Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Asymptotically Unbiased Average Consensus Under Measurement Noises and Fixed Topologies ⋆

Tao Li *

* *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China (E-mail: litao@ amss.ac.cn).*

Abstract: This paper is concerned with average-consensus control under directed topologies and random measurement noises. To attenuate the measurement noises, time-varying consensus gains are introduced in the protocol. It is shown that under the protocol designed, all agents' states converge to a common Gaussian random variable, whose mathematical expectation is just the average of the initial states, and the mean square static error vanishes as the number of agents increases to infinity under certain topologies. In addition, for the noise-free case, necessary and sufficient conditions are given on the network topology and consensus gains to achieve average-consensus; and for the noisy measurement case, by combining algebraic graph theory and stochastic analysis, necessary and sufficient conditions are given on the consensus gains to achieve asymptotically unbiased mean square average-consensus.

1. INTRODUCTION

Recently, distributed coordination for multi-agent systems has become a hot topic in the area of systems and control (Bauso *et al.*, 2006; Olfati-Saber *et al.*, 2007; Li & Zhang, 2008), which is characterized by a basic requirement that without central control stations, the whole group can achieve consensus on the shared data only through local communications. Consensus control generally means to design a network protocol such that all agents asymptotically reach an agreement on their states. In some cases, the common value to which the states converge is also required to be the average of the initial states, which is often called average-consensus and has wide applications in various areas such as formation control (Sinha & Ghose, 2006), distributed filtering (Olfati-Saber, 2005), multi-senor data fusion (Xiao *et al.*, 2005) and distributed computation (Lynch, 1996).

Real networks are often in uncertain communication environments due to many random factors such as channel noises, output quantization, and the limit to channel capacity, therefore, distributed coordination in uncertain environments is a key set of problems for the study of large classes of multi-agent systems. As a first step, consensus problems under random measurement noises have attracted the attention of some researchers (Huang & Manton, 2006; Carli *et al.*, 2006; Ren *et al.*, 2005; Kingston *et al.*, 2005). However, for average-consensus problems under random measurement noises, there is still lack of good results comparable with those obtained in the noise-free cases (Olfati-Saber & Murray, 2004; Kingston & Beard, 2006), even if the network topology is fixed. Ren, Beard and Kingston (2005) and Kingston, Ren and Beard (2005) introduced time-varying consensus gains and designed consensus protocols based on a Kalman filter structure. They proved that consensus can be achieved under those protocols for noise-free cases and the closed-loop system is input-to-state stable from measurement noises to consensus errors. Huang and Manton (2006) considered the first-order discrete-time consensus control under fixed topologies. They introduced decreasing consensus gains $a(k)$ (where k is the discrete time instant), and proved that, if the network topology is a strongly connected circulant graph, and $a(k) = O(1/k^{\gamma}), \gamma \in (0.5, 1]$, then the static mean square error between the individual state and the average of the initial states of all agents is in the same order as the variance of the measurement noises.

The common features of the above literature include that (i) balanced graphs play an important role in the average-consensus protocols for noise-free cases. (ii) time-varying consensus gains are introduced to attenuate measurement noises; (iii) due to random measurement noises, the static error of the closed-loop system is not zero as the noise-free cases in Olfati-Saber and Murray (2004). These naturally give rise to the following questions: (a) can we give a necessary and sufficient condition on the consensus gains and the network topology to ensure average-consensus? (b) what is the relationship between the performances of the closed-loop system (such as static error, convergence rate) and the designed parameters (such as the weighted adjacency matrix of the topology graph, the consensus gains and the number of agents)? Can we give a quantitative characterization on this relationship? These are all fundamental problems to be investigated for distributed coordination in uncertain environments. The purpose of this paper is to answer these questions.

We consider the average-consensus control for networks of continuous-time integrator agents under fixed and directed topologies. The control input of each agent can use only its local state and the states of its neighbors corrupted by

⋆ This work was supported by the National Natural Science Foundation of China under grants 60221301 and 60674038.

white noises. Inspired by Ren, Beard and Kingston (2005) and Kingston, Ren and Beard (2005) and Huang and Manton (2006), we introduce time-varying consensus gains in our network protocol to attenuate the measurement noises, which leads to a time-varying stochastic differential equation of the closed-loop system. The state matrix of the closed-loop equation is a time-varying Laplacian matrix of a digraph. Different from the cases of undirected and circulant graphs (which are both special cases of digraphs), this kind of state matrices, generally speaking, is neither symmetric nor diagonalizable, which is the difficulty in the convergence analysis.

We combine stochastic analysis and algebraic graph theory together, by introducing the concept and tools of symmetrized graph (Olfati-Saber & Murray, 2004) in stochastic Lyapunov analysis. Firstly, for the noise-free case, necessary and sufficient conditions are given on the network topology and consensus gains to achieve average-consensus. Then for the noisy measurement case, necessary and sufficient conditions are given on the consensus gains to achieve asymptotically unbiased mean square average-consensus.

We prove that under the protocol designed, the state of each agent converges in mean square to a common Gaussian random variable, whose mathematical expectation is just the average of the initial states. It is shown that the variance of the random variable, which gives the static maximum mean square error between the individual state and the average of the initial states, vanishes as the number of agents increases to infinity. We use the decibel changing rate of the mean square consensus error as the transient performance index and give the estimates of the upper and lower bounds of the transient performance index, respectively. When the protocol designed here degenerates to the time-invariant protocol given for noise-free systems by Olfati-Saber and Murray (2004), the estimate of the upper bound degenerates to that of the convergence rate given by Olfati-Saber and Murray (2004) correspondingly. It is quantitatively shown that enlarging consensus gains can speed up the convergence rate of consensus, but will worsen the static error in the meantime. Therefore, the key point of the design of a consensus protocol under measurement noises consists in a trade-off between the transient and static performances by choosing the consensus gains properly.

It is worth pointing out that Moreau and Belgium (2004) studied the stability of linear continuous-time systems whose state matrices are time-varying Laplacian matrices and gave a sufficient condition to ensure the state components to achieve consensus. In this paper, for the noise-free case, the closed-loop equation degenerates to a special case of those studied by Moreau and Belgium (2004), however, we give necessary and sufficient conditions on the network topology and the consensus gains to ensure average-consensus.

The remainder of this paper is organized as follows. In section 2, some concepts in graph theory and the problem to be investigated are formulated. In section 3 and 4, the convergence and performances of the closed-loop system are analyzed respectively. In section 5, two numerical examples are given to illustrate our results. Due to the space limit, only the proofs of Theorems 2-3 are presented, which are put in Appendix A.

The following notations will be used throughout this paper: $\mathbf{1}$ denotes a column vector with all ones. I_m denotes the m dimensional identity matrix. For a given set $\mathcal{S}$, $|S|$ denotes its number of elements. For a given vector or (square) matrix A, A^T denotes its transpose; $tr(A)$ denotes its trace. For a given random variable X, $E(X)$ denotes its mathematical expectation; $Var(X)$ denotes its variance. For a given positive number x, $\log(x)$ denotes the common logarithm of x. For a family of random variables (r.v.s) $\{\xi_\lambda, \lambda \in \Lambda\}$, $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ-algebra $\sigma(\{\xi_\lambda \in B\}, B \in \mathcal{B}, \lambda \in \Lambda)$, where $\mathcal{B}$ denotes the 1-dimensional Borel sets. For a σ-algebras $\mathcal{F}$ and a r.v. ξ, we say ξ is adapted to $\mathcal{F}$, if ξ is $\mathcal{F}$ measurable.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminary Concepts in Graph Theory

Let $\mathcal{G}=\{\mathcal{V},\mathcal{E},\mathcal{A}\}$ be a weighted digraph, where $\mathcal{V}=\{1,2,...,N\}$ is the set of nodes, node i represents the ith agent; $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (j,i). $(j,i)\in\mathcal{E}$ if and only if the jth agent can send information to the ith agent directly. In this case, j is called the parent of i, and i is called the child of j. The neighborhood of the ith agent is denoted by $N_i = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$, which is the set of all parents of i. i is called a source, if it has no parent but only children.

$\mathcal{A}=[a_{ij}] \in \mathbb{R}^{N\times N}$ is called the weighted adjacency matrix of $\mathcal{G}$. For any i, $j\in \mathcal{V}$, $a_{ij}\geq 0$, and $a_{ij}>0 \Leftrightarrow j\in N_i$. $deg_{in}(i) = \sum_{j=1}^N a_{ij}$ is called the in-degree of i; $deg_{out}(i) = \sum_{j=1}^N a_{ji}$ is called the out-degree of i; $L_{\mathcal{G}} = \mathcal{D} - \mathcal{A}$ is called the Laplacian matrix of $\mathcal{G}$, where $\mathcal{D} = diag(deg_{in}(1), ..., deg_{in}(N))$.

$\mathcal{G}$ is called a balanced digraph, if $deg_{in}(i) = deg_{out}(i)$, $\forall i \in \mathcal{V}$. $\mathcal{G}$ is called an undirected graph, if $\mathcal{A}$ is a symmetric matrix. It is easily shown that an undirected graph must be a balanced digraph. $\mathcal{G}$ is called a regular graph, if it is an undirected graph and $|N_i| = |N_j|$, $\forall i, j \in \mathcal{V}$.

A sequence (i_1, i_2), (i_2, i_3), ..., (i_{k-1}, i_k) of edges is called a directed path from node i_1 to node i_k. $\mathcal{G}$ is called a strongly connected digraph, if for any $i, j \in \mathcal{V}$, there is a directed path from i to j. A directed tree is a digraph, where every node, except the root, has exactly one parent, and the root is a source. A spanning tree of $\mathcal{G}$ is a directed tree, whose node set is $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}$.

2.2 Consensus Protocols

In this paper, we consider the average-consensus control for a network of continuous-time integrator agents with the dynamics

$$\dot{x}_i(t) = u_i(t),\ i = 1, 2, ..., N, \tag{1}$$

where $x_i(t)\in\mathbb{R}^1$ is the state of the ith agent, and $u_i(t)\in\mathbb{R}^1$ is the control input. The initial state $x_i(0)$ is deterministic. Denote $X(t) = [x_1(t), ..., x_N(t)]^T$.

The ith agent can receive information from its parents:

$$y_{ji}(t) = x_j(t) + \sigma_{ji} n_{ji}(t),\ j \in N_i, \qquad (2)$$

where $y_{ji}(t)$ denotes the measurement of the jth agent's state $x_j(t)$ by the ith agent. $\{n_{ji}(t),\ i,j{=}1,2,...,N\}$ are independent standard white noises (Øksendal, 2000), where $\sigma_{ji} \geq 0$ is the noise intensity. $(\mathcal{G}, X)$ is usually called a dynamic network (Olfati-Saber & Murray, 2004).

We call the group of controls $\mathcal{U}{=}\{u_i,\ i{=}1,2...,N\}$ a measurement-based distributed protocol, if $u_i(t)$ is adapted to $\sigma(x_i(s),\ \cup_{j\in N_i} y_{ji}(s),\ 0{\leq}s{\leq}t),\ \forall t{\geq}0,\ i = 1,2,...,N$. The so-called average-consensus control means to design a measurement-based distributed protocol for the dynamic network $(\mathcal{G}, X)$, such that the states of all the agents converge towards the value $\frac{1}{N}\sum_{j=1}^N x_j(0)$ in some sense, when $t \to \infty$. Below we give the definition of average-consensus protocol in mean square for stochastic systems.

Definition 1. A distributed protocol $\mathcal{U}$ is called an asymptotically unbiased mean square average-consensus protocol if it renders the system (1)-(2) has the following properties: for any given $X(0) \in \mathbb{R}^n$, there is a r.v. x^*, such that $E(x^*) = \frac{1}{N}\sum_{j=1}^N x_j(0)$, $Var(x^*) < \infty$, and

$$\lim_{t\to\infty} E[x_i(t) - x^*]^2 = 0,\ i = 1,2,...,N.$$

Remark 1. Here the term "asymptotically unbiased" is borrowed from mathematical statistics, since average-consensus can be viewed as a distributed estimation problem for the group decision value $\frac{1}{N}\sum_{j=1}^N x_j(0)$. If $\mathcal{U}$ is an asymptotically unbiased mean square average-consensus protocol, then $x_i(t)$ is the asymptotically unbiased estimate of $\frac{1}{N}\sum_{j=1}^N x_j(0)$, that is,

$$\lim_{t\to\infty} E[x_i(t)] = \frac{1}{N}\sum_{j=1}^{N} x_j(0),\ i = 1,2,...,N.$$

If there is no measurement noise, and $\mathcal{U}$ is an asymptotically unbiased mean square average-consensus protocol, then $Var(x^*) = 0$, that is, $x^* = \frac{1}{N}\sum_{j=1}^N x_j(0)$. In this case, Definition 1 is equivalent to the definition of average-consensus protocol for deterministic systems given by Olfati-Saber and Murray (2004).

For the dynamic network $(\mathcal{G}, X)$, we propose the distributed protocol as

$$u_i(t) = \begin{cases} 0, & |N_i| = 0, \\ a(t)\displaystyle\sum_{j\in N_i} a_{ij}[y_{ji}(t) - x_i(t)], & |N_i| > 0, \end{cases} \qquad (3)$$

where $a(\cdot) : [0,\infty) \to (0,\infty)$ is piecewise continuous, called consensus-gain function.

Comparing with the protocol given by Olfati-Saber and Murray (2004), in protocol (3), the measurement noises are explicitly considered and the time-varying consensus gain $a(\cdot)$ is introduced. In this paper, we will prove that under mild conditions, the control law (3) is an asymptotically unbiased mean square average-consensus protocol.

3. CONVERGENCE ANALYSIS

Denote the ith row of the matrix $\mathcal{A}$ by α_i, and $\Sigma_i \overset{\triangle}{=} diag(\sigma_{1i}, ..., \sigma_{Ni})$, $i{=}1,...,N$, where $\sigma_{ji}{=}0$, $j{\notin}N_i$. $\Sigma \overset{\triangle}{=} diag(\alpha_1^T\Sigma_1, ..., \alpha_N^T\Sigma_N)$ is an $N{\times}N^2$ dimensional block diagonal matrix. $n_i(t) \overset{\triangle}{=} [n_{1i}(t),...n_{Ni}(t)]^T$, $\eta(t) \overset{\triangle}{=} [n_1^T(t),...,n_N^T(t)]^T$. Substituting protocol (3) into system (1) leads to

$$\frac{dX(t)}{dt} = [-a(t)L_{\mathcal{G}}X(t)] + a(t)\Sigma\eta(t). \qquad (4)$$

It is a system driven by an N^2 dimensional standard white noise, which can be written in the form of the Itô stochastic differential equation

$$dX(t) = [-a(t)L_{\mathcal{G}}X(t)]dt + a(t)\Sigma dW(t), \qquad (5)$$

where $W(t) \overset{\triangle}{=} [W_{11}(t),...W_{N1}(t),....,W_{1N}(t),...,W_{NN}(t)]^T$ is an N^2 dimensional standard Brownian motion.

To get the main results, we need the following assumptions.

A1) Network Topology: $\mathcal{G}$ is a balanced digraph containing a spanning tree.

A2) Convergence Condition: $\int_0^\infty a(s)ds = \infty$.

A3) Robustness Condition: $\int_0^\infty a^2(s)ds < \infty$.

Remark 2. It can be seen that if there are constants $\beta_1 \leq 1$, $\beta_2 > -0.5$, $\gamma_1 \leq 1$, $\gamma_2 > 0.5$, $C_1 > 0$, $C_2 > 0$ such that $\frac{C_1}{t^{\gamma_1}[\log(t)]^{\beta_1}} \leq a(t) \leq \frac{C_2[\log(t)]^{\beta_2}}{t^{\gamma_2}}$ for sufficiently large t, then A2)-A3) hold.

For simplicity of problem formulation, we introduce the following assumption:

A4) In the dynamic network $(\mathcal{G}, X)$, there is an edge $(j,i) \in \mathcal{E}$ such that $\sigma_{ji} > 0$.

The intuitive meaning of Assumption A4) is that, there is at least one noisy communication channel in the dynamic network. The negative proposition of A4) is given by

A4′) In the dynamic network $(\mathcal{G}, X)$, for any $(j,i) \in \mathcal{E}$, we have $\sigma_{ji} = 0$.

When A4′) holds, the dynamic network degenerates to the noise-free case, and protocol (3) can be written as

$$u_i(t) = \begin{cases} 0, & |N_i| = 0, \\ a(t)\displaystyle\sum_{j\in N_i} a_{ij}[x_j(t) - x_i(t)], & |N_i| > 0. \end{cases} \qquad (6)$$

Firstly, we will give a necessary and sufficient condition to ensure average-consensus under protocol (3) for the noise-free case. Denote $J \overset{\triangle}{=} \frac{1}{N}\mathbf{1}\mathbf{1}^T$, $\delta(t) \overset{\triangle}{=} X(t) - JX(t)$, and $V(t) \overset{\triangle}{=} \delta^T(t)\delta(t)$. $\delta(t)$ is called the consensus error, and $V(t) = \frac{1}{N}\sum_{1\leq i<j\leq N}(x_i(t)-x_j(t))^2$ is the energy function of consensus error. These concepts are widely used in the relevant literature (Olfati-Saber & Murray, 2004; Xiao *et al.*, 2007). We have the following theorem.

Theorem 1. Apply the protocol (3) to the system (1)-(2) and suppose that Assumption A4′) holds. Then,

$$\lim_{t\to\infty} \|X(t) - JX(0)\| = 0,\ \forall X(0) \in \mathbb{R}^N, \qquad (7)$$

if and only if A1)-A2) hold.

Remark 3. Substituting the protocol (6) into the system (1), we get the closed-loop system for the noise-free case:

$$\dot{X}(t) = -L_{\mathcal{G}(t)}X(t),\ t \geq 0, \tag{8}$$

where $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}, a(t)\mathcal{A}\}$ is a digraph with the time-varying weighted adjacency matrix $a(t)\mathcal{A}$. The system (8) can be regarded as a special case of a kind of time-varying systems described by

$$\dot{X}(t) = -L_{\mathcal{G}(t)}X(t),\ t \geq 0, \tag{9}$$

where $\mathcal{G}(t)$=$\{\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)\}$ is a digraph with time-varying topologies.

The convergence properties of the system (9) and its corresponding discrete-time model have been widely studied (Moreau & Belgium, 2004; Tsitsiklis *et al.*, 1986; Moreau, 2005; Blondel *et al.*, 2005). Moreau and Belgium (2004) gave a sufficient condition to guarantee that all state components of the system (9) converge to a common value as time goes on. From Theorem 1, it can be seen that, the condition given by Moreau and Belgium (2004) is not necessary. It is easy to verify that if A1) is satisfied, and $a(t) = \frac{1}{t+1}$, then by Theorem 1, all state components of (8) converge to $\frac{1}{N}\sum_{j=1}^{N} x_j(0)$, but the condition given by Moreau and Belgium (2004) is not satisfied (See Theorem 1 of Moreau and Belgium (2004)).

Remark 4. From Theorem 1, it can be seen that for the fixed topology case, when there is no measurement noise, Assumption A1) is the weakest condition on the network topology for protocol (3) to achieve average-consensus. For A1), containing a spanning tree is to ensure the connectivity of the network to some extent, such that different agents may asymptotically agree on their states; while the balance of the digraph is to make the state average be a constant, such that the final group decision value is the average of the initial states.

Assumption A2) is to ensure that the consensus error converges to zero with a certain rate. In fact, when $a(t) \equiv 1$, the protocol (6) degenerates to the time-invariant protocol (A.1) in Olfati-Saber and Murray (2004) where A2) holds naturally, and the consensus error converges to zero exponentially. Therefore, we call A2) the convergence condition on consensus gains.

Below we will prove that under Assumptions A1)-A3), the control law (3) is an asymptotically unbiased mean square average-consensus protocol. The following theorem, in which we combine stochastic analysis and algebraic graph theory together, is a key result to prove Theorem 3.

Theorem 2. Applying the protocol (3) to the system (1)-(2), if Assumptions A1)-A3) hold, then

$$\lim_{t\to\infty} E[V(t)] = 0. \tag{10}$$

Remark 5. Theorem 2 is to say that under Assumptions A1)-A3), protocol (3) leads to

$$\lim_{t\to\infty} E[x_i(t) - x_j(t)]^2 = 0,\ \forall\, i, j = 1, 2, ..., N.$$

So it is a mean square weak consensus protocol (Huang & Manton, 2006).

Theorem 3. Applying the protocol (3) to the system (1)-(2), if Assumptions A1)-A3) hold, then

$$\lim_{t\to\infty} \max_{1\leq i\leq N} E[x_i(t) - x^*]^2 = 0,$$

where x^* is a Gaussian random variable whose mathematical expectation is $\frac{1}{N}\sum_{j=1}^{N} x_j(0)$, and variance is $N^{-2}\sum_{j=1}^{N}\sum_{j\in N_i} \sigma_{ji}^2 a_{ij}^2 \int_0^\infty a^2(s)ds$, that is, (3) is an asymptotically unbiased mean square average-consensus protocol.

Theorem 4. Apply the protocol (3) to the system (1)-(2) and suppose that Assumptions A1) and A4) hold. Then, (3) is an asymptotically unbiased mean square average-consensus protocol if and only if A2)-A3) hold.

Remark 6. Combing Theorem 3 and Theorem 4, one can see the important role played by A3). When there is no measurement noise, to achieve average-consensus, it is only required that the consensus gains satisfy the condition A2). However, in the noisy environment, the state average of the closed-loop system is not a constant any more, and A2) itself is no longer sufficient. A3) ensures that the state average of the closed-loop system converges in mean square rather than diverges. Theorem 3 and Theorem 4 also tell us that the time-invariant protocol (A.1) proposed by Olfati-Saber and Murray (2004) is not robust with respect to Gaussian noises. The purpose of the introduction of time-varying consensus gains and Assumption A3) is just to attenuate the measurement noises, such that the consensus protocol is robust with respect to measurement noises. We call A3) the robustness condition on consensus gains.

4. PERFORMANCE ANALYSIS

For single-agent systems, the performance indexes can be roughly divided into two categories: transient indexes and static indexes. A transient index (such as rising or setting time for a unit step response) often reflects the convergence rate of the closed-loop system to the steady state, while a static index often reflects the final error between the steady state of the system and the target state of the control objective. For average-consensus problems under measurement noises, we can also choose some transient or static indexes to evaluate the system performances. A transient index should reflect the rate of different agents converging to the agreement on their states, while a static index should reflect the final error between the steady state of each agent and the average of the initial states of all agents.

4.1 Static Performance Analysis

Definition 2. Applying the distributed protocol $\mathcal{U}$ to the system (1)-(2),

$$J_s(\mathcal{U}, N) \triangleq \limsup_{t\to\infty} \max_{1\leq i\leq N} E[x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(0)]^2$$

is called the static maximum mean square error for average-consensus.

Theorem 5. Applying the distributed protocol $\mathcal{U}$ to the system (1)-(2), if the conditions of Theorem 3 hold, then

$$J_s(\mathcal{U}, N) = Var(x^*), \tag{11}$$

where x^* and $Var(x^*)$ are given by Theorem 3.

Remark 7. From Theorem 5 it is known that under the conditions of Theorem 3, $Var(x^*)$ can also be viewed as a static performance index of the system.

In some application of the information fusion of wireless sensor networks, the number N of network nodes is usually quite large. This gives rise to investigating the impact of N on the information fusion and the asymptotic property of the system when N increases to infinity. In this case, we have the following theorems.

Theorem 6. Applying the protocol (3) to the system (1)-(2), if $\max_{ij} a_{ij} = O(1)$ and $\max_i |N_i| = o(N), N \to \infty$, then under Assumptions A1)-A3), we have

$$\lim_{t\to\infty} \max_{1\le i\le N} E[x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(0)]^2 = o(1),\ N \to \infty.$$

Theorem 7. Applying the protocol (3) to the system (1)-(2), if $\mathcal{G}$ is an equally weighted and connected regular graph, then under Assumptions A2)-A3), we have

$$\lim_{t\to\infty} \max_{1\le i\le N} E[x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(0)]^2 = O(N^{-1}),\ N \to \infty.$$

Remark 8. Theorem 6 and Theorem 7 say that, under Assumptions A1)-A3), if the weighted adjacency matrix and the network degree do not diverge too fast with respect to N, then the more the network nodes are, the better the effect of the information fusion is. Especially, for equally weighted regular networks, the static error of the system is inversely proportional to the number of network nodes. Though more nodes can be added, a large number of nodes will result in a high cost for running and maintenance of the whole network, so the choice of N is a trade-off between the fusion accuracy and the cost.

4.2 Transient Performance Analysis

The mean square consensus error $E[V(t)]$ can be used to represent the degree of deviation between agents' states, so the rate of $E[V(t)]$ converging to zero can be viewed as a transient performance index of the system.

Definition 3. Applying the distributed protocol $\mathcal{U}$ to the system (1)-(2),

$$J_t(\mathcal{U}, s_0, h) \triangleq \frac{10\log(E(V(s_0+h))) - 10\log(E(V(s_0)))}{h}$$

is called the average decibel changing rate of the mean square consensus error from s_0 to $s_0{+}h$, where $s_0{>}0$, $h{>}0$.

Theorem 8. Applying the protocol (3) to the system (1)-(2), if $V(0) > 0$, then under Assumption A1), we have

$$\limsup_{0<h\to 0} J_t(\mathcal{U}, t, h) \le \frac{(10\log(e))C_0 a^2(t)\exp\{2\lambda_N(\widehat{L}_{\mathcal{G}})\int_0^t a(s)ds\}}{V(0)}$$

$$-20(\log(e))\lambda_2(\widehat{L}_{\mathcal{G}})a(t),\ \forall\ t \ge 0, \tag{12}$$

$$\liminf_{0<h\to 0} J_t(\mathcal{U}, t, h) \ge -20(\log(e))\lambda_N(\widehat{L}_{\mathcal{G}})a(t),\ \forall\ t \ge 0, \tag{13}$$

where $C_0 = tr[(I-J)^2\Sigma\Sigma^T]$, $\lambda_N(\widehat{L}_{\mathcal{G}})$ is the largest eigenvalue of $\widehat{L}_{\mathcal{G}}$, and $\lambda_2(\widehat{L}_{\mathcal{G}})$ is the second largest eigenvalue of $\widehat{L}_{\mathcal{G}}$.

Remark 9. When there is no measurement noise in the network, we have $C_0 = 0$. Take $a(t) \equiv 1$. Then, protocol (6) degenerates to the time-invariant protocol (A.1) proposed by Olfati-Saber and Murray (2004), and (12) in Theorem 8 becomes

$$\limsup_{0<h\to 0} J_t(\mathcal{U}, t, h) \le -20\log(e)\lambda_2(\widehat{L}_{\mathcal{G}}).$$

This is nothing but the estimate of the upper bound of the convergence rate obtained in Olfati-Saber and Murray (2004).

Remark 10. Theorem 3 and Theorem 8 give quantitative characterizations on the relationship between the performances of the closed-loop system and the designed parameters. From Theorem 8, it can be seen that for average-consensus protocol design under measurement noises, enlarging consensus gains can decrease the transient index, or is helpful to speed up the convergence rate of consensus. Unfortunately, from Theorem 3, one can see that the static error $Var(x^*)$ is proportional to $\int_0^\infty a^2(s)ds$, which means that enlarging consensus gains will inevitably worsen the static performance of the system. Therefore, the key point of the consensus protocol design consists in a trade-off between the transient and static performances by choosing the consensus gains properly. This is in agreement with the similar classical observation made for single-agent feedback systems.

5. NUMERICAL EXAMPLES

Example 1. In this example we investigate the necessity of A3) when there are measurement noises by a two-agent interacting system with topology graph $\mathcal{G}_1 = \{\{1,2\},\{(1,2),(2,1)\},\mathcal{A}_1 = [a_{ij}]_{2\times 2}\}$, where $\mathcal{A}_1$ is a 2×2 nonnegative matrix with positive element $a_{12} = a_{21} = 1$. The intensity of the measurement noises $\sigma_{21} = \sigma_{12} = 1$, and the initial states of the agents are $x_1(0) = 1$ and $x_2(0) = -1$, respectively. The consensus-gain function $a(t)$ is taken as $a(t) \equiv 1$, $\forall t \ge 0$. In this case, Assumptions A1) and A2) hold, but A3) does not hold. Under the control of the protocol (3), the states of the closed-loop system are shown in Fig. 1. It can be seen that the closed-loop system is divergent and so the protocol (A.1) in Olfati-Saber and Murray (2004) fails for the case under measurement noises.

Example 2. Consider a dynamic network under the topology of a strongly connected balanced digraph $\mathcal{G}_2 = \{\{1,2,3\},\{(1,2),(2,3),(3,1)\},\mathcal{A}_2 = [a_{ij}]_{3\times 3}\}$, where $\mathcal{A}_2$ is a 3×3 nonnegative matrix with positive element $a_{13} = a_{32} = a_{21} = 1$. The intensity of the measurement noises $\sigma_{12} = \sigma_{23} = \sigma_{31} = 1$. The initial states of agents are given by $x_1(0) = -2$, $x_2(0) = -4$ and $x_3(0) = 6$, respectively. The consensus-gain function $a(t)$ is taken as

$a(t) = \frac{\log(t+2)}{t+2}$, $t \geq 0$. Both A2) and A3) hold. Under the protocol (3), the states of the closed-loop system are shown in Fig. 2. It can be seen that as time goes on, the states of the group asymptotically achieve consensus, and approach the average of the initial states of all agents.

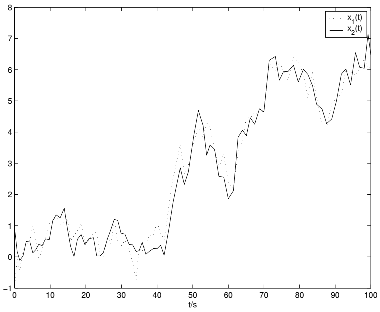

Fig. 1. Curves of states of Example 1

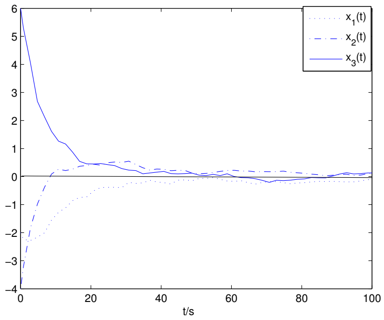

Fig. 2. Curves of states of Example 2

REFERENCES

Bauso, D., Giarré, L., & Pesenti, R. (2006). Non-linear protocols for optimal distributed consensus in networks of dynamics agents, *Systems and Control Letters*, 55(11), 918-928.

Blondel, V. D., Hendrickx, J. M., Olshevsky, A., & Tsitsiklis, J. N. (2005). Convergence in multiagent coordination, consensus, and flocking, *Proc. of 44th IEEE Conference on Decision and Control and the European Control Conference 2005*, Seville, Spain, December 12-15.

Carli, R., Fagani, F., Speranzon, A., & Zampieri, S. (2006). Communication constraints in coordinated consensus problems, *Proc. of the 2006 American Control Conference*, Minneapolis, Minnesota, June, 14-16.

Friedman, A. (1975). *Stochastic Differential Equations and Applications*, Volumn 1, New York: Academic Press.

Godsil, C., & Royle, G. (2001). *Algebraic Graph Theory*, New York: Springer.

Huang, M., & Manton, J. H. (2006). Coordination and consensus of networked agents with noisy measurement: stochastic algorithms and asymptotic behavior, submitted to *SIAM Journal on Control and Optimization:* Special issue on Control and Optimization in Cooperative Networks, 2006.

Kingston, D. B., & Beard, R. W. (2006). Discrete-time average-consensus under switching network topologies, *Proc. of the 2006 American Control Conference*, Minneapolis, Minnesota, June 14-16.

Kingston, D. B., Ren, W., & Beard, R. W. (2005). Consensus algorithm are input-to-state stable, *Proc. of the 2005 American Control Conference*, Portland, OR, June, 8-10.

Li, T., & Zhang, J. F. (2008). Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash-equilibria, *Automatica*, 44(3).

Lynch, N. (1996). *Distributed Algorithms*, San Matero, CA: Morgan Kaufmann.

Michel, A. N., & Miller, R. K. (1977). *Qualitative Analysis of Large Scale Dynamical Systems.* New York: Academic Press.

Moreau, L. (2005). Stability of multi-agent systems with dependent communication links, *IEEE Transactions on Automatic Control*, 50(2), 169-182.

Moreau, L., & Belgium, S. G. (2004). Stability of continuous-time distributed consensus algorithms, *Proc. of the 43rd IEEE Conference on Decision and Control*, Atlantis, paradise Island, Bahamas, December 14-17, 3998-4003.

Olfati-Saber, R. (2005). Distributed Kalman filter with embedded consensus filters, *Proc. of the 44th IEEE Conference on Decision and Control and the European Control Conference 2005*, Seville, Spain, December, 12-15.

Olfati-Saber, R., Fax, J. A., & Murry, R. M. (2007). Consensus and cooperation in networked multi-agent systems, *Proc. of the IEEE*, 95(1), 215-233.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problem in networks of agents with switching topology and time-delays, *IEEE Trans. on Automatic Control*, 49(9), 1520-1533.

Ren, W. , Beard, R. W. & Kingston, D. B. (2005). Multi-agent Kalman consensus with relative uncertainty, *Proc. of the 2005 American Control Conference*, Portland, OR, June, 8-10.

Sinha, A., & Ghose, D. (2006). Generalization of linear cyclic pursuit with application to rendezvous of multiple autonomous agents, *IEEE Trans. on Automatic Control*, 51(11), 1819-1824.

Tsitsiklis, J. N., Bertsekas, D. P., & Athans, M. (1986). Distributed asynchronous deterministic and stochastic gradient optimization algorithms, *IEEE Trans. on Automatic Control*, 31(9), 803-812.

Xiao, L., Boyd, S., & Lall, S. (2005). A scheme for robust distributed sensor fusion based on average consensus, *Proc. of the 4th International Symposium on Information Processing in Sensor Networks* , April, 15.

Xiao, L., Boyd, S. & Kim, S. J. (2007). Distributed average consensus with least-mean square deviation, *Journal of Paralell and Distributed Computing*, 67(1), 33-46.

Øksendal, B. (2000). *Stochastic Differential Equations: An Introduction with Applications*, fifth edition. Berlin: Springer.

Appendix A. PROOFS OF THEOREMS 2-3

To prove Theorems 2-3, we need the following results.

Lemma A.1. (Olfati-Saber & Murray, 2004) Let $L_{\mathcal{G}}$ be the Laplacian matrix of digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$. Then,

$\frac{L_{\mathcal{G}}+L_{\mathcal{G}}^T}{2}$ is the Laplacian matrix of $\widehat{\mathcal{G}}$, the symmetrized graph of $\mathcal{G}$, if and only if $\mathcal{G}$ is a balanced digraph.

Lemma A.2. (Olfati-Saber & Murray, 2004) $\mathcal{G}=\{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is a balanced digraph if and only if $\mathbf{1}^T L_{\mathcal{G}} = 0$.

Lemma A.3. (Godsil & Royle, 2001) If $\mathcal{G}=\{\mathcal{V},\mathcal{E},\mathcal{A}\}$ is a strongly connected undirected graph, then $L_{\mathcal{G}}$ is a symmetric matrix, and has N real eigenvalues, in an ascending order:

$$0 = \lambda_1(L_{\mathcal{G}}) < \lambda_2(L_{\mathcal{G}}) \le ... \le \lambda_N(L_{\mathcal{G}}) \le 2\Delta,$$

and

$$\min_{x\neq 0, \mathbf{1}^T x=0} \frac{x^T L_{\mathcal{G}} x}{\|x\|^2} = \lambda_2(L_{\mathcal{G}}),$$

where $\Delta = \max_{1\le i\le N} deg_{in}(i)$, and $\lambda_2(L_{\mathcal{G}})$ is called the algebraic connectivity of $\mathcal{G}$.

Lemma A.4. Applying the protocol (3) to the system (1)-(2), if Assumption A1) holds, then

$$E\int_{t_0}^t a(s)\delta^T(s)(I-J)\Sigma dW(s) = 0, \ \forall t \ge t_0. \quad \text{(A.1)}$$

Due to the space limit, the proof of Lemmas A.4 is omitted. We now begin to prove Theorems 2-3.

Proof of Theorem 2 Denote $\widehat{L}_{\mathcal{G}} = \frac{L_{\mathcal{G}}+L_{\mathcal{G}}^T}{2}$. Then, by A1) and Lemma A.1, $\widehat{L}_{\mathcal{G}}$ is the Laplacian matrix of the symmetrized graph $\widehat{\mathcal{G}}$, and $\widehat{\mathcal{G}}$ is strongly connected. Thus, by Lemma A.3, $\lambda_2(\widehat{L}_{\mathcal{G}}) > 0$.

From (5), A1) and Lemma A.2 it follows that

$$\begin{aligned} dJX(t) &= -a(t)J\Sigma dW(t), \\ d\delta(t) &= [-a(t)L_{\mathcal{G}}X(t)]dt - a(t)(I-J)\Sigma dW(t) \\ &= -a(t)L_{\mathcal{G}}\delta(t)dt - a(t)(I-J)\Sigma dW(t). \end{aligned}$$

Thus, by A1), Lemma A.3 and the Itô formula, we have

$$\begin{aligned} dV(t) &= \{-2a(t)\delta^T(t)L_{\mathcal{G}}\delta(t) + a^2(t)C_0\}dt \\ &\quad -2a(t)\delta^T(t)(I-J)\Sigma dW(t) \\ &= [-2a(t)\delta^T(t)\widehat{L}_{\mathcal{G}}\delta(t) + a^2(t)C_0]dt \\ &\quad -2a(t)\delta^T(t)(I-J)\Sigma dW(t) \\ &\le [-2\lambda_2(\widehat{L}_{\mathcal{G}})a(t)V(t) + a^2(t)C_0]dt \\ &\quad -2a(t)\delta^T(t)(I-J)\Sigma dW(t), \end{aligned} \quad \text{(A.2)}$$

which together with Lemma A.4 gives

$$\begin{aligned} E[V(t)] - E[V(t_0)] &\le -2\lambda_2(\widehat{L}_{\mathcal{G}})\int_{t_0}^t a(s)E[V(s)]ds \\ &\quad + C_0\int_{t_0}^t a^2(s)ds, \\ &\quad \forall t \ge t_0 \ge 0. \end{aligned} \quad \text{(A.3)}$$

Denote

$$\begin{aligned} I_1(t) &= \int_0^t \exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_s^t a(u)du\}a^2(s)ds, \\ I_2(t) &= V(0)\exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_0^t a(s)ds\}. \end{aligned}$$

Then, by (A.3) and the comparison theorem (Michel & Miller, 1977) we have

$$E[V(t)] \le C_0 I_1(t) + I_2(t). \quad \text{(A.4)}$$

From A2) and $\lambda_2(\widehat{L}_{\mathcal{G}}) > 0$, one can get $\lim_{t\to\infty} I_2(t) = 0$. Thus, to prove (10), we need only to show $\lim_{t\to\infty} I_1(t) = 0$. In fact, for any given $\epsilon > 0$, by Assumption A3), there is $s_0 > 0$ such that $\int_{s_0}^\infty a^2(s)ds < \epsilon$. Therefore,

$$\begin{aligned} & I_1(t) \\ &= \int_0^{s_0} \exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_s^t a(u)du\}a^2(s)ds \\ &\quad + \int_{s_0}^t \exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_s^t a(u)du\}a^2(s)ds \\ &\le \exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_{s_0}^t a(u)du\}\int_0^\infty a^2(s)ds + \int_{s_0}^\infty a^2(s)ds \\ &\le \exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_{s_0}^t a(u)du\}\int_0^\infty a^2(s)ds \\ &\quad + \epsilon, \forall\, t \ge s_0. \end{aligned} \quad \text{(A.5)}$$

From A2) and $\lambda_2(\widehat{L}_{\mathcal{G}})>0$ it follows that $\lim_{t\to\infty}\exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_{s_0}^t a(u)du\}=0$. Thus, by the arbitrariness of ϵ and (A.5), we have $\lim_{t\to\infty} I_1(t) = 0$. □

Proof of Theorem 3 By (5), Assumption A1) and Lemma A.2, we have $d(\frac{1}{N}\sum_{j=1}^N x_j(t))=a(t)\frac{1}{N}\mathbf{1}^T\Sigma dW(t)$, or equivalently,

$$\frac{1}{N}\sum_{j=1}^N x_j(t) = \frac{1}{N}\sum_{j=1}^N x_j(0) + \frac{\mathbf{1}^T\Sigma}{N}\int_0^t a(s)dW(s). \quad \text{(A.6)}$$

From $\int_0^\infty a^2(s)ds < \infty$, we know that $\int_0^\infty a(s)dW(s)$ is well defined (Friedman, 1975). Let $x^*=\frac{1}{N}\sum_{j=1}^N x_j(0) + \frac{\mathbf{1}^T\Sigma}{N}\int_0^\infty a(s)dW(s)$. Then, by (A.6) and the Itô equality (Friedman, 1975), we have

$$\begin{aligned} & \lim_{t\to\infty} E(\frac{1}{N}\sum_{j=1}^N x_j(t) - x^*)^2 \\ &= \lim_{t\to\infty} E(\frac{1}{N}\mathbf{1}^T\Sigma\int_t^\infty a(s)dW(s))^2 \\ &= \frac{tr(\Sigma\Sigma^T)}{N^2}\int_t^\infty a^2(s)ds = o(1),\ t\to\infty. \end{aligned} \quad \text{(A.7)}$$

Noticing that

$$\begin{aligned} E(x^*) &= \frac{1}{N}\sum_{j=1}^N x_j(0), \\ Var(x^*) &= E(\frac{1}{N}\mathbf{1}^T\Sigma\int_0^\infty a(s)dW(s))^2 \\ &= \frac{\sum_{j=1}^N\sum_{j\in N_i}\sigma_{ji}^2 a_{ij}^2}{N^2}\int_0^\infty a^2(s)ds, \end{aligned}$$

from (A.7) and Theorem 2, we can get the conclusion of Theorem 3. □