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Formulas for Discrete Time LQR, LQG, LEQG and Minimax LQG Optimal Control Problems ⋆

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Abstract: The purpose of this paper is to provide a unified presentation of the formulas arising in the discrete-time finite-horizon linear Linear Quadratic Regulator problem, the Linear Quadratic Gaussian problem, the Linear Exponential of Quadratic Gaussian problem, and the minimax Linear Quadratic Gaussian problem. For these classes of optimal control problems, the paper presents formulas for optimal policies and optimal cost. This allows for a comparison between these different optimal control problems.

Keywords: Discrete-time systems, linear quadratic regulators, Gaussian noise, optimal control, linear output feedback, Riccati equations, uncertain linear systems.

1. INTRODUCTION

We present formulas for the optimal controller and optimal cost for four classes of discrete-time finite horizon linear optimal control problems: (i) the linear quadratic regulator (LQR), (ii) the linear quadratic Gaussian (LQG), (iii) the linear exponential-of-quadratic Gaussian (LEQG), and (iv) the minimax LQG. In all cases, the formulas are presented for the most general case of the problems under consideration allowing for cross terms in the quadratic cost function and for correlation in the noise covariances.

Our motivation for providing such a uniform presentation of these formulas is that it allows for an easy comparison between these different optimal control problems and thus provides a way of understanding the relationship between these methods. Also, for the general discrete time problems being considered, the formulas being presented can become quite complicated and by providing a comparison between the different classes of problems, we can minimize the chances of errors occurring in the formulas. The LQR and LQG results presented can be found in standard linear optimal control references such as Kwakernaak and Sivan (1972); Whittle (1990). The LEQG and minimax LQG results presented are extensions of results which can be found in the references Whittle (1981, 1990); Collings et al. (1996); Petersen et al. (2000a).

Throughout this paper, $k = 0, 1, \cdots, T-1$ is the time horizon. The transpose of a matrix X is denoted by X', its determinant is denoted by $|X|$, its spectral radius is denoted by $\rho(X)$, and its trace by $\mathrm{Tr}(X)$. The system state is $x \in \mathbf{R}^n$, the n-dimensional Euclidean space, the control input is $u \in \mathbf{R}^m$, the measured output is $y \in \mathbf{R}^p$.

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2. LQR PROBLEM

The system under consideration evolves according to the recursion

$$x_{k+1} = Ax_k + Bu_k; \quad k = 0, 1, \cdots, T-1. \tag{1}$$

The cost functional for the LQR problem is

$$J(u) := \sum_{k=0}^{T-1} c(x_k, u_k) + x_T' M_T x_T, \tag{2}$$

where

$$c(x, u) := x'Mx + u'Nu + 2x'Su. \tag{3}$$

Hypothesis 1.

$$\begin{bmatrix} M & S \\ S' & N \end{bmatrix} \geq 0, \ M_T \geq 0, \ N > 0. \qquad \square$$

We define the *value function* or the optimal cost as a function of the initial state and time by

$$F_T(x) := x' M_T x,$$

and for $k = T-1, \cdots, 1, 0,$

$$F_k(x) := \inf_u \Big[c(x, u) + F_{k+1}(Ax + Bu) \Big]. \tag{4}$$

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Then, we have the result (e.g., see Chapter 3 of Whittle (1990)):

Theorem 2. For the LQR problem defined by (1) and (2): (i). The optimal control is given by

$$u_k^* = K_k x_k; \quad K_k := -(N + B'P_{k+1}B)^{-1}(S' + B'P_{k+1}A), \tag{5}$$

where P_k is given by the Riccati recursion:

$$P_k = M + A'P_{k+1}A - (S' + B'P_{k+1}A)'(N + B'P_{k+1}B)^{-1}(S' + B'P_{k+1}A); \quad P_T = M_T. \tag{6}$$

(ii). The value function has the form $F_k(x) = x'P_kx$. □

Remark: The recursion (6) can be written in the compact form:

$$P_k = f(P_{k+1}),$$

where

$$f(P) := M + A'PA - (S' + B'PA)' \times (N + B'PB)^{-1}(S' + B'PA). \tag{7}$$

It can also be established that

$$f(P) = M - SN^{-1}S' + (A - BN^{-1}S')' \times (BN^{-1}B' + P^{-1})^{-1}(A - BN^{-1}S'),$$
$$K_k = -N^1S' - N^{-1}B' \times (BN^{-1}B' + P_{k+1}^{-1})^{-1}(A - BN^{-1}S'). \tag{8}$$

Note that a version of (8) holds even if P_k is singular. If P_k is singular, we can write $P(I + BN^{-1}B'P)^{-1}$ in place of $(BN^{-1}B' + P^{-1})^{-1}$. Hence, with this convention, we might write the formulas in (8), even if P is singular. We refer to Chapter 3 in Whittle (1990) for more details on the results stated in this section.

3. LQG PROBLEM

3.1 State feedback case

The system is governed by the recursion

$$x_{k+1} = Ax_k + Bu_k + v_{k+1}; \quad k = 0, 1, \cdots, T-1. \tag{9}$$

The system is defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows, $\mathbf{E}$ will denote expectation w.r.t. the probability measure $\mathbb{P}$. The sequence $\{v_k\}$ is i.i.d., with $v_k \sim N(0, \Sigma)$. Furthermore it is assumed that, for each k, x_k is independent of v_{k+1}.

The cost function for the LQG problem is

$$J(u) := \mathbf{E}\Big[\sum_{k=0}^{T-1} c(x_k, u_k) + x_T'M_Tx_T\Big]. \tag{10}$$

If x_0 is normal, so is x_k for $k = 1, 2, \cdots, T$. Let $x_k \sim N(\mu_k, R_k)$. Then, from (9), we obtain

$$\mu_{k+1} = A\mu_k + Bu_k,$$

$$R_{k+1} = \Sigma + AR_kA'.$$

Theorem 3. (See Chapter 3 in Whittle (1990)) For the state feedback LQG problem defined by (9) and (10):
(i). The optimal control is

$$u_k^* = K_kx_k,$$

where K_k is as in (5);
(ii). The value function is

$$\bar{F}_k(x) = x'P_kx + r_k,$$

where P_k is defined by (6), $r_T = 0$, and

$$r_k := \sum_{j=k+1}^{T} \mathrm{Tr}(\Sigma P_j); \quad k = 0, 1, \cdots, T-1. \qquad \square$$

3.2 Output feedback case

The state equation is (9). Let $x_0 \sim N(\mu_0, R_0)$. The observation equation is

$$y_{k+1} = Cx_k + w_{k+1}. \tag{11}$$

The covariance matrix of $[v_k, w_k]'$ is $\Delta = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & \Gamma \end{pmatrix}$.

Hypothesis 4. The matrix $\Gamma > 0$. □

The cost function is again defined as in (10). In this subsection, by μ_k, we mean the conditional expectation of x_k given $y_1, \cdots, y_k$, $u_0, \cdots, u_{k-1}$. The associated conditional covariance matrix is denoted by R_k. Also let $\chi_k = (\mu_k, R_k)$. We then have the Kalman filter recursion (See Chapter 4 in Whittle (1990)):

$$\mu_{k+1} = A\mu_k + Bu_k + H_{k+1}(y_{k+1} - C\mu_k), \tag{12}$$

and the following Riccati recursion for the covariance matrices R_k:

$$R_{k+1} = \Sigma + AR_kA' - (\Upsilon + AR_kC')(\Gamma + CR_kC')^{-1} \times (\Upsilon + AR_kC')', \tag{13}$$

where

$$H_{k+1} := (\Upsilon + AR_kC')(\Gamma + CR_kC')^{-1}. \tag{14}$$

Theorem 5. (See Chapter 4 in Whittle (1990)) For the LQG output feedback problem defined by (9), (11) and(10):
(i). The optimal control is

$$u_k^* = K_k\mu_k;$$

(ii). The value function is a function of the information state $\chi = (\mu, R)$, and is given by

$$\bar{F}_k(\chi_k) = \mu_k'P_k\mu_k + s_k$$

where $s_T = \mathrm{Tr}(R_TM_T)$, and for $k = 0, 1, \cdots, T-1$,

$$s_k = s_T + \sum_{j=k}^{T-1} \mathrm{Tr}\big[R_jM + (\Sigma + AR_jA' - R_{j+1})P_{j+1}\big].$$

In this theorem, K_k, μ_k, P_k, R_k are given by (5), (12), (6), (13) respectively. □

Remark: We can write the recursion (13) in the compact form:

$$R_{k+1} = p(R_k),$$

where

$$p(R) = \Sigma + ARA' - (\Upsilon + ARC')(\Gamma + CRC')^{-1}(\Upsilon + ARC')'.$$

We also have the alternate forms:

$$p(R) = \Sigma - \Upsilon\Gamma^{-1}\Upsilon' + (A - \Upsilon\Gamma^{-1}C) \times (C'\Gamma^{-1}C + R^{-1})^{-1}(A - \Upsilon\Gamma^{-1}C)',$$
$$H_{k+1} = \Upsilon\Gamma^{-1} + (A - \Upsilon\Gamma^{-1}C)(C'\Gamma^{-1}C + R^{-1})^{-1}C'\Gamma^{-1}. \tag{15}$$

In a similar fashion to the remark at the end of Section 2, a version of this alternate form is valid even if R is singular. In this case, one can replace $(C'\Gamma^{-1}C + R^{-1})^{-1}$ with $(RC'\Gamma^{-1}C + I)^{-1}R$. For more details, we refer to Chapter 4 in Whittle (1990).

4. LEQG PROBLEM

4.1 State feedback case

The state equation is (9). The cost functional for the LEQG problem is

$$\gamma_\theta(u) := \frac{2}{\theta}\log \mathbf{E}e^{\frac{\theta}{2}\{\sum_{k=0}^{T-1} c(x_k,u_k) + x_T' M_T x_T\}}. \tag{16}$$

Here θ is the risk-sensitive parameter. Note that $\gamma_0(\cdot) = J(\cdot)$, where J is the cost function of the LQG problem as defined in (10).

As for the LQG Problem, if $x_k \sim N(\mu_k, R_k)$, then from the state equation (9), it follows that

$$\mu_{k+1} = A\mu_k + Bu_k,$$

$$R_{k+1} = \Sigma + AR_kA'.$$

The control Riccati recursion for the LEQG Problem takes the form:

$$P_{\theta,k} = M + A'\tilde{P}_{\theta,k+1}A - (S' + B'\tilde{P}_{\theta,k+1}A)' \times (N + B'\tilde{P}_{\theta,k+1}B)^{-1}(S' + B'\tilde{P}_{\theta,k+1}A),$$
$$\tilde{P}_{\theta,k+1} = (P_{\theta,k+1}^{-1} - \theta\Sigma)^{-1}; \ \ k = 0,1,\cdots,T-1,$$
$$P_{\theta,T} = M_T. \tag{17}$$

Hypothesis 6. The *risk-resistance condition* is satisfied; that is, for all $k = 0,1,\cdots,T-1$,

$$P_{\theta,k+1}^{-1} - \theta\Sigma > 0. \qquad \square$$

Theorem 7. (Chapter 7 in Whittle (1990)) For the LEQG state feedback problem defined by (9) and (16):
(i). The optimal control is

$$u_{\theta,k}^* = K_{\theta,k}x_k,$$

where

$$K_{\theta,k} := -(N + B'\tilde{P}_{\theta,k+1}B)^{-1}(S' + B'\tilde{P}_{\theta,k+1}A);$$

(ii). The value function is

$$F_{\theta,k}(x) = x'P_{\theta,k}x + r_{\theta,k},$$

where $P_{\theta,k}$ is given by the recursion (17), $r_{\theta,T} = 0$, and

$$r_{\theta,k} = -\frac{1}{\theta}\sum_{j=k+1}^{T}\log|I - \theta\Sigma P_{\theta,j}|.; \ \ k = 0,1,\cdots,T-1.$$

$\square$

Remark: The expression for $K_{\theta,k}$ and the recursion (17) have alternative representations in a similar way to those in Sections 2 and 3. More precisely, we can write

$$P_{\theta,k} = M - SN^{-1}S' + (A - BN^{-1}S')' \times (BN^{-1}B' + \tilde{P}_{\theta,k+1}^{-1})^{-1}(A - BN^{-1}S'),$$
$$K_{\theta,k} = -N^1S' - N^{-1}B'(BN^{-1}B' + \tilde{P}_{\theta,k+1}^{-1})^{-1} \times (A - BN^{-1}S'). \tag{18}$$

4.2 Output feedback case

The state equation is (9), and the observation equation is (11). Let $x_0 \sim N(\mu_0, R_0)$. The cost function is (16).

The LEQG estimation Riccati recursion has the from:

$$R_{\theta,k+1} = \Sigma + A\tilde{R}_{\theta,k}A' - (\Upsilon + A\tilde{R}_{\theta,k}C') \times (\Gamma + C\tilde{R}_{\theta,k}C')^{-1}(\Upsilon + A\tilde{R}_{\theta,k}C')',$$
$$\tilde{R}_{\theta,k} := (R_{\theta,k}^{-1} - \theta M)^{-1}; \ \ k = 0,1,\cdots,T-1,$$
$$R_{\theta,0} = R_0. \tag{19}$$

Hypothesis 8. A4 For all $k = 0,1,\cdots,T-1$,

$$R_{\theta,k}^{-1} - \theta M > 0. \qquad \square$$

We then have the filter equation (See Chapter 8 in Whittle (1990)):

$$\mu_{\theta,k+1} = A\tilde{\mu}_{\theta,k} + Bu_k + H_{\theta,k+1}(y_{k+1} - C\tilde{\mu}_{\theta,k}),$$
$$\tilde{\mu}_{\theta,k} := \tilde{R}_{\theta,k}(R_{\theta,k}^{-1}\mu_{\theta,k} + \theta S u_{\theta,k}), \tag{20}$$

where

$$H_{\theta,k+1} := (\Upsilon' + C\tilde{R}_{\theta,k}A')'(\Gamma + C\tilde{R}_{\theta,k}C')^{-1}.$$

We require:
Hypothesis 9. For every k, $\rho(R_{\theta,k}P_{\theta,k}) < \frac{1}{\theta}$. $\square$
Theorem 10. (See Shaiju and Petersen (2007) and Chapters 7, 8 in Whittle (1990)) For the LEQG output feedback problem defined by (9), (11) and (16):
(i). The optimal control is given by

$$u_{\theta,k}^* = K_{\theta,k}(I - \theta R_{\theta,k}P_{\theta,k})^{-1}\mu_{\theta,k};$$

(ii). The value function is a function the information-state $\chi = (\mu_\theta, R_\theta)$, and is given by

$$\bar{F}_{\theta,k}(\mu_{\theta,k}, R_{\theta,k}) = \mu_{\theta,k}'P_{\theta,k}(I - \theta R_{\theta,k}P_{\theta,k})^{-1}\mu_{\theta,k} - \frac{1}{\theta}\log\frac{|R_{\theta,k}||I - \theta R_{\theta,T}M_T|}{|R_{\theta,T}|} - \frac{1}{\theta}\sum_{j=k}^{T-1}\log|\Delta|.|G_{\theta,j}|.|\Psi_{\theta,j}|,$$

where

$$G_{\theta,j} = \tilde{R}_{\theta,j}^{-1} + C'\Gamma^{-1}C + (A - \Upsilon\Gamma^{-1}C)'(\Sigma - \Upsilon\Gamma^{-1}\Upsilon')^{-1}(A - \Upsilon\Gamma^{-1}C),$$

$$\Psi_{\theta,j}^{-1} = \Gamma + C\tilde{R}_{\theta,j}C' - (\Upsilon + A\tilde{R}_{\theta,j}C')' \times (-\frac{\tilde{P}_{j+1}^{-1}}{\theta} + A\tilde{R}_{\theta,j}A')^{-1}(\Upsilon + A\tilde{R}_{\theta,j}C'). \qquad \square$$

Remark: The expression for $H_{\theta,k+1}$ and the recursion (20) have alternate forms:

$$R_{\theta,k+1} = \Sigma - \Upsilon\Gamma^{-1}\Upsilon' + (A - \Upsilon\Gamma^{-1}C) \times (C'\Gamma^{-1}C + \tilde{R}_{\theta,k}^{-1})^{-1}(A - \Upsilon\Gamma^{-1}C)', \tag{21}$$

$$H_{\theta,k+1} = \Upsilon\Gamma^{-1} + (A - \Upsilon\Gamma^{-1}C) \times (C'\Gamma^{-1}C + \tilde{R}_{\theta,k}^{-1})^{-1}C'\Gamma^{-1}. \tag{22}$$

For more details, we refer to Chapter 8 in Whittle (1990) $\square$

5. MINIMAX LQG PROBLEM

5.1 State feedback case

The uncertain system model is

$$x_{k+1} = Ax_k + Bu_k + B_0\xi_k + B_0w_{k+1},$$
$$z_k = C_1x_k + D_1u_k. \tag{23}$$

Here ξ_k is the uncertainty input, and z_k is the uncertainty output (Formally, as in Petersen et al. (2000b), the uncertainties are in probability measures). The noise $w_k \sim N(0, I)$.

Definition 11. An uncertainty $\xi = (\xi_0, \cdots, \xi_{T-1})$ is *admissible* if the following Sum Quadratic Constraint (SQC) is satisfied:

$$\mathbf{E}\sum_{k=0}^{T-1}\|\xi_k\|^2 \le \frac{1}{2}\Big(\mathbf{E}\sum_{k=0}^{T-1}\|z_k\|^2 + d\Big). \tag{24}$$

Here d is a given positive constant. We denote the set of all admissible uncertainties, for a controller $u(\cdot)$, by Ξ_u. $\square$

The cost functional for the minimax LQG problem is

$$L(u(\cdot), \xi(\cdot)) := \frac{1}{2}\mathbf{E}\Big[x_T'M_Tx_T + \sum_{k=0}^{T-1}[x_k'Mx_k + +2x_k'Su_k + u_k'Nu_k]\Big]. \tag{25}$$

Let

$$V := \inf_{u(\cdot)}\sup_{\xi(\cdot)\in\Xi_u} L(u(\cdot), \xi(\cdot))$$

be the minimax cost and for $\tau > 0$ define

$$L_\tau(u(\cdot), \xi(\cdot)) := L(u(\cdot), \xi(\cdot)) + \frac{\tau}{2}\Big(\mathbf{E}\sum_{k=0}^{T-1}[\|z_k\|^2 - 2\|\xi_k\|^2] + d\Big).$$

We note that

$$L_\tau(u(\cdot), \xi(\cdot)) = \frac{1}{2}\mathbf{E}\Big(\sum_{k=0}^{T-1} c_\tau(x_k, u_k) + x_T'M_Tx_T\Big) - \tau\mathbf{E}\sum_{k=0}^{T-1}\|\xi_k\|^2 + \frac{\tau}{2}d,$$

where

$$c_\tau(x, u) := x'M_\tau x + 2x'S_\tau u + u'N_\tau u,$$
$$M_\tau := M + \tau C_1'C_1,$$
$$S_\tau := S + \tau C_1'D_1,$$
$$N_\tau := N + \tau D_1'D_1.$$

Using the duality between relative entropy and free energy (see e.g. Petersen et al. (2000b,c), we can establish that

$$\sup_{\xi(\cdot)\in\mathcal{P}} L_\tau(u(\cdot), \xi(\cdot)) = \frac{1}{2}\Big[\gamma_{1/\tau}(u(\cdot)) + \tau d\Big].$$

Note that here the sup is over all uncertainties in $\mathcal{P}$ and not just admissible uncertainties in Ξ_u. Also note that $\gamma_{1/\tau}$ is the cost function of the risk-sensitive control problem with state given given by

$$x_{k+1} = Ax_k + Bu_k + B_0w_{k+1} \tag{26}$$

and cost functional given by

$$\gamma_{1/\tau}(u(\cdot)) = 2\tau\log\mathbf{E}e^{\frac{1}{2\tau}\Big\{\sum_{k=0}^{T-1} c_\tau(x_k,u_k) + x_T'M_Tx_T\Big\}}. \tag{27}$$

If

$$V_\tau := \inf_{u(\cdot)}\sup_{\xi(\cdot)\in\mathcal{P}} L_\tau(u(\cdot), \xi(\cdot))$$

then, we obtain

$$V_\tau = \frac{1}{2}\Big[\inf_{u(\cdot)}\gamma_{1/\tau}(u(\cdot)) + \tau d\Big].$$

Let

$$\mathcal{T} := \Big\{\tau > 0 \,:\, V_\tau < \infty\Big\}.$$

Hypothesis 12. For every non-anticipating control $u(\cdot)$,

$$\sup_{\xi(\cdot)\in\mathcal{P}} L(u(\cdot), \xi(\cdot)) = \infty \qquad \square$$

As in Section 8.4 of Petersen et al. (2000c) (where the analogous continuous time result is proved), we can prove the following theorem.

Theorem 13. (i). V is finite if and only if $\mathcal{T} \neq \emptyset$.
(ii) If $\mathcal{T} \neq \emptyset$, then $V = \inf_{\tau\in\mathcal{T}} V_\tau$.
(iii). Assume $\mathcal{T} \neq \emptyset$. Let $V = V_{\tau^*}$ and $u^*(\cdot)$ is the optimal control for the risk-sensitive problem defined by (26) and (27) with parameter $\frac{1}{\tau^*}$. Then $u^*(\cdot)$ is the minimax controller for the constrained stochastic optimal control problem.
(iii). If $u^\tau(\cdot)$ is the optimal control for the risk-sensitive problem defined by (26) and (27) with parameter $\frac{1}{\tau}$, then

$$\sup_{\xi(\cdot)\in\Xi_{u^\tau}} L(u^\tau(\cdot), \xi(\cdot)) \le V_\tau. \qquad \square$$

As a consequence, we have the result:

Corollary 14. Let $\check{x}_0 := \mathbf{E}x_0$ and $Y_0 := \mathbf{E}(x_0 - \check{x}_0)(x_0 - \check{x}_0)'$. Assume that,for some $\tau > 0$, the following Riccati recursion has nonnegative definite solution:

$$\begin{aligned} X_k &= M_\tau + A'\tilde{X}_{k+1}A \\ &\quad -(S'_\tau + B'\tilde{X}_{k+1}A)(N_\tau + B'\tilde{X}_{k+1}B)^{-1} \\ &\quad \times(S'_\tau + B'\tilde{X}_{k+1}A); \\ \tilde{X}_{k+1} &:= (X_{k+1}^{-1} - \frac{1}{\tau}B_0B_0')^{-1}, \;\; X_T = M_T. \quad (28) \end{aligned}$$

Suppose we apply the following linear feedback controller (denoted by K) to the uncertain system (23).

$$u_k = K_k^c x_k;$$

where

$$K_k^c := -(N_\tau + B'\tilde{X}_{k+1}B)^{-1}(S'_\tau + B'\tilde{X}_{k+1}A). \quad (29)$$

Then, for every admissible uncertainty $\xi(\cdot)$, we have the cost-bound:

$$\begin{aligned} L(K,\xi(\cdot)) \le V_\tau = \frac{1}{2}\Big[&\check{x}_0'X_0\check{x}_0 + \tau d \\ &-\tau\sum_{k=1}^{T}\log\Big(|I - \frac{1}{\tau}B_0B_0'X_k|\Big)\Big]. \quad \Box \end{aligned}$$

Remark: As in previous sections, the Riccati recursion (28) and the controller (29) can be replaced by their alternate forms given by

$$\begin{aligned} X_k &= M_\tau - S_\tau N_\tau^{-1}S'_\tau + (A - BN_\tau^{-1}S'_\tau)' \times \\ &\quad (BN_\tau^{-1}B' + \tilde{X}_{k+1}^{-1})^{-1}(A - BN_\tau^{-1}S'_\tau), \quad (30) \\ K_k^c &= -N_\tau^{1}S'_\tau - N_\tau^{-1}B'(BN_\tau^{-1}B' + \tilde{X}_{k+1}^{-1})^{-1} \times \\ &\quad (A - BN_\tau^{-1}S'_\tau). \quad (31) \end{aligned}$$

5.2 Output feedback case

The uncertain system model is

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + B_0\xi_k + B_0w_{k+1}, \\ z_k &= C_1x_k + D_1u_k, \\ y_{k+1} &= Cx_k + B_2\xi_k + B_2w_{k+1}. \quad (32) \end{aligned}$$

Here ξ_k is the uncertainty input, and z_k is the uncertainty output (Formally, as in Petersen et al. (2000b), the uncertainties are in probability measures). The noise $w_k \sim N(0, I)$.

The admissible uncertainties $\xi \in \Xi_u$ are defined by the SQC (24). The cost functional is (25). With the minimax cost V, $c_\tau(x,u)$ and L_τ are defined in a similar fashion as in the previous subsection, we again have the result:

$$\sup_{\xi(\cdot)\in\mathcal{P}} L_\tau(u(\cdot),\xi(\cdot)) = \frac{1}{2}\Big[\gamma_{1/\tau}(u(\cdot)) + \tau d\Big].$$

Note that here the sup is over all uncertainties in $\mathcal{P}$ and not just the admissible uncertainties in Ξ_u. Also note that $\gamma_{1/\tau}$ is the cost function of the risk-sensitive control problem with state and observation equations given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + B_0w_{k+1}, \\ y_{k+1} &= Cx_k + B_2w_{k+1}, \quad (33) \end{aligned}$$

and

$$\gamma_{1/\tau}(u(\cdot)) = 2\tau\log\mathbf{E}e^{\frac{1}{2\tau}\Big\{\sum_{k=0}^{T-1}c_\tau(x_k,u_k) + x_T'M_Tx_T\Big\}}. \quad (34)$$

If

$$V_\tau := \inf_{u(\cdot)}\sup_{\xi(\cdot)\in\mathcal{P}} L_\tau(u(\cdot),\xi(\cdot))$$

then, we obtain

$$V_\tau = \frac{1}{2}\Big[\inf_{u(\cdot)}\gamma_{1/\tau}(u(\cdot)) + \tau d\Big].$$

Let

$$\mathcal{T} := \Big\{\tau > 0 \,:\, V_\tau < \infty\Big\}.$$

Hypothesis 15. For every non-anticipating control $u(\cdot)$,

$$\sup_{\xi(\cdot)\in\mathcal{P}} L(u(\cdot),\xi(\cdot)) = \infty. \qquad \Box$$

As in Section 8.4 of Petersen et al. (2000c) (where the analogous continuous time result is proved), we can prove the following theorem.

Theorem 16. (i). V is finite if and only if $\mathcal{T} \neq \emptyset$.
(ii) If $\mathcal{T} \neq \emptyset$, then $V = \inf_{\tau\in\mathcal{T}} V_\tau$.
(iii). Assume $\mathcal{T} \neq \emptyset$. Let $V = V_{\tau^*}$ and $u^*(\cdot)$ is the optimal control for the risk-sensitive problem defined by (33) and (34) with parameter $\frac{1}{\tau^*}$. Then $u^*(\cdot)$ is the minimax controller for the constrained stochastic optimal control problem.
(iii). If $u^\tau(\cdot)$ is the optimal control for the risk-sensitive problem defined by (33) and (34) with parameter $\frac{1}{\tau}$, then

$$\sup_{\xi(\cdot)\in\Xi_{u^\tau}} L(u^\tau(\cdot),\xi(\cdot)) \le V_\tau. \qquad \Box$$

Corollary 17. Let $\check{x}_0 := \mathbf{E}x_0$ and $Y_0 := \mathbf{E}(x_0 - \check{x}_0)(x_0 - \check{x}_0)'$. Let the matrices Δ (defined in (39)) and B_2B_2' be positive definite. Assume that for some $\tau > 0$, the following two Riccati recursions have nonnegative definite solutions:

$$\begin{aligned} Y_{k+1} &= B_0B_0' + A\tilde{Y}_kA' - (B_0B_2' + A\tilde{Y}_kC') \times \\ &\quad (B_2B_2' + C\tilde{Y}_kC')^{-1}(B_0B_2' + A\tilde{Y}_kC')'; \\ \tilde{Y}_k &:= (Y_k^{-1} - \frac{1}{\tau}M_\tau)^{-1}. \quad (35) \end{aligned}$$

$$\begin{aligned} X_k &= M_\tau + A'\tilde{X}_{k+1}A - (S'_\tau + B'\tilde{X}_{k+1}A) \times \\ &\quad (N_\tau + B'\tilde{X}_{k+1}B)^{-1}(S'_\tau + B'\tilde{X}_{k+1}A); \\ \tilde{X}_{k+1} &:= (X_{k+1}^{-1} - \frac{1}{\tau}B_0B_0')^{-1}, \;\; X_T = M_T. \quad (36) \end{aligned}$$

Also assume that $\rho(Y_kX_k) < \tau$, for $k = 0, 1, \cdots, T-1$.

Consider applying the following controller (denoted by K) to the uncertain system (32).

$$\begin{aligned} x_{k+1}^c &= A\tilde{x}_k^c + Bu_k + H_{k+1}^c(y_{k+1} - C\tilde{x}_k^c); \;\; x_0^c = \check{x}_0, \\ u_k &= K_k^c(I - \frac{1}{\tau}Y_kX_k)^{-1}x_k^c; \quad (37) \end{aligned}$$

where

$$H^c_{k+1} := (B_0B_2' + A\tilde{Y}_kC')(B_2B_2' + C\tilde{Y}_kC')^{-1},$$
$$\tilde{x}^c_k = \tilde{Y}_k(Y_k^{-1}x^c_k + \frac{1}{\tau}S_\tau u_k),$$
$$K^c_k := -(N_\tau + B'\tilde{X}_{k+1}B)^{-1}(S_\tau' + B'\tilde{X}_{k+1}A). \quad (38)$$

Then, for every admissible uncertainty $\xi(\cdot)$, we have the cost-bound:

$$\begin{aligned} L(K,\xi(\cdot)) &\le V_\tau \\ &= \frac{1}{2}\Big[\breve{x}_0'X_0(I - \frac{1}{\tau}Y_0X_0)^{-1}\breve{x}_0 + \tau d \\ &\quad -\tau\log\Big(\frac{|Y_0|.|I - \frac{1}{\tau}Y_TX_T|}{|Y_T|}\Big) \\ &\quad -\tau\sum_{k=0}^{T-1}\log\Big(|\Delta|.|G_k|.|\Psi_k|\Big)\Big], \end{aligned}$$

where

$$\Delta = \begin{bmatrix} B_0B_0' & B_0B_2' \\ B_2B_0' & B_2B_2' \end{bmatrix},$$
$$\begin{aligned} G_k = \tilde{Y}_k^{-1} &+ C'(B_2B_2')^{-1}C + (A - B_0B_2'(B_2B_2')^{-1}C)' \times \\ &\Big(B_0(I - B_2'(B_2B_2')^{-1}B_2)B_0'\Big)^{-1} \times \\ &(A - B_0B_2'(B_2B_2')^{-1}C), \end{aligned}$$
$$\begin{aligned} \Psi_k^{-1} = \Gamma &+ C\tilde{Y}_kC' - (B_0B_2' + A\tilde{Y}_kC')' \times \\ &(-\frac{\tilde{X}_{k+1}^{-1}}{\theta} + A\tilde{Y}_kA')^{-1}(B_0B_2' + A\tilde{Y}_kC'). \end{aligned}$$

□

Remark: The recursion (36) and K^c_k have alternate forms (30) and (31) respectively. Analogously the recursion (35) and (38) have the alternate forms:

$$\begin{aligned} Y_{k+1} = B_0&\Big(I - B_2'(B_2B_2')^{-1}B_2\Big)B_0' \\ &+(A - B_0B_2'(B_2B_2')^{-1}C)(C' \times \\ &(B_2B_2')^{-1}C + \tilde{Y}_k^{-1})^{-1}(A - B_0B_2'(B_2B_2')^{-1}C)', \\ &\tilde{Y}_k := (Y_k^{-1} - \frac{1}{\tau}M_\tau)^{-1}. \end{aligned}$$

$$\begin{aligned} H^c_{k+1} = B_0B_2'(B_2B_2')^{-1} &+ (A - B_0B_2'(B_2B_2')^{-1}C) \times \\ &(C'(B_2B_2')^{-1}C + \tilde{Y}_k^{-1})^{-1}C'(B_2B_2')^{-1}. \end{aligned}$$ □

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