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Second order sliding mode and adaptive observers for a chaotic system: a comparative study

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Abstract: In this paper two nonlinear observers for a chaotic system are compared. Moreover, the left invertible problem and the observability singularity are discussed. Thus, after a presentation of both observers, a comparison of the two proposed methods and a discussion are done on the basis of simulations results. The last part highlights the fact that the finite time observer is more sensible to the singularity observations, but less sensible to parameter uncertainties and noise in the output of the system.

1. INTRODUCTION

The synchronization of chaotic systems was studied since 20 years ago PC90 [90], HV92 [92], and many articles have been published from a theoretical point of view NM97 [97] as well as from an applicative one PCKHS92 [92]. The aim of this manuscript is to compare two observer approaches of chaotic synchronization, and this in order to highlight some difficulties introduced by a modified G.Y. Qi system Qi [05]. The first difficulty is the observability singularity (or bifurcation) BBBB06 [06], the second one is the left invertibility problem when the observability matching condition is not verified Barbot et al. [05], and the third one is the difficulty to design an observer when the system is not linearizable by output injection KI83 [83], KR85 [85] (see also SRCH01 [01] for the case of chaotic synchronization). The last difficulty is due to the original Qi system has large state amplitude with dynamics varying from slow to fast behavior. The proposed comparison is non exhaustive and many other observers dedicated to chaotic systems may be used.

2. PROBLEM STATEMENT

Consider the following modified Qi chaotic system

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - x_1(t)) + x_2(t)\, x_3(t) + m_1(t) \\ \dot{x}_2(t) = b\,(x_1(t) + x_2(t)) - x_1(t)\, x_3(t) \\ \dot{x}_3(t) = -cx_3(t) - ex_4(t) + x_1(t)\, x_2(t) + m_1(t) \\ \dot{x}_4(t) = fx_3(t) - dx_4(t) + x_1(t)\, x_3(t) + m_2(t) \\ y_1(t) = x_1(t) \\ y_2(t) = x_2(t) \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ are the states of the system, m_1 and m_2 represent the messages, which for the observation problem are considered as the unknown inputs, and y_1 and y_2 represent the outputs of the system.

Assumption. 2.1. Firstly, the signals m_1 and m_2 are assumed to be sufficiently small and with slow variations in order to preserve the chaotic behavior and, secondly, the signals must be undetectable by a simple frequency analysis. Consequently, the state vector is in an open bounded set $\mathcal{D} \subset \mathbb{R}^4$.

The goal of both observers to be proposed is to reconstruct m_1 and m_2. This is a left invertible problem R90 [90] for a system with relative degree strictly smaller than the state dimension. This problem was studied recently by two complementary approaches Barbot et al. [05] and Fliess [07]. In the first one, a constructive algorithm was given in order to determine if the system is left invertible and a state and output transformation was given in order to express the system in a well adapted observer form. In the second one, an algebraic approach, introduced in F89 [89], was used and all the states and inputs were expressed in function of the outputs and their derivatives. The purpose of this paper is not to determine if the system (1) is left invertible or not because it was designed in order to be left invertible thanks to the algorithm of Barbot et al. [05]. The objective is to proposed two observers and compare them with respect to the fact that the system (1) has observation singularities in the point $(x_1 = 0)$, which, in this case, implies left invertibility singularities in the same point. The left invertibility problem of (1) was studied in BFF07 [07] in an algebraic frame Fliess [07] and a numerical differentiation was used in Mboup [07]. Moreover, the observation error dynamic is not linearizable by output injection. Two observers were chosen: an observer based on the Super-Twisting Algorithm (STA) and an adaptive observer. The first one basically allows to obtain information from the outputs and their derivatives and,

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consequently, to reconstruct the states and the messages (unknown inputs), this is a very closed way to that of the algebraic approach and the algorithm proposed in Barbot et al. [05]. The second observer allows to overcome the observability singularity difficulty if a condition of persistent excitation is verified Hammouri [90], Ghanes [06].

3. SUPER-TWISTING OBSERVER

The following assumptions are requested by the STA.

Assumption. 3.1. a) The states of the system are bounded for any bounded unknown input; b) the unknown inputs m_1 and m_2 are bounded and with bounded derivatives and their bounds are known.

Considering (1) and Assumption 2.1, it follows that Assumption 3.1 is verified.

3.1 Reconstruction of x_3

To generate a new output, namely the variable x_3, we reconstruct x_1x_3 in the following way

$$\begin{cases} \dot{x}_{a,1} = b(x_1+x_2) + \hat{x}_{1,3} + \lambda_1 |s_1|^{1/2} \operatorname{sign} s_1 \\ \dot{\hat{x}}_{1,3} = \alpha_1 \operatorname{sign} s_1 \\ s_1 = x_2 - x_{a,1} \end{cases} \tag{2}$$

In this way, the derivative of s_1 takes the form

$$\dot{s}_1 = -x_1x_3 - \hat{x}_{1,3} - \lambda_1 |s_1|^{1/2} \operatorname{sign} s_1 \tag{3}$$

Choosing the gains $\lambda_1 \geq \frac{(\alpha_1+M_1)(1+\theta)}{1-\theta}\sqrt{\frac{2}{\alpha_1-M_1}}$ and $\alpha_1 > M_1 \geq \left|\frac{d}{dt}(x_1x_3)\right|$ $(0<\theta<1)$, we get, according to Levant [93, 98], Davila [05], a second order sliding motion, that is, $s_1(t) = 0$, $\dot{s}_1(t) = 0$ after some finite time T_1. Therefore, from (3), we get

$$\hat{x}_{1,3}(t) \equiv -x_1(t)\,x_3(t) \ \text{ for all } t \geq T_1. \tag{4}$$

Thus, the state x_3 can be obtained from (4) provided $x_1(t) \neq 0$. Thus, the observer for x_3 is designed in the form

$$\hat{x}_3(t) = \begin{cases} -\frac{\hat{x}_{1,3}(t)}{x_1(t)} & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{x}_3(t-\tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \tag{5}$$

where τ and ε are enough small constants [1]. Thus, we get the identity

$$\hat{x}_3(t) \equiv x_3(t) \ \text{ for all } t \geq T_1 \text{ and } |x_1(t)| \geq \varepsilon.$$

Remark. 3.1. At a glance, it seems that, in (5), it is enough to use $x_1(t) \neq 0$ instead of $|x_1(t)| \geq \varepsilon$. However, it is known that, in the practical situation, s_1 and $\dot{s}_1$ are very close to zero, but they are not exactly zero; hence, instead of having (4), we have the equality $\hat{x}_{1,3}(t) = -x_1(t)\,x_3(t) + \Delta(t)$, where Δ represents the estimation error, which does not tend to zero. Then, the equality $\frac{\hat{x}_{1,3}}{x_1} = x_3 - \frac{\Delta}{x_1}$ induced an error between $\frac{\hat{x}_{1,3}}{x_1}$ and x_3 of the form $O(1/x_1)$. This justifies that in a vicinity of the point $(x_1 = 0)$ the value of $\hat{x}_3$ is chosen as in (5).

3.2 Reconstruction of x_4

Now, the definition of $\bar{y}_3(t) := x_3(t)$ allows to rewrite (1) as a linear system with output injection and unknown inputs, that is,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix} -a & a & 0 & 0\\ b & b & 0 & 0\\ 0 & 0 & -c & -e\\ 0 & 0 & f & -d\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}y_2\bar{y}_3\\ -y_1\bar{y}_3\\ y_1y_2\\ y_1\bar{y}_3\end{bmatrix}}_{\phi(y_1,y_2,\bar{y}_3)} + \underbrace{\begin{bmatrix}1&0\\0&0\\1&0\\0&1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}m_1\\ m_2\end{bmatrix}}_{w}$$

$$\underbrace{\begin{bmatrix}y_1\\ y_2\\ \bar{y}_3\end{bmatrix}}_{\bar{y}} = \underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}}_{C} x \tag{6}$$

Now, let z be defined by the solution of the following differential equation

$$\dot{z} = Az + \phi(y_1, y_2, \bar{y}_3)$$

Thus, defining $e_z = x - z$ we obtain the dynamic equation for e_z

$$\begin{aligned} \dot{e}_z(t) &= Ae_z(t) + Dw(t) \\ y_z &= Ce_z \end{aligned} \tag{7}$$

Then, for the reconstruction of x_4 we follow in essence, but with a little modification, the algorithm proposed in Bejarano [07]. That is, we will find an algebraic expression for e_z in terms of the output y_z and its derivatives. Thus, we get the equalities [2]

$$y_z = Ce_z$$

$$\frac{d}{dt}(CD)^{\perp} y_z = (CD)^{\perp} CAe_z$$

which yields the following algebraic expression for e_z [3]

$$e_z(t) = \begin{bmatrix} C \\ (CD)^{\perp} CA\end{bmatrix}^{+}\begin{bmatrix} y_z(t) \\ \frac{d}{dt}(CD)^{\perp} y_z(t)\end{bmatrix} \tag{8}$$

After denoting $e_{z,4}$ as the four state of e_z, we obtain

$$e_{z,4} = \frac{1}{e}\left[\dot{e}_{z,1} - \dot{e}_{z,3} + a(e_{z,1} - e_{z,2}) - ce_{z,3}\right] \tag{9}$$

By expanding the terms in (9), we get,

$$\begin{aligned} x_4 - z_4 = \frac{1}{e}&\left[a(y_1-y_2) + y_2\bar{y}_3 + y_1y_2 - c\bar{y}_3\right] \\ &+\frac{1}{e}\left(\dot{y}_1 - \dot{\bar{y}}_3\right) - z_4 \end{aligned} \tag{10}$$

The equation (10) gives an algebraic expression for $x_4 - z_4$ in terms of the output $\bar{y}$ and its derivatives.

As $\dot{y}_1$ and $\dot{\bar{y}}_3$ are not measurable directly, to obtain these derivatives the STA will be used again. Before

[1] The constant τ is chosen enough small but bigger than the sampling time used during the realization of the observer. The constant ε should be chosen sufficiently big to avoid the singularity, but also should be notice that any estimation in this zone can not be considered as an acceptable estimation.

[2] $Y^{\perp}$ is a full row rank matrix such that $Y^{\perp}Y = 0$. The matrix $Y^{\perp}$ is not unique, but this not affect the final result and any of those matrices can be used.

[3] The matrix X^{+} is defined to be the pseudo-inverse of X. The extended matrix considered in (8) belongs to the sort of matrices of full column rank, and , in such a case, $X^{+} = \left(X^TX\right)^{-1}X^T$.

reconstructing x_4, to reduce the fast dynamics and have smaller gains in the ST algorithm, we design the following like-linear observer whose dynamics is governed by the following differential equation

$$\begin{aligned} \dot{\tilde{x}} &= \underbrace{\begin{bmatrix} -c & -e \\ f & -d \end{bmatrix}}_{\bar{A}} \tilde{x} + \begin{bmatrix} y_1 y_2 \\ y_1 \bar{y}_3 \end{bmatrix} + L\left(\hat{x}_3 - \tilde{y}_3\right) \\ \tilde{y}_3 &= \underbrace{[\,1\ 0\,]}_{\bar{C}} \tilde{x} \end{aligned} \qquad (11)$$

The matrix L is chosen so that the eigenvalues of the matrix $\left(\bar{A} - L\bar{C}\right)$ have negative real part. The dynamic equations for $\bar{e} := [x_3\ x_4]^T - \tilde{x}$ are

$$\dot{\bar{e}} = \left(\bar{A} - LC\right)\bar{e}(t) + w(t)$$

where w is defined in (6). Hence some upper bounds for the norm of $\bar{e}$ and for the norm of $\dot{\bar{e}}$ are obtained by

$$\|\bar{e}(t)\| \leq \gamma \exp(-\lambda t)\|\bar{e}(0)\| + \mu\|w(t)\|$$

Where γ, λ, μ are positive constants. Hence, $\bar{e}$ is constrained to remain in a zone depending on the amplitude of m_1 and m_2.

The expression in the right side of (10) must appear on the derivative of the sliding surface; that is why, we design the finite time observer for x_4 in the following form

$$\begin{cases} \dot{x}_{a,2} = \dfrac{1}{e}\left[a(x_2 - x_1) + c\hat{x}_3 - x_1 x_2 + x_2 \hat{x}_3\right] \\ \qquad + \tilde{x}_4 + v_2 + \lambda_2 |s_2|^{1/2} \operatorname{sign} s_2 \\ \dot{v}_2 = \alpha_2 \operatorname{sign} s_2 \\ s_2 = \dfrac{1}{e}(x_1 - x_3) - x_{a,2} \\ \hat{x}_4(t) = \begin{cases} \tilde{x}_4 + v_2 & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{x}_4(t-\tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \end{cases}$$

where $\tilde{x}_4$ is the second component of the vector $\tilde{x}$ defined in (11). Then, the time derivative of s_2 is

$$\begin{aligned} \dot{s}_2 = &\frac{1}{e}(\dot{x}_1 - \dot{x}_3) - \frac{1}{e}\left[a(x_2 - x_1) + c\hat{x}_3 - x_1 x_2 + x_2 \hat{x}_3\right] \\ &- \tilde{x}_4 - v_2 - \lambda_2 |s_2|^{1/2} \operatorname{sign} s_2 \end{aligned}$$

Thus, from the (10), $\dot{s}_2$ takes the form

$$\dot{s}_2 = x_4 - (\tilde{x}_4 + v_2) - \lambda_2 |s_2|^{1/2} \operatorname{sign} s_2$$

Thus, for $\lambda_2 \geq \dfrac{(\alpha_2 + M_2)(1+\theta)}{1-\theta}\sqrt{\dfrac{2}{\alpha_2 - M_2}}$ and $\alpha_2 > M_2 \geq \left|\dfrac{d}{dt}(x_4 - \tilde{x}_4)\right|$ $(0 < \theta < 1)$, after some finite time $T_2 > T_1$, we get $s_2 = 0$ and $\dot{s}_2 = 0$; therefore,

$$\hat{x}_4 \equiv x_4 \text{ for } |x_1(t)| \geq \varepsilon.$$

3.3 Messages reconstruction

The reconstruction of m_1 is made in the following way

$$\begin{cases} \dot{x}_{a,3} = a(x_2 - x_1) + x_2 \hat{x}_3 + v_3 + \lambda_3 |s_3|^{1/2} \operatorname{sign} s_3 \\ \dot{v}_3 = \alpha_3 \operatorname{sign} s_3 \\ \hat{m}_1 = \begin{cases} v_3 & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{m}_1(t-\tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \\ s_3 = x_1 - x_a^3 \end{cases}$$

Thus, for $\lambda_3 \geq \dfrac{(\alpha_3 + M_3)(1+\theta)}{1-\theta}\sqrt{\dfrac{2}{\alpha_3 - M_3}}$ and $\alpha_3 > M_3 \geq |\dot{m}_1|$ $(0 < \theta < 1)$, there exists a finite time T_3 so that the equalities $s_3 = 0$, $\dot{s}_3 = 0$ hold. Thus, we get the equality

$$\hat{m}_1 \equiv m_1, \text{ for } |x_1(t)| \geq \varepsilon$$

The reconstruction of m_2 is made in a similar way, that is,

$$\begin{cases} \dot{x}_{a,4} = b(x_1 + x_2) + f x_3 - d\hat{x}_4 + v_4 + \lambda_4 |s_4| \operatorname{sign} s_4 \\ \dot{v}_4 = \alpha_4 \operatorname{sign} s_4 \\ \hat{m}_2 = \begin{cases} v_4 & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{m}_2(t-\tau) & \text{if } |x_1(t)| < \varepsilon \end{cases} \\ s_4 = x_2 + \hat{x}_4 - x_{a,4} \end{cases}$$

Thus, taking into account (1) and the derivative of $x_{a,4}$, the gains α_4 and λ_4 are chosen to satisfy the inequalities $\lambda_4 \geq \dfrac{(\alpha_4 + M_4)(1+\theta)}{1-\theta}\sqrt{\dfrac{2}{\alpha_4 + M_4}}$ and $\alpha_4 > M_4 \geq |\dot{m}_2|$ $(0 < \theta < 1)$ (see, Levant [93, 98], Levant [98]). After some finite time $T_4 > T_2 > T_1$ the equalities $s = 0$ and $\dot{s} = 0$ are verified. Therefore, from the equation of $\dot{s}$, we obtain

$$\hat{m}_2 \equiv m_2 \text{ for } |x_1(t)| \geq \varepsilon$$

4. INTERCONNECTED ADAPTIVE OBSERVER

Before introducing the observer, the following assumptions are requested.

Assumption. 4.1. 1. The outputs of chaotic system (1) y_1 and y_2 are regularly persistent (see more details in Hammouri [90], Ghanes [06]).
2. The system (1) is Lipschitz in the considered domain $\mathcal{D}$ defined in section 2 with respect to the states x_i, $i = 1, 4$.

The assumption 4.1.1 is satisfied due to the chaotic behavior of (1). y_1 and y_2 are equal to zero during a very short time.
The assumption 4.1.2 is verified by the fact that the state of the chaotic system (1) stays in the domain D.

Assumption. 4.2. The messages m_1 and m_2 are considered constants and unknown parameters.

This is the only extra assumption requested by this method.

Now, we present an observer design for the chaotic system (1) that is based on the interconnection between two observers, and satisfying a particular property called input persistency introduced in assumption 4.1. This persistency condition is sufficient to guarantee the observer design. Furthermore, the idea of interconnected observers is to design a set of observers for the whole system, from the separate synthesis of set observers for each subsystem. The key is assuming that, for each of these separate set of observers, the states of the other subsystem are available.

The chaotic system (1) can be rewritten in the following interconnected extended ($x_5 = m_1$, $x_6 = m_2$) compact form

$$\begin{cases} \dot{X}_1 = A_1(y_2)X_1 + g_1(y, X_1, X_2) \\ y_1 = C_1 X_1 \end{cases} \qquad (12)$$

$$\begin{cases} \dot{X}_2 = A_2(y_1)X_2 + g_2(y, X_2, X_1) \\ y_2 = C_2 X_2 \end{cases} \qquad (13)$$

where $X_1 = (x_2, x_3, x_5)^T$ is the state of the first subsystem, $X_2 = (x_1, x_3, x_4, x_6)^T$ is the state of the second subsystem. $y = [x_1, x_2]^T$ are the output of the whole system, and

$$A_1(y_2) = \begin{pmatrix} 0 & -y_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ A_2(y_1) = \begin{pmatrix} 0 & y_1 & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$g_1(y, X_1, X_2) = \begin{pmatrix} b(y_1 + y_2) \\ -cx_3 - ex_4 + y_1 y_2 \\ 0 \end{pmatrix}$$

$$g_2(y, X_1, X_2) = \begin{pmatrix} a(y_1 - y_2) + x_5 \\ -ex_4 + y_1 y_2 + x_5 \\ fx_3 - dx_4 + y_1 x_3 \\ 0 \end{pmatrix}$$

$$C_1 = (\ 1\ 0\ 0\ ), C_2 = (\ 1\ 0\ 0\ 0\ ).$$

Remark. 4.1. 1. The same state x_3 is used in the subsystems (12) and (13). From this, the subsystems (12) and (13) are on the affine system form. It is for this class of nonlinear systems that we interest for which an observer can be designed under some hypothesis such that the property of inputs persistency Hammouri [90].
2. The choice of the variables of each subsystem has been considered in order to be able to consider m_1 (respectively m_2) as parameter to adapt for subsystem (12) (respectively for subsystem (13)).

Next, let us introduce an adaptive observer in order to estimate the system's state and the unknown parameters simultaneously.
At first, let us introduce the following assumptions in order to establish the results concerning the adaptive observer design (see more details in Ghanes [06]).

Assumption. 4.3. -$g_1(y, X_1, X_2)$ is Lipschitz in the considered domain $\mathcal{D}$ with respect to X_1 and X_2.
-$g_2(y, X_1, X_2)$ is Lipschitz in the considered domain $\mathcal{D}$ with respect to X_1 and X_2.
This assumption is satisfied from the assumption 4.1.1.

Then, an adaptive observer for interconnected subsystems (12) and (13) estimating the state and unknown parameters is given by

$$\begin{cases} \dot{Z}_1 = A_1(y_2)Z_1 + g_1(y, Z_1, Z_2) + S_1^{-1}C_1^T(y_1 - \hat{y}_1) \\ \dot{S}_1 = -\theta_1 S_1 - A_1^T(y_2)S_1 - S_1 A_1(y_2) + C_1^T C_1 \\ \hat{y}_1 = C_1 Z_1 \end{cases} \tag{14}$$

$$\begin{cases} \dot{Z}_2 = A_2(y_1)Z_1 + g_2(y, Z_2, Z_1) + S_2^{-1}C_2^T(y_2 - \hat{y}_2) \\ \dot{S}_2 = -\theta_2 S_2 - A_2^T(y_1)S_2 - S_2 A_2(y_1) + C_2^T C_2 \\ \hat{y}_2 = C_2 Z_2 \end{cases} \tag{15}$$

where $Z_1 = (\hat{x}_2, \hat{x}_3, \hat{x}_5)^T$, $Z_2 = (\hat{x}_1, \hat{x}_3, \hat{x}_4, \hat{x}_6)^T$; $S_i = S_i^T > 0$, $i = 1, 2$. Note that $S_1^{-1}C_1^T$ and $S_2^{-1}C_2^T$ are the gains of the observers (14) and (15), respectively. θ_1 and θ_2 are positive constant parameters of the observer convergence tuning.

Remark. 4.2. It is worth noticing that $\|S_1\|$ and $\|S_2\|$ are such that $0 < \alpha_{i,2} \leq \|S_i\| \leq \alpha_{i,1}, i = 1, 2$ for θ_1 and θ_2 large enough due to the persistency of input considered in assumption 4.1.1. $\alpha_{i,1}$ and $\alpha_{i,2}$ are positive constants, $i = 1, 2$.

Now, in order to guarantee the convergence of the proposed observer, sufficient conditions are established in the following result. Denote the estimation errors:

$$\epsilon_1 = X_1 - Z_1 \ \text{ and } \ \epsilon_2 = X_2 - Z_2$$

whose dynamics are given by

$$\begin{cases} \dot{\epsilon}_1 = [A_1(y_2) - S_1^{-1}C_1^T C_1]\epsilon_1 \\ \quad + g_1(y, X_1, X_2) - g_1(y, Z_1, Z_2) \end{cases}$$

$$\begin{cases} \dot{\epsilon}_2 = [A_2(y_1) - S_2^{-1}C_2^T C_2]\epsilon_2 \\ \quad + g_2(y, X_2, X_1) - g_2(y, Z_2, Z_1). \end{cases}$$

Proposition. 4.1. Under the assumptions 4.1 and 4.3, the system (14)-(15) is an exponential observer for system (12)-(13).

Proof. Consider the following Lyapunov function candidate: $V_o := V_1 + V_2 = \epsilon_1^T S_1 \epsilon_1 + \epsilon_2^T S_2 \epsilon_2$. From assumption 4.3 and remark 4.2, the following inequalities hold

$$\|S_1\| \leq \alpha_{1,1}$$
$$\|\{g_1(y, X_1, X_2) - g_1(y, Z_1, Z_2)\}\| \leq k_1 \|\epsilon_1\| + k_2 \|\epsilon_2\|$$
$$\|S_2\| \leq \alpha_{2,1}$$
$$\|\{g_2(y, X_2, X_1) - g_2(y, Z_2, Z_1)\}\| \leq k_3 \|\epsilon_1\| + k_4 \|\epsilon_2\|$$

Computing the time derivative of V_o, by using the above inequalities it follows that

$$\begin{aligned} \dot{V}_o \leq - \quad & \theta_1 \epsilon_1^T S_1 \epsilon_1 + 2\mu_1 \|\epsilon_1\| \|\epsilon_1\| + 2\mu_2 \|\epsilon_1\| \|\epsilon_2\| \\ & - \theta_2 \epsilon_2^T S_2 \epsilon_2 + 2\mu_3 \|\epsilon_2\| \|\epsilon_1\| + 2\mu_4 \|\epsilon_2\| \|\epsilon_2\| \end{aligned} \tag{16}$$

where $\mu_1 = \alpha_{1,1}k_1$, $\mu_2 = \alpha_{1,1}k_2$, $\mu_3 = \alpha_{2,1}k_3$, and $\mu_4 = \alpha_{2,1}k_4$. k_1, k_2, k_3 and k_4 are positive constants.

Now, the following inequalities are satisfied

$$\lambda \min(S_i) \|\epsilon_i\|^2 \leq \|\epsilon_i\|_{S_i}^2 \leq \lambda \max(S_i) \|\epsilon_i\|^2, \ i = 1, 2$$

where $\lambda \min(S_i)$ and $\lambda \max(S_i)$ are respectively the minimal and maximal eigenvalues of S_i, $i = 1, 2$ and

$$\|\epsilon_i\|_{S_i}^2 = \epsilon_i^T S_i \epsilon_i, \ i = 1, 2.$$

By writing (16) in terms of the functions V_1 and V_2, it follows

$$\begin{aligned} \dot{V}_o \leq - \quad & (\theta_1 - 2\tilde{\mu}_1)V_1 - (\theta_2 - 2\tilde{\mu}_2)V_2 \\ & + 2(\tilde{\mu}_3 + \tilde{\mu}_4)\sqrt{V_1}\sqrt{V_2} \end{aligned}$$

where $\tilde{\mu}_1 = \frac{\mu_1}{\lambda_{min}(S_1)}$, $\tilde{\mu}_2 = \frac{\mu_2}{\lambda_{min}(S_2)}$,
$\tilde{\mu}_3 = \frac{\mu_3}{\sqrt{\lambda_{min}(S_1)}\sqrt{\lambda_{min}(S_2)}}$, and $\tilde{\mu}_4 = \frac{\mu_4}{\sqrt{\lambda_{min}(S_1)}\sqrt{\lambda_{min}(S_2)}}$.

Next, by using the following inequality $\sqrt{V_1}\sqrt{V_2} \leq \frac{\upsilon}{2}V_1 + \frac{1}{2\upsilon}V_2$, $\forall \upsilon \in]0, 1[$, one get

$$\begin{aligned} \dot{V}_o \leq - \quad & (\theta_1 - 2\tilde{\mu}_1)V_1 + (\tilde{\mu}_3 + \tilde{\mu}_4)\upsilon V_1 \\ & - (\theta_2 - 2\tilde{\mu}_2)V_2 + \frac{(\tilde{\mu}_3 + \tilde{\mu}_4)}{\upsilon}V_2. \end{aligned}$$

Then

$$\begin{aligned} \dot{V}_o \leq - \quad & [\theta_1 - (2\tilde{\mu}_1 + (\tilde{\mu}_3 + \tilde{\mu}_4)\upsilon)]V_1 \\ & - [\theta_2 - (2\tilde{\mu}_2 + \frac{(\tilde{\mu}_3 + \tilde{\mu}_4)}{\upsilon})]V_2. \end{aligned}$$

Finally, by choosing θ_1 and θ_2 such that the following inequalities (17) and (18) are satisfied

$$\delta_1 = \theta_1 - (2\tilde{\mu}_1 + (\tilde{\mu}_3 + \tilde{\mu}_4)\upsilon) > 0 \tag{17}$$

and

$$\delta_2 = \theta_2 - (2\tilde{\mu}_2 + \frac{(\tilde{\mu}_3 + \tilde{\mu}_4)}{\upsilon}) > 0 \tag{18}$$

and by taking δ such that $\delta = min(\delta_1, \delta_2)$ one has

$$\dot{V}_o \leq -\delta V_o.$$

Thus the estimation error converges exponentially to zero as t tends to ∞. □

With respect to the original system (1), the proposition 4.1 gives as a result the following corollary.

Corollary 4.1. Under the assumptions 4.1 and 4.3 and for θ_1 and θ_2 satisfying the inequalities (17) and (18) respectively, the system (14)-(15) is an exponential observer for system (1). Furthermore, the observer (14)-(15) exponentially reconstruct the unknown parameters m_1 and m_2 of the system (1).

5. NUMERICAL EXAMPLE AND DISCUSSIONS

In the simulations, we use the parameters $a = 42.5$, $b = 24$, $c = 13$, $d = 20$, $e = 50$, and $f = 40$, which ensure a chaotic behavior of the system (1) (see Qi [05]). The parameters used for the observer with the STA are $\alpha_1 = 7 \times 10^7$, $\lambda_1 = 7 \times 10^3$, $\alpha_2 = 300$, $\lambda_2 = 100$, $\alpha_3 = 1000$, $\lambda_3 = 200$, $\alpha_4 = 600$, and $\lambda_4 = 300$. For the adaptive observer, the parameters are $\theta_1 = \theta_2 = 600$. The messages m_1 and m_2 are considered to have a slow dynamics for satisfying the assumption persistency condition required by the AO.

The chaotic behavior of the system is shown in the 3-D portrait 1.

The sampling period used in the simulations is 10^{-6} seconds. Due to the fast behavior of the system, a larger sampling time yields an unacceptable estimation of the messages when the super-twisting observer is used.

Figure 2 shows the trajectories of the state x_3 as well as the ones of $\hat{x}_3$ for both the super-twisting and the adaptive observers. There one can see that the trajectories of the ST observer converge much faster than the ones of the AO.

The message m_1 and its estimation $\hat{m}_1$ are depicted in Fig. 3, where we can note that the singularity in the point $(x_1 = 0)$ (see, e.g., on $t \simeq 1.42$, 2.65, 8.15) affects more the estimation done with the SA than the one done with the AO. This is clear due to the explanation of Remark 3.1. The adaptive observer performs worse than the super-twisting observer if their high gains are insufficiently large, i.e., the persistency condition is not satisfied.

In order to test both observers with respect to parameter uncertainties, we introduce a variation of 0.01% into the nominal parameters. The figure 4 illustrates how this uncertainty in the parameters affects the estimation of the message m_1. In the case of the adaptive observer, the parameter uncertainty destroys completely the estimation of the messages. Obviously, in the case of the observer based on the STA, the parameter uncertainty increases the sensibility of the observer to the singularities.

To test both observers with respect to the noise appearing in the outputs of the system, we added to the each output of the system a chirp signal. A signal with initial frequency of 10 Hz, frequency of target time of 12 Hz, and an amplitude of 0.015 for the first output, and 12 Hz to 15 Hz and an amplitude of 0.001 for the second output. The figure 5 compares both observers by the estimation of m_2. In this case we see that the adaptive observer is quite sensible to the noises, while the STA observer is sensible to the noises, but we can consider that it still estimates (perhaps poorly) the message.

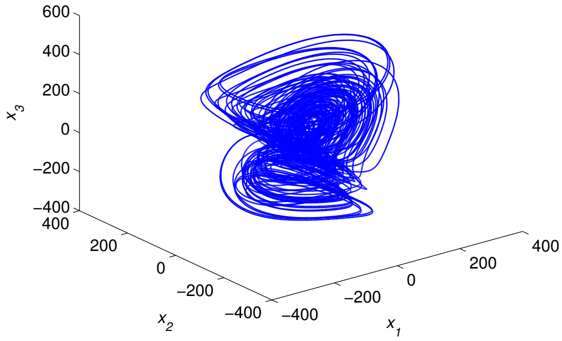

Fig. 1. 3-D portrait showing the chaotic behavior of the system.

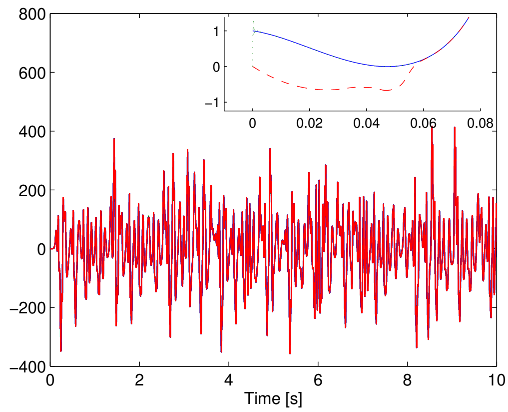

Fig. 2. x_3 (solid line) and its estimation $\hat{x}_3$ using the super-twisting (dot line) and adaptive (dash line) observers.

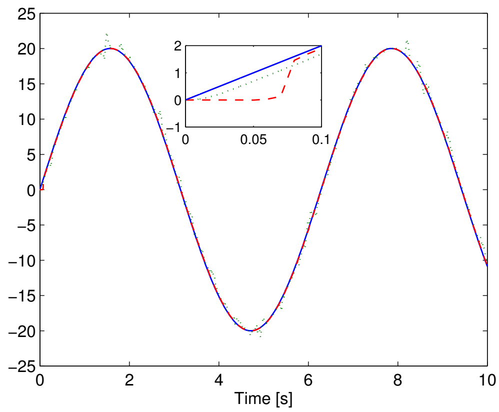

Fig. 3. Message m_1 (solid line) and its estimation $\hat{m}_1$ using the super-twisting (dot line) and adaptive (dash line) observers.

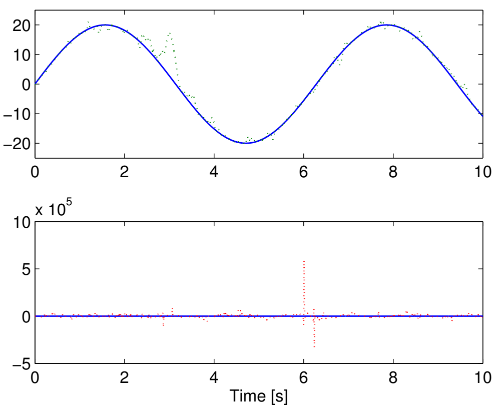

Fig. 4. Message m_1 (solid line) and its estimation $\hat{m}_1$ (dot line) for the system with 0.01% of uncertainty in the parameters. Above with the super-twisting observer, below with the adaptive observer.

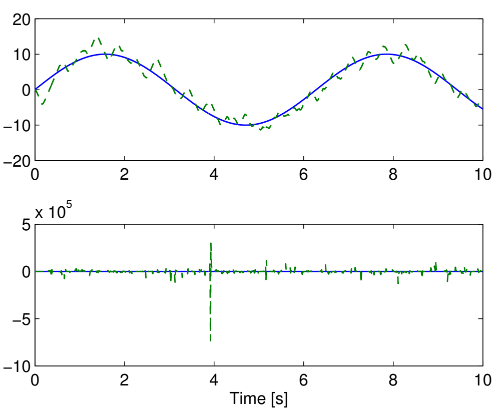

Fig. 5. Message m_2 (solid line) and its estimation $\hat{m}_2$ (dashed line) for the system with noise in the output. Above with the super-twisting observer, below with the adaptive observer.

REFERENCES

J.P. Barbot, I. Bemouhoub and L. Boutat-Baddas, 2006, 'Observability Bifurcations: Application to Cryptography', *Chaos in Automatic Control, Taylor & Francis, ed W. Perruquetti and J-P Barbot* Chapter 9, pp 370-410.

J.P. Barbot, D. Boutat and T. Floquet, 2005, 'A new observation algorithm for nonlinear systems with unknown inputs', *44th IEEE CDC and ECC, Seville, Spain.*

J.P. Barbot, M. Fliess and L. T. Floquet, 2007, 'An algebraic framework for design of nonlinear observer with unknown inputs', *46th IEEE CDC, New Orleans, United States.*

F.J. Bejarano, L. Fridman and A. Poznyak, 2007, 'Exact state estimation for linear systems with unknown inputs based on hierarchical super-twisting algorithm', *International Journal of Robust and Nonlinear Control*, vol. 17, no. 18, pp 1734-1753.

J. Davila, L. Fridman and A. Levant, 2005, 'Second-order sliding-mode observer for mechanical systems', *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1785-1789.

M. Fliess, 1989, 'Automatique et corps différentiels', *Forum Math.*, Vol 1, pp 227-237.

M. Fliess, C. Join, H. Sira-Ramrez, 'Non-linear estimation is easy', *Int. J. Modelling Identification Control*, to appear.

M. Ghanes, J. De Leon and A. Glumineau, 'Observability study and observer-based interconnected form for sensorless induction motor', *Proc 45th IEEE Conference on Decision and Control*, San Diego, CA, December, USA, 2006, 13-15 December, USA.

H. Hammouri, J. De Leon, 'Observer synthesis for state-affine systems' *Proc 29th IEEE Conference on Decision and Control*, Honolulu, Hawaii. 1990.

R. He and P.G. Vaidya, 1992, 'Analysis and synthesis of synchronous periodic and chaotic system', *Phys. Rev. A.* Vol. 46, pp. 7387-7392.

A. Krener and A. Isidori, 1983, 'Linearization by output injection and nonlinear observer', *Systems & Control Letters*, Vo 3, pp 47-52.

A. Krener and W. Respondek, 1985, 'Nonlinear observer with linearizable error dynamics', *SIAM J. Control and Optimization*, Vol 30, No 6, pp 197-216.

A. Levant, 1996, 'Sliding order and sliding accuracy in sliding mode control', *International Journal of Control*, vol. 58, no. 6, pp. 1247-1263.

A. Levant, 1998, 'Robust exact differentiation via sliding mode technique', *International Journal of Control*, vol. 34, no. 3, pp. 379-384.

M. Mboup, C. Join, M. Fliess, 'A revised look at numerical differentiation with an application to nonlinear feedback control', *Proc.* 15^{th} *Mediterrean Conf. Control Automation – MED'2007*, Athens, 2007.

H. Nijmeijer and I.M.Y. Mareels, 1997, 'An observer looks at synchronization', *IEEE Trans. Circuits and Systems I* Vol. 44 (11), pp 882-891.

U. Parlitz, L.O. Chua, Lj. Kocarev, K. S. Halle and A. Sang, 1992, 'Transmission of digital signals by chaotic synchronization', *Int. Journal of Bifurcations and Chaos* Vol. 2, pp 973-977.

L.M. Pecora and T.L Caroll, 1990, 'Synchronization in chaotic system', *Phys. Rev. Lett.* Vol. 64, pp 821-824.

G.Y. Qi, S.Z. Du, G.R. Chen et al., 2005, 'On a four-dimensional chaotic system', *Chaos, Solutions and Fractals*, vol. 23, no. 5, pp. 1671-1682.

W. Respondek, 1990, 'Right and left invertibility of nonlinear control systems', *in Nonlinear controllability and optimal control, H.J. Sussmann ED., Marcel Dekker*, pp 133-176.

H. Sira-Ramirez and C. Cruz-Hernandez, 2001, 'Synchronization of chaotic system: A generalized hamiltonian systems approach', *Int. Journal of Bifurcations and Chaos*, Vol 11, No 5, pp 1381-1395.