturbances. As we have discussed, either control approach (pure tracking or pure synchronisation) is insufficient to achieve the control goal hence, the novelty of the controller and what actually makes it work is the design of the *supervisor* –*cf.* (Morse 1995). For clarity, we start by recalling pure tracking and synchronisation controllers that achieve exponential stability in the absence of disturbances, the supervisor is presented next.

Pure tracking control. We choose to use the well-known Slotine and Li algorithm –*cf.* (Slotine and Li 1988) in its non-adaptive version, that is, let

$$\begin{aligned} u_i = {} & M(q_i)\,\ddot{q}_d + g(q_i) - [K_1 + K_2\,\Lambda]\,(q_i - q_d) - \\ & -[K_2 + M(q_i)\,\Lambda]\,(\dot{q}_i - \dot{q}_d) + \\ & +C(q_i, \dot{q}_i)\,[\dot{q}_d - \Lambda\,(q_i - q_d)]\,, \quad i \in [1, N] \end{aligned} \tag{4}$$

where K_1, K_2, Λ are positive definite diagonal matrices. The closed-loop system is then given by

$$\begin{aligned} M(q_i)\,\dot{e}_{i,2} = {} & -[K_1 + K_2\,\Lambda]\,e_{i,1} - [K_2 + M(q_i)\,\Lambda]\,e_{i,2} - \\ & -C(q_i, \dot{q}_i)\,[e_{i,2} + \Lambda\,e_{i,1}] + d_i, \quad i \in [1, N]\,. \end{aligned} \tag{5}$$

where we defined $e_i = [e_{i,1}\; e_{i,2}]^T$, $e_{i,1} = q_i - q_d$ and $e_{i,2} = \dot{q}_i - \dot{q}_d$. For this system, it can be shown that the origin is uniformly globally exponentially stable via a direct Lyapunov analysis, along the lines of *e.g.*, (Spong *et al.* 1990, Loría *et al.* 2005).

2. Pure synchronisation control. First, it is convenient to note that the dimension of the error δ is $N-1$ since the errors are relative to the first robot to the remaining robots, indexed $i \in [2, N]$. Hence, for simplicity we assume that the control task for the first (master) robot is limited to pure tracking[2] In other words, we address master-slave synchronisation for systems (3). For the master system we apply the controller defined by (4) that is, the synchronisation error for the master robot takes the form (5) with $i = 1$. For the remaining systems we use a variant of controller (4) by replacing $q_d, \dot{q}_d, \ddot{q}_d$ with the corresponding variables of the "master" system $q_1, \dot{q}_1, \ddot{q}_1$ hence, for all $i \in [2, N]$ we have:

$$\begin{aligned} u_i = {} & M(q_i)\,\ddot{q}_1 + g(q_i) - [K_1 + K_2\,\Lambda]\,(q_i - q_1) - \\ & -[K_2 + M(q_i)\,\Lambda]\,(\dot{q}_i - \dot{q}_1) + \\ & +C(q_i, \dot{q}_i)\,[\dot{q}_1 - \Lambda\,(q_i - q_1)]\,. \end{aligned}$$

Since the variable $\ddot{q}_1$ is not available for measurement we use (as it is customary in related literature) the expression $\ddot{q}_1 = M(q_1)^{-1}[u_1 - C(q_1, \dot{q}_1)\,\dot{q}_1 - g(q_1)]$ so the synchronisation controller takes the form

$$\begin{aligned} u_1 = {} & M(q_1)\,\ddot{q}_d + g(q_1) - [K_1 + K_2\,\Lambda]\,(q_1 - q_d) - \\ & -[K_2 + M(q_1)\,\Lambda]\,(\dot{q}_1 - \dot{q}_d) + \\ & +C(q_1, \dot{q}_1)\,[\dot{q}_d - \Lambda\,(q_1 - q_d)]\,; \\ u_i = {} & M(q_i)\,\Big[\,M(q_1)^{-1}[u_1 - C(q_1, \dot{q}_1)\,\dot{q}_1 - g(q_1)]\,\Big] + \\ & +g(q_i) - [K_1 + K_2\,\Lambda]\,(q_i - q_1) - \\ & -[K_2 + M(q_i)\,\Lambda]\,(\dot{q}_i - \dot{q}_1) + \\ & +C(q_i, \dot{q}_i)\,[\dot{q}_1 - \Lambda\,(q_i - q_1)]\,, \; i \in [2, N] \end{aligned} \tag{6}$$

[2] We stress that, in view of the discussion in Section 2 there is no loss of generality.

Next, let $\delta_{i,1} = q_{i+1} - q_1$, $\delta_{i,2} = \dot{q}_{i+1} - \dot{q}_1$ with $i \in [1, N-1]$ then, the relations $\varepsilon_1 = -e_1$, $\varepsilon_i = \delta_{i-1}$ with $i \in [2, N]$, hold for the master-slave synchronisation errors ε_i. The closed-loop dynamics with the controller (6) becomes

$$\begin{aligned} M(q_1)\,\dot{\varepsilon}_{1,2} = {} & -[K_1 + K_2\,\Lambda]\,\varepsilon_{1,1} - [K_2 + M(q_i)\,\Lambda]\,\varepsilon_{1,2} - \\ & -C(q_1, \dot{q}_1)\,[\varepsilon_{1,2} + \Lambda\,\varepsilon_{1,1}] - d_1\,; \\ M(q_i)\,\dot{\varepsilon}_{i,2} = {} & -[K_1 + K_2\,\Lambda]\,\varepsilon_{i,1} - [K_2 + M(q_i)\,\Lambda]\,\varepsilon_{i,2} - \\ & -C(q_i, \dot{q}_i)\,[\dot{\varepsilon}_{i,2} + \Lambda\,\varepsilon_{i,1}] + \tilde{d}_i, \\ & \tilde{d}_1 = d_1,\; \tilde{d}_i = d_i - M(q_i)\,M(q_1)^{-1}\,d_1\,. \end{aligned} \tag{7}$$

where $i \in [2, N]$, which is similar to Equations (5) and the disturbances $\tilde{d}_i$ are of class M_{R^ν} since $d \in M_{R^\nu}$ and in view of the properties of M. Stability of the closed-loop system may be obtained along similar proof-lines as for system (5).

3. Mutual synchronisation. This can also be solved in a similar manner, using the controller

$$\begin{aligned} u_i = {} & M(q_i)\,\ddot{q}_{ri} + g(q_i) - [K_1 + K_2\,\Lambda]\,(q_i - q_{ri}) - \\ & -[K_2 + M(q_i)\,\Lambda]\,(\dot{q}_i - \dot{q}_{ri}) + \\ & +C(q_i, \dot{q}_i)\,[\dot{q}_{ri} - \Lambda\,(q_i - q_{ri})]\,, \end{aligned} \tag{8}$$

where

$$q_{ri} = q_d - \sum_{j=1}^{N, i\neq j} k_{i,j}[q_i - q_j];\; \dot{q}_{ri} = \dot{q}_d - \sum_{j=1}^{N, i\neq j} k_{i,j}[\dot{q}_i - \dot{q}_j];$$

$$\ddot{q}_{ri} = q_d - \sum_{j=1}^{N, i\neq j} k_{i,j}[\ddot{q}_i - \dot{q}_j]; \quad i \in [1, N].$$

With control (8) the mutual synchronisation errors $\zeta_i = [\zeta_{i,1}\; \zeta_{i,2}]^T$ obey the following differential equations ($\zeta_{i,1} = q_i - q_{ri}$, $\zeta_{i,2} = \dot{q}_i - \dot{q}_{ri}$; $i \in [1, N]$):

$$\begin{aligned} M(q_i)\,\dot{\zeta}_{i,2} = {} & -[K_1 + K_2\,\Lambda]\,\zeta_{i,1} - [K_2 + M(q_i)\,\Lambda]\,\zeta_{i,2} - \\ & -C(q_i, \dot{q}_i)\,[\zeta_{i,2} + \Lambda\,\zeta_{i,1}] + \widehat{d}_i, \\ \widehat{d}_i = {} & d_i + \sum_{j=1}^{N, i\neq j} k_{i,j}[d_i - M(q_i)\,M(q_j)^{-1} d_j], \quad i \in [1, N]. \end{aligned} \tag{9}$$

where the disturbances $\widehat{d}_i$ are of class M_{R^ν}.

To summarise and for the sake of making explicit the stability bounds, that we shall use in our main result, we present the following proposition for systems (5), (7) and (9).

Proposition 1 Consider the system

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \boldsymbol{M}(t)\,\dot{e}_2 &= -[K_1 + K_2\,\Lambda]\,e_1 - [K_2 + \boldsymbol{M}(t)\,\Lambda]\,e_2 \\ & \quad -\boldsymbol{C}(t)\,[e_2 + \Lambda\,e_1] + d(t) \end{aligned}$$

where $e_1 \in R^\nu$, $e_2 \in R^\nu$; $\boldsymbol{M}(\cdot)$ and $\boldsymbol{C}(\cdot)$ are absolutely continuous and satisfy: $0 < M_{\min} \leq |\boldsymbol{M}(t)| \leq M_{\max} < +\infty$; $\dot{\boldsymbol{M}}(\cdot) = C(\cdot) + C(\cdot)^T$, K_1, K_2, Λ are positive definite diagonal. Then, for any $e_{1\circ} \in R^\nu$, $e_{2\circ} \in R^\nu$, $t_\circ \geq 0$ and $d \in R_{R^\nu}$ we have

$$\begin{aligned} |\,[e_1(t), e_2(t)]\,| \leq {} & \sqrt{2\,\chi_{\max}\,\chi_{\min}^{-1}}\,|\,(e_{1\circ}, e_{2\circ})\,|\,e^{-0.5\,\chi\,t} + \\ & +2\sqrt{\chi^{-1}\chi_{\min}^{-1}\,\lambda_{\min}^{-1}(K_2)}\,||d||\,, \quad t \geq t_0, \end{aligned}$$

where $\chi = \min\{ 2\,\lambda_{\min}(\Lambda), \lambda_{\min}(K_2)M_{\max}^{-1} \}$,

$$\chi_{\min} = 0.5\,\min\{ \lambda_{\min}(K_1) + M_{\min}\lambda_{\min}(\Lambda^2), M_{\min}, \lambda_{\min}(K_1)\,[1 + \lambda_{\max}(\Lambda^{-2})] \},$$

$$\chi_{\max} = \max\{ \lambda_{\max}(\Lambda^2)M_{\max}, M_{\max}, 0.5\,\lambda_{\max}(K_1) \}$$

and $\lambda_{\min}(K_1)$, $\lambda_{\max}(K_1)$ denote the minimal and maximal eigen-values of matrix K_1. □

As a corollary of Proposition 1 it follows that, for each robot in closed loop with the controller (4) the dynamics (5) is IOS. That is, for the vector of tracking errors $e = [e_1^T ... e_N^T]^T$ we have for any $e(t_0) \in R^{2\,\nu\,N}$, $t_0 \geq 0$

$$|\,e(t)\,| \;\leq \sqrt{2\,N\,\chi_{\max}\,\chi_{\min}^{-1}}\,|\,e(t_0)\,|\,e^{-0.5\,\chi\,t} + 2\sqrt{N\,\chi^{-1}\chi_{\min}^{-1}\,\lambda_{\min}^{-1}(K_2)}\,||d||\;,\;\; t \geq t_0 \geq 0\,, \tag{10}$$

where χ, $\chi_{\min}$, $\chi_{\max}$ correspond to gains K_1, K_2, Λ used in (4). Similarly, under the controller (6) the synchronisation errors $\delta = (\delta_1^T ... \delta_{N-1}^T)^T$ satisfy, for any $\delta(t_0) \in R^{2\,\nu\,(N-1)}$,

$$|\,\delta(t)\,| \;\leq \sqrt{2\,(N-1)\,\chi'_{\max}\,\chi'^{-1}_{\min}}\,|\,\delta(t_0)\,|\,e^{-0.5\,\chi'\,t} + 2\sqrt{N\,\chi'^{-1}\chi'^{-1}_{\min}\,\lambda_{\min}^{-1}(K_2')}\,||\tilde{d}||\;,\;\; t \geq t_0 \geq 0\,, \tag{11}$$

where constants χ', $\chi'_{\min}$, $\chi'_{\max}$ corresponds to gains K_1', K_2', Λ' used in control (6) (note that $||\tilde{d}_{[t_0,\infty)}|| \leq (1 + M_{\max}M_{\min}^{-1})||d||_{[t_0,\infty)}$). Additionally this control provides the same estimate for "master-slave" synchronisation error for any $\varepsilon(t_0) \in R^{2\,\nu\,N}$:

$$|\,\varepsilon(t)\,| \;\leq \sqrt{2\,N\,\chi'_{\max}\,\chi'^{-1}_{\min}}\,|\,\varepsilon(t_0)\,|\,e^{-0.5\,\chi'\,t} + 2\sqrt{N\,\chi'^{-1}\chi'^{-1}_{\min}\,\lambda_{\min}^{-1}(K_2')}\,||\tilde{d}||_{[t_0,\infty)}\;,\;\; t \geq t_0 \geq 0\,.$$

Due to transformation (2) it also implies the corresponding estimate for tracking the error under controller (6):

$$|\,e(t)\,| \;\leq \rho_{\max}\,\rho_{\min}^{-1}\sqrt{2\,N\,\chi'_{\max}\,\chi'^{-1}_{\min}}\,|\,e(t_0)\,|\,e^{-0.5\,\chi'\,t} + 2\,\rho_{\max}\sqrt{N\,\chi'^{-1}\chi'^{-1}_{\min}\,\lambda_{\min}^{-1}(K_2')}\,||\tilde{d}_{[t_0,\infty)}||\;,\;\; t \geq t_0 \geq 0\,, \tag{12}$$

where $\rho_{\min}$ and $\rho_{\max}$ are maximal and minimal singular values of matrix Υ^{-1} in (2).

Thus, the estimates (10)–(12) describe the stability properties obtained under controllers (4) and (6) for pure tracking and pure synchronisation respectively.

3.0.1 The supervisor

The overall *switched* controller has the general form

$$U(t) = U_{i(t)}(t, q(t), \dot{q}(t)), \tag{13}$$

where $q = [q_1^T ... q_N^T]^T$, $\dot{q} = [\dot{q}_1^T \ldots \dot{q}_N^T]^T$, $i : R_+ \to \{1, 2\}$ is a piecewise-constant function, the controller $U_1 \in R^{\nu\,N}$ is defined by (4) and $U_2 \in R^{\nu\,N}$ is defined by (6). The switching function $t \mapsto i$ is defined by the following algorithm:

$$t_{j+1} = \begin{cases} \arg\inf\limits_{t \geq t_j}\;(q(t), \dot{q}(t)) \notin X_2 & \text{if } i(t_j) = 1\,; \\ \arg\inf\limits_{t \geq t_j + \tau_D}\;([q(t), \dot{q}(t)) \in X_1 & \text{if } i(t_j) = 2\,, \end{cases}$$

$$i(t_{j+1}) = \begin{cases} 1, & \text{if } (q(t_{j+1}), \dot{q}(t_{j+1})) \in X_1\,; \\ 2, & \text{if } (q(t_{j+1}), \dot{q}(t_{j+1})) \notin X_2\,, \end{cases} \tag{14}$$

$i(t) = i(t_j)$ for $t \in [t_j, t_{j+1})$;

$$t_0 \in R_+, \quad i(t_0) = \begin{cases} 1, & \text{if } (q(t_0), \dot{q}(t_0)) \in X_1\,; \\ 2, & \text{otherwise,} \end{cases}$$

$$X_1 = \{ (q, \dot{q}) : |\delta| \leq \vartheta \}, \quad X_2 = \{ (q, \dot{q}) : |\delta| < \Delta \},$$

where t_j, $j = 0, 1, 2, ...$ are switching instants, j indexes the last switch; $\tau_D > 0$ is a constant dwell-time and the value of the threshold $\vartheta > 0$ will be specified later. The controller (13) consists in the tracking controller (4) for trajectories in the set X_1 (that is when the synchronisation error δ is smaller than some ϑ) and in the synchronisation controller (6) for trajectories in the set $R^n \backslash X_2$ (*i.e.*, when $|\delta| \geq \Delta$ so a switch to synchronisation control occurs). The signal $i(t)$ takes constant values in the set $N = X_2 \backslash X_1$ thereby introducing *hysteresis*. Since the set N is not necessarily compact, to avoid chattering, a dwell-time is applied; however, this is used only during the synchronisation regime, whereas switching from the tracking regime is done without dwell-time that is, if the trajectories leave the set X_2 (the synchronisation errors become bigger than the given tolerance Δ), then synchronisation control is turned on immediately.

Our main result establishes IOS for the closed-loop system under supervisory control: denote $a_1 = \sqrt{2\,N\,\chi_{\max}\,\chi_{\min}^{-1}}$, $c_1 = 0.5\,\chi$, $c_2 = 0.5\,\chi'$, $a_2 = \rho_{\max}\,\rho_{\min}^{-1}\sqrt{2\,N\,\chi'_{\max}\,\chi'^{-1}_{\min}}$, $a_3 = \sqrt{2\,(N-1)\,\chi'_{\max}\,\chi'^{-1}_{\min}}$, $b_1 = 2\sqrt{N\,\chi^{-1}\chi_{\min}^{-1}\,\lambda_{\min}^{-1}(K_2)}$, $b_2 = 2\,\rho_{\max}\sqrt{N\,\chi'^{-1}\chi'^{-1}_{\min}\,\lambda_{\min}^{-1}(K_2')}(1 + M_{\max}M_{\min}^{-1})$, $b_3 = 2\sqrt{N\,\chi'^{-1}\chi'^{-1}_{\min}\,\lambda_{\min}^{-1}(K_2')}(1 + M_{\max}M_{\min}^{-1})$ then, for $i(t) = 1$, $t \geq t_0 \geq 0$, the inequality

$$|\,e(t)\,| \;\leq a_1\,|\,e(t_0)\,|\,e^{-c_1\,(t-t_0)} + b_1\,||d||_{[t_0,t)}, t \geq t_0 \tag{15}$$

holds and for the case $i(t) = 2$ we have

$$|\,e(t)\,| \;\leq\; a_2\,|\,e(t_0)\,|\,e^{-c_2\,(t-t_0)} + b_2\,||d||_{[t_0,t)}, \tag{16a}$$

$$|\,\delta(t)\,| \;\leq\; a_3\,|\,\delta(t_0)\,|\,e^{-c_2\,(t-t_0)} + b_3\,||d||_{[t_0,t)}, \tag{16b}$$

for all $t \geq t_0 \geq 0$.

Theorem 1 For any positive definite gains $K_1, K_2, \Lambda, K_1', K_2'$, Λ' in controllers (4) and (6) define the supervisor parameters as:

$$\vartheta = \lambda\,a_3\,\Delta\,(a_2\,a_1)^{-1}, \tag{17}$$

$$0 < \lambda < \min\{ 1, a_1\,a_2\,a_3^{-1}), \quad \sum_{k=0}^{\infty} \lambda^k \leq \Lambda < \infty,$$

$$\tau_D = -c_2^{-1}\,\ln[\vartheta\,(a_3\,\Delta)^{-1}]. \tag{18}$$

Then, for all $t \geq t_0 \geq 0$ and $(q(t_0), \dot{q}(t_0)) \in R^{2\,\nu N}$, the trajectories of the system (3), (13), (14) satisfy

$$|\,e(t)\,| \;\leq a_1\,a_3\,\lambda^{-2}\,|\,e(t_0)\,|\,e^{-\,\ln(\lambda)\,(t-t_0)/T} + [2\,\Lambda + 1]\,(b_1 + b_2)\,||d||_{[t_0,t)}\,, \tag{19}$$

$$T = 3\,\max\{ \tau_D, \tau_1 \}, \quad \tau_1 = c_1^{-1}[\tau_D\,c_2 - \,\ln(a_2)];$$

$$|\,\delta(t)\,| \;\leq a_3\,\max\{ \Delta, \delta(t_0)\,| \} + b_3\,||d||_{[t_0,\infty)}, \tag{20}$$

where $d = [d_1^T \,...\, d_N^T]^T$. □

The switched synchronisation approach given in Theorem 1 allows to design, *independently*, two control algorithms for tracking and synchronisation. The computation of the supervisor parameters under appropriate thresholds (dwell-time, tolerance, etc) ensures the desired tracking control objective under the constraint of an admissible synchronisation error. Correspondingly, when a given tolerance on synchronisation error is maintained the control system operates in tracking control, hence in *decentralised* mode; this is a particularly significant advantage in cases when the systems cooperate through networks since the influence of disturbances and communication delays on the robots' performance is diminished.

We wrap up the section with a numerical example and simulation results that illustrate the superiority of supervisory control over pure tracking or pure synchronisation control as is customary in the literature.

3.1 Example

Let $\nu = 2$, $N = 2$, and consider the 2-link planar robot model from (Berghuis 1993). For this robot we have the following numerical values: $M_{\min} = 1$, $M_{\max} = 25$, $k = 6$, $K_1 = 5$, $K_2 = 10$, $\Lambda = 2$, $K_1' = 9$, $K_2' = 18$, $\Lambda' = 2$, $k_{12} = k_{21} = 0.5$, all initial conditions for the first robot equal 0.5, for the second robot -0.5. For switching synchronisation we choose $\Delta = 0.1$. Controller (4), (6), (8) for tracking, master-slave and mutual synchronisations are applied with values K_1, K_2, Λ. For switched synchronisation for the tracking case ($i = 1$) control (4) is applied with gains K_1, K_2, Λ, while for the case of emergency synchronisation ($i = 2$) control (6) with gains K_1', K_2', Λ' is used. In this case we have the following values of parameters, calculated for chosen gains and threshold Δ: $\chi = 0.4$, $\chi' = 0.72$, $\chi_{\min} = {\chi'}_{\min} = 0.5$, $\chi_{\max} = {\chi'}_{\max} = 100$, $\rho_{\max} = 2.618$, $\rho_{\min} = 0.382$, $a_1 = 2$, $a_2 = 193.863$, $a_3 = 20$, $c_1 = 0.2$, $c_2 = 0.36$, $b_1 = 2$, $b_2 = 75.632$, $b_3 = 28.889$. Then, for $\lambda = 0.5$ with $\Lambda = 2$ we have $\vartheta = 0.0025$, $\tau_D = 18.482$, $\tau_1 = 6.931$, $T = 55.445$.

To compare the performances under controllers (4), (6), (8) and (13) we define

$$J_d(t_e) = \int_0^{t_e} [q_1(t) - q_d(t)]^T [q_1(t) - q_d(t)]\ dt + \\ + \int_0^{T} [q_2(t) - q_d(t)]^T [q_2(t) - q_d(t)]\ dt\,,$$

$$J_e(t_e) = \int_0^{t_e} [q_1(t) - q_2(t)]^T [q_1(t) - q_2(t)]\ dt,$$

$$J_u(t_e) = \int_0^{t_e} u(t)^T u(t)\ dt,$$

which correspond to the integral square synchronisation error and to the amount of control energy. The limit $t_e \geq 0$ determines the length of simulations time interval: $[0, t_e]$ *i.e.*, $t_0 = 0$. Two series of simulations were performed: firstly, using

$$d_1 = \begin{bmatrix} 2.5\ \sin(\pi^{-1} t) \\ -2\ \sin(2\,\pi^{-1}\, t) \end{bmatrix}; \quad d_2 = \begin{bmatrix} 1.5\ \sin(0.5\,\pi^{-1} t) \\ -3\ \sin(1.5\,\pi^{-1}\, t) \end{bmatrix}$$

and secondly, without disturbances. The superiority of the proposed supervisor control is clear from the data organised in following tables:

Table 1. Values of performance indexes without disturbances

	Performance indexes		
	$J_d(t_e)$	$J_e(t_e)$	$J_u(t_e)$
Independent tracking controller (4)	1.080	1.727	2572.9
Master-slave synch. controller (6)	1.677	2.071	1603.8
Mutual synchronisation controller (8)	1.080	1.728	2573.1
Switched synchronisation controller (13)	0.701	1.001	2599.1

Table 2. Values of performance indexes with disturbances

	Performance indexes		
	$J_d(t_e)$	$J_e(t_e)$	$J_u(t_e)$
Independent tracking controller (4)	1.704	2.351	3818.8
Master-slave synch. controller (6)	2.661	3.030	2869.6
Mutual synchronisation controller (8)	1.704	2.351	3819.1
Switched synchronisation controller (13)	1.477	1.680	3147.4

4 Conclusion

The problem of nonlinear dynamical systems synchronisation with simultaneous tracking of a reference signal under acting disturbances is considered. The analytical comparison of tracking versus master-slave synchronisation and mutual synchronisation shows that, under the absence of disturbances, the problems are equivalent up to a transformation. However, the main drawback of conventional master-slave or mutual synchronisation is that the effects of external disturbances propagate through all the systems thereby degrading performance and demanding increasingly control efforts. Such problem can be effectively solved via switching control.

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A Proof of main result

From (14) it follows that the system has average dwell-time $\tau_D/2$. In this case on any time interval $[t_s, t_e)$ with $t_e \geq t_s \geq 0$ the number of switches $N_{[t_s,t_e)}$ is bounded as $N_{[t_s,t_e)} \leq 2[1 + (t_e - t_s)/\tau_D]$. Since $\vartheta < \Delta$ then for any $d \in M_{R^{\nu N}}$ there exists a time delay between switches and the switching signal $i(t)$ is right-continuous hence, so is the right-hand side of the closed-loop system (3), (13). Continuity of the system solutions follows and there is no chattering. Since for $i(t) = 1$ or $i(t) = 2$ by Proposition 1 the solutions of the system are bounded, the switched system (3), (13), (14) is forward complete.

Let us consider different possible scenarios: firstly, assume that system undergoes infinitely many switches (equivalently, either control regime lasts a finite time). Let the system solutions start under tracking control regime *i.e.*, $(q(t_0), \dot{q}(t_0)) \in X_1$; in this case, there exists a time instant $t_1 > t_0$ such that $|\delta(t_1)| = \Delta$ and for all $t \in [t_0, t_1)$

$$|\delta(t)| \leq \Delta; \quad |e(t)| \leq a_1 |e(t_0)| + b_1 ||d||_{[t_0,t)}.$$

At t_1 the synchronisation controller (6) becomes active and lasts for a finite number of time units, but larger than the dwell time τ_D *i.e.*, there exists a finite time instant $t_2 \geq t_1 + \tau_D$ such that $|\delta(t_2)| \leq \vartheta$ and, for all $t \in [t_1, t_2)$,

$$|\delta(t)| \leq a_3 \Delta + b_3 ||d||;$$

$$|e(t)| \leq a_2 |e(t_1)| e^{-c_2 (t-t_1)} + b_2 ||d||_{[t_1,t)}.$$

hence

$$|e(t)| \leq a_2 (a_1 |e(t_0)| + b_1 ||d||_{[t_0,t_1)}) + b_2 ||d||_{[t_1,t)}$$

and, from (17) and (18),

$$\begin{aligned} |e(t_2)| &\leq a_2 (a_1 |e(t_0)| + b_1 ||d||_{[t_0,t_1)}) e^{-c_2 \tau_D} + \\ &+ b_2 ||d||_{[t_1,t)} \leq \lambda |e(t_0)| + \lambda b_1 ||d||_{[t_0,t_1)} + b_2 ||d||_{[t_1,t_2)}. \end{aligned}$$

By the same arguments, let $t_3 > t_2$ and $t_4 \geq t_3 + \tau_D$ be such that $|\delta(t_3)| = \Delta$, $|\delta(t_4)| \leq \vartheta$ and

$$|\delta(t)| \leq \Delta, \quad |e(t)| \leq a_1 |e(t_2)| + b_1 ||d||_{[t_2,t)}, \quad t \in [t_2, t_3);$$

$$|\delta(t)| \leq a_3 \Delta + b_3 ||d||,$$

$$|e(t)| \leq a_2 |e(t_3)| e^{-c_2 (t-t_3)} + b_2 ||d||_{[t_3,t)}, \quad t \in [t_3, t_4)$$

hence,

$$\begin{aligned} |e(t_4)| \leq \lambda^2 |e(t_0)| + \lambda^2 b_1 ||d||_{[t_0,t_1)} + \lambda b_2 ||d||_{[t_1,t_2)} + \\ + \lambda b_1 ||d||_{[t_2,t_3)} + b_2 ||d||_{[t_3,t_4)}, \ t \in [t_3, t_4). \end{aligned}$$

Proceeding by induction we conclude that, for any $j \geq 1$,

$$\begin{aligned} |e(t_{2j})| &\leq \lambda^j |e(t_0)| + \lambda^j b_1 \|d\|_{[t_0,t_1)} \\ &\quad + \sum_{k=1}^{j-1} \lambda^{j-k} (b_2 \|d\|_{[t_{2k-1},t_{2k})} \\ &\quad + b_1 \|d\|_{[t_{2k},t_{2k+1})}) + b_2 \|d\|_{[t_{2j-1},t_{2j})} \\ &\leq \lambda^j \|e(t_0)\| + 2\Lambda(b_1 + b_2) \|d\|_{[t_0,t_{2j})}, \end{aligned}$$

$$|e(t)| \leq a_1 a_3 \lambda^{j-1} |e(t_0)| + [2\Lambda + 1](b_1 + b_2) ||d||_{[t_0,t_{2j})}, \tag{21}$$

for all $t \in [t_{2j-2}, t_{2j})$. That is, over $[t_{2j-2}, t_{2j})$ for any $j \geq 1$, $|e(t)|$ decreases by a factor of λ. To obtain the desired IOS bound we compute a *constant* length interval over which the error trajectories take a decrease of a factor of λ: if $i = 1$, that is, when the system is under tracking regime, according to (15) the error trajectories decrease by a factor of λ over an interval of length $\tau_1 = c_1^{-1}[\tau_D c_2 - \ln(a_2)]$; on the other hand, if $i = 2$ the errors satisfy (16) and (21) over τ_D units of time. It follows that over any interval of length $T = 3 \max\{\tau_D, \tau_1\}$, $e(t)$ decreases at least by a factor of λ *i.e.*,

$$\begin{aligned} |x(t_0 + (j-1)T + \tau)| \leq \lambda^{j-1} |x(t_0)| + [2\Lambda + 1](b_1 + b_2) \\ \times \|d\|_{[t_0,t_0+(j-1)T+\tau)} \end{aligned}$$

for all $j \geq 1$ and $\tau \in [0, T)$. A direct calculation (setting $t = t_0 + (j-1)T + \tau$ yields the estimate (19).

Let us consider now that the system undergoes infinitely many switches, starting off from initial conditions $t_0 \in R_+$, $[q(t_0)^T, \dot{q}(t_0)^T]^T \notin X_1$ (*i.e.*, in synchronisation regime) then, there exists a time instant $t_1 \geq t_0 + \tau_D$ such, that $|\delta(t_1)| \leq \vartheta$ and

$$|\delta(t)| \leq a_3 |\delta(t_0)| + b_3 ||d||;$$

$$|e(t)| \leq a_2 |e(t_0)| e^{-c_2 (t-t_0)} + b_2 ||d||_{[t_0,t)}.$$

From (17) we have

$$|e(t_1)| \leq \lambda |e(t_0)| + b_2 ||d||_{[t_0,t)}$$

and starting from t_1 the proof for this case follows similar guidelines as for the previous case with "new" initial conditions t_1 and $[q(t_1)^T, \dot{q}(t_1)^T]^T \in X_1$). The estimate (19) again holds (the additional dependence of initial conditions on $b_2 ||d||_{[t_0,t)}$ is already taken into account in summation Λ).

Finally, consider the case when the system undergoes a finite number of switches *i.e.*, from a finite time instant $t = t_0 + t^*$, the system remains either under tracking or synchronisation regimes. From the arguments above, the system trajectories satisfy (19) for all $t \in [t_0, t_0 + t^*)$; for all $t \geq t_0 + t^*$ either (15) or (16) hold therefore, the estimate (19) holds for all $t \geq t_0$. Furthermore, the worst-case estimate (20) holds for δ.