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An off-line MPC strategy for nonlinear systems based on SOS programming

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Abstract: A novel moving horizon control strategy for input-saturated nonlinear polynomial systems is proposed. The control strategy makes use of the so called sum-of-squares (SOS) decomposition, i.e. a convexification procedure able to give sufficient conditions on the positiveness of polynomials. The complexity of SOS-based numerical methods is polynomial in the problem size and, as a consequence, computationally attractive. SOS programming is used here to derive an "off-line" model predictive control (MPC) scheme and analyze in depth his properties. Such an approach may lead to less conservative MPC strategies than most existing methods based on the global linearization approach. An illustrative example is provided to show the benefits of the proposed SOS-based algorithm.

Keywords: Sum of Squares, Nonlinear Systems, Predictive Control, Convex Relaxations, Constrained Systems.

1. INTRODUCTION

Model Predictive Control (MPC) in an optimization based control strategy able to efficiently deal with plant constraints. At each control interval, the MPC algorithm computes an open-loop sequence of inputs by minimizing, compatibly with prescribed constraints, a cost index based on future plant predictions. The first input of the optimal sequence is applied to the plant and the entire optimization procedure is repeated at subsequent time instants.

Though almost all processes are inherently nonlinear, the vast majority of MPC applications and results are based on linear or uncertain linear dynamic models (see (Kothare et al. (1996); Mayne et al. (2000); Imsland et al. (2004); Kouvaritakis et al. (2000); Angeli et al. (2001)) and references therein). One of the main reason for this trend is probably related to the huge on-line computational burdens usually resulting from the utilization of nonlinear, in some cases non-convex, programming techniques (Allgöwer et al. (1999); Qin and Badgwell (2000); Chen and Allgöwer (1998); Magni et al. (2001)).

Although such difficulties have been recently ameliorated by the introduction of specialized solvers for nonlinear MPC applications (Bock et al. (2005)), alternative approaches based on convex embedding have been investigated in the past for their simplicity. In particular, uncertain linear models can be identified, as an example, by linearizing the plant at different operating points under the hypothesis that the nonlinear trajectory is embedded inside the tube obtained by evaluating each linear plant vertex (linear embedding, see (Wan and Kothare (2003b)). As a consequence, the use of an uncertain linear model and a quadratic/linear objective function makes it possible to obtain a robust MPC framework which takes the form of a highly structured semidefinite/QP/LP programming problem, for which reliable algorithms and software can easily be found (see (Scherer and Hol (2006); Björnberg and Diehl (2006); Pluymers et al. (2005))).

Nevertheless, there are cases when nonlinear effects are significant enough to justify the use of direct nonlinear MPC (NMPC) technologies. These include at least two broad categories of applications:

- regulation problems where the plant is highly nonlinear and subject to large frequent disturbances;
- servo problems where the set point changes frequently and spans a sufficiently wide range of nonlinear process dynamics.

The purpose of this paper is to consider a particular class of nonlinear plants and constraints described by means of polynomials. The formulation of the MPC problem in such a case gives rise to polynomial optimization problems for which solution efficient numerical methods have been proposed in the literature: Gröbner bases, cylindrical algebraic decomposition etc. (see (Parrillo (2000); Scherer and Hol (2006); Henrion and Garulli (2005); Henrion et al. (2003))).

In particular, SOS decomposition and semidefinite programming (Prajna et al. (2004); Fotiou et al. (2006); Jarvis-Wloszek et al. (2003)) techniques will be used here, whose computational complexity is polynomial in the problem size. Strictly speaking, the SOS-based approach is a powerful convexification method which generalizes the well-known S-procedure (Parrillo (2000)) by searching for polynomial multipliers. As one of its major merits, the SOS-based approach provides less conservative results than most available methods.

On the other hand, the huge computational burden of a SOS program advises against an implementation of the related optimization procedure at each sampling instant. Nonetheless, if the SOS relaxation polynomial problem is off-line moved, the on-line controller could be relatively easier to set up.

Then, we propose an off-line formulation of a Receding Horizon Control (RHC) problem for polynomial systems based on the computation of a nested sequence of asymptotically stable invariant sets (see Wan and Kothare (2003a) where a similar algorithm is detailed for uncertain linear plants). With this off-line approach, the SOS computation time is not a limiting factor and increased control performance can be achieved also for fast processes and large scale nonlinear systems.

NOTATION

With the term $p \in \mathbb{R}[x]$ we denote a multivariate scalar polynomial $p(x)$, in the unknown $x \in \mathbb{R}^n$.

With the term $s \in \Sigma[x]$, where

$$\Sigma[x] := \left\{ s \in \mathbb{R}[x] |\, \exists M < \infty, \exists \{p_i\}_{i=1}^M, p_i \in \mathbb{R}[x], \text{ s.t. } s = \sum_{i=1}^M p_i^2 \right\}$$

we denote a multivariate Sum-of-Squares $s(x)$, in the unknown $x \in \mathbb{R}^n$.

With the term $p \in \mathbb{R}^{n \times m}[x]$ we denote a multivariate n-rows, m-columns matrix polynomial i.e, $p_{ij} \in \mathbb{R}[x]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

2. PROBLEM FORMULATION

Consider the following nonlinear system with polynomial vector field

$$x(t+1) = f(x(t)) + g(x(t))\, u(t) \tag{1}$$

where $f \in \mathbb{R}^n[x]$, $g \in \mathbb{R}^{n \times m}[x]$, with $x \in \mathbb{R}^n$, denoting the state and $u \in \mathbb{R}^m$ denoting the control input subject to the following component-wise saturation constraints

$$u \in \mathcal{U} : \quad |u_i| \le \bar{u}_i, \; i = 1, \ldots, m. \tag{2}$$

It assumed in this paper that $0_x \in \mathbb{R}^n$ is an equilibrium point for (1) with $u = 0$ (Chen and Allgöwer (1998)). The aim is to find a state feedback regulation strategy

$$u(t) = g(x(t)) \tag{3}$$

which asymptotically stabilizes (1) to the origin under (2). We introduce a cost function that penalizes the deviation of the state and control action from zero

$$J(u, x(0)) \triangleq \sum_{t=0}^{\infty} \left(\|x(t)\|^2_{\Psi_x} + \|u(t)\|^2_{\Psi_u} \right) \tag{4}$$

$\Psi_x = \Psi_x^T \ge 0$, $\Psi_u = \Psi_u^T > 0$, and we look for guaranteed cost conditions, under which a state feedback regulation strategy (3) can be derived. It has been proved in (Jarvis-Wloszek et al. (2003)) that if, at a given time instant t, a SOS $V(x(t))$ and a polynomial state-dependent control law $u(t) = K(x(t))$, compatible with (2), are found so that $J(K(x(t)), x(t)) \le V(x(t))$ then the set $\mathcal{E} := \{x \in \mathbb{R}^n | V(x) \le 1\}$ is a positive invariant region for the regulated input constrained plant.

3. COMPUTATION OF $(V(X), K(X))$

The computation of a couple $(V(x), K(x))$ can be accomplished using standard machinery taken from the semi-algebraic sets theory (Cox et al. (2004)). To this end, let P_β an inner polynomial approximation of $\mathcal{E}$, viz. $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \le \beta\}$, $P_\beta \subseteq \mathcal{E}$, for some given positive definite polynomial $p \in \mathbb{R}[x]$. Then, the positive invariant set $\mathcal{E}$ can be derived by finding a value of β such that all points in P_β converge to the origin under the control law K and saturation constraints (2). Using a Hamilton-Jacobi-Bellman (HJB) inequality argument, guaranteed cost conditions ensuring satisfaction of input constraints and closed-loop stability can be stated as: *given an initial state $x_0 \in \mathbb{R}^n$, find a scalar β, a SOS $V \in \Sigma[x]$ and polynomial $K \in \mathbb{R}[x]$ such that the following set of inclusions hold*

$$\begin{cases} V(x) > 0 \,\forall\, x \in \mathbb{R}^n \setminus \{0\} \text{ and } V(0) = 0 \\ \{x \in \mathbb{R}^n | p(x) \le \beta\} \subseteq \{x \in \mathbb{R}^n | V(x) \le 1\} \\ \{x \in \mathbb{R}^n | V(x) \le 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | V(x(t+1)) - V(x(t)) \\ \quad + x'\,\Psi_x\, x + K(x)'\,\Psi_u\, K(x) < 0\} \\ \{x \in \mathbb{R}^n | V(x) \le 1\} \subseteq \{x \in \mathbb{R}^n | K_i(x) \le \bar{u}_i, \; i = 1, \ldots, m\} \\ \{x \in \mathbb{R}^n | V(x) \le 1\} \subseteq \{x \in \mathbb{R}^n | K_i(x) \ge -\bar{u}_i, i = 1, \ldots, m\} \\ \left\{x \in \mathbb{R}^n | (x - x_0)^T (x - x_0) \le \varepsilon \right\} \subseteq \{x \in \mathbb{R}^n | V(x) \le 1\} \end{cases} \tag{5}$$

The above implications respectively show that V is positive definite, P_β is contained in a level set of V, the HJB difference: $V(x(t+1)) - V(x(t)) + x'\,\Psi_x\, x + K(x)'\,\Psi_u\, K(x)$ is strictly negative on all points contained in the level set aside from $x = 0_x$ and the saturation input constraints are satisfied. The last inclusion in the previous collection rephrases the belonging of the initial state x_0 to the invariant region $V(x) \le 1$, to be computed as a relaxed semialgebraic subset for a sufficiently small scalar $\varepsilon > 0$.

By resorting to a "Positivstellensatz" (P-satz) (Cox et al. (2004)) argument, verification of (5) is equivalent to emptyness of

$$\begin{cases} \{x \in \mathbb{R}^n | V(x) \le 0, \; l_1(x) \ne 0\} \\ \{x \in \mathbb{R}^n | p(x) \le \beta, \; V(x) \ge 1, \; V(x) \ne 1\} \\ \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{c} V(x) \le 1, \; l_2(x) \ne 0, \\ V(x(t+1)) - V(x(t)) + x'\Psi_x x + K(x)'\Psi_u K(x) \ge 0 \end{array} \right\} \\ \{x \in \mathbb{R}^n | K_i(x) \ge \bar{u}_i, \; i = 1, \ldots, m, \; V(x) \le 1, \} \\ \{x \in \mathbb{R}^n | K_i(x) \le -\bar{u}_i, \; i = 1, \ldots, m, \; V(x) \le 1, \} \\ \left\{x \in \mathbb{R}^n | (x - x_0)^T (x - x_0) \le \varepsilon, V(x) \ge 1 \right\} \end{cases} \tag{6}$$

where l_1, $l_2 \in \mathbb{R}[x]$ are positive definite polynomials. Moreover, the inclusions inside (6) can be removed by using the S-procedure and the related conditions can be modified as: *find $\beta \in \mathbb{R}, K \in \mathbb{R}[x]$, $V \in \Sigma[x]$, k_1, k_2, $k_3 \in \mathbb{Z}^+, s_1, \ldots, s_{10}$, $s_{11,i}$, $s_{12,i}$, $s_{14} \in \Sigma[x]$, $i = 1, \ldots, m$ such that*

$$\begin{cases} s_1 - V\,s_2 + l_1^{2\,k_1} = 0 \\ s_3 + (\beta - p)\,s_4 + (V-1)\,s_5 + (\beta - p)\,(V-1)\,s_6 \\ +(V-1)^{2\,k_2} = 0 \\ s_7 + (1-V)\,s_8 + (V(f(x,K(x)) - V(x) + x'\,\Psi_x\,x \\ +K(x)'\,\Psi_u\,K(x))\,s_9 + (1-V)\,(V(f(x,K(x)) - V(x) \\ +x'\,\Psi_x\,x + K(x)'\,\Psi_u\,K(x))\,s_{10} + l_2^{2\,k_3} = 0 \\ (\bar{u}_i - K_i) - (1-V)\,s_{11,i} = 0,\ i = 1,\ldots,m \\ (\bar{u}_i + K_i) - (1-V)\,s_{12,i} = 0,\ i = 1,\ldots,m \\ (-\varepsilon + (x-x_0)^T\,(x-x_0))\,s_{14} + (1-V) = 0 \end{cases} \quad (7)$$

Because (7) cannot be treated with a semidefinite programming procedure (Jarvis-Wloszek et al. (2003)), we will employ an iterative approach to solve the above SOS conditions in the unknowns V and K, consisting of holding one of these polynomials fixed while adjusting the other. Nonetheless, (7) can be further simplified by removing the multipliers s_i, $i = 1,\ldots,5$ and s_7 (see (Jarvis-Wloszek et al. (2003))). The following proposition summarizes the previous discussion.

Proposition 1. A pair $(V(x), K(x))$ satisfying conditions (5) is determinable if there exist $\beta \in \mathbb{R}, K \in \mathbb{R}[x]$, $V \in \Sigma[x]$, k_1, k_2, $k_3 \in \mathbb{Z}^+, s_6, s_8, s_9, s_{10}, s_{11,i}, s_{12,i}, s_{14} \in \Sigma[x]$, $i = 1,\ldots,m$ can be found such that

$$V - l_1 \in \Sigma[x] \quad (8)$$

$$-\left((\beta - p)\,s_6 + (V-1)\right) \in \Sigma[x] \quad (9)$$

$$-\Big((1-V)\,s_8 + (V(f(x,K(x)) - V(x) + x'\,\Psi_x\,x + K(x)'\,\Psi_u\,K(x))\,s_9 + l_2^{2\,k_3}\Big) \in \Sigma[x] \quad (10)$$

$$(\bar{u}_i - K_i) - (1-V)\,s_{11,i} \in \Sigma[x], \quad i = 1,\ldots,m \quad (11)$$

$$(\bar{u}_i + K_i) - (1-V)\,s_{12,i} \in \Sigma[x], \quad i = 1,\ldots,m \quad (12)$$

$$-\varepsilon + (x - x0)^T (x - x0)\,s_{14} + (1-V) \in \Sigma[x] \quad (13)$$

under the following necessary requirements on the degrees of the involved polynomials

$$\begin{cases} \partial\,(V) = \partial\,(l_1) \\ \partial\,(p\,s_6) \geq \partial\,(V) \\ \max\left(\partial\,(V\,s_8)\,,\partial\,(V\,s_9)\right) \geq \\ \max\left(\partial\,(V(f(x),K(x))s_9)\,,\partial\left(x^T\Psi_x x s_9\right),\partial\left(K(x)^T\Psi_u K(x) s_9\right)\right) \\ \partial\,(V\,s_{11,i}) \geq \partial\,(K_i)\,,\ i = 1,\ldots,m \\ \partial\,(s_{14}) + 2 \geq \partial\,(V) \end{cases}$$

Proof - By referring to standard SOS arguments (Parrillo (2000)) and collecting the above derivations. □

Note that the decision polynomials s_6, s_8, s_9, $s_{11,i}$, $s_{12,i}$, $s_{14} \in \Sigma[x]$, $i = 1,\ldots,m$ do not enter linearly in the constraints, so we will employ an iterative algorithm to achieve a feasible solution.

4. A SOS PROGRAMMING ALGORITHM FOR THE COMPUTATION OF $(V(X), K(X))$

In what follows, we propose a numerical bisection procedure to derive an admissible pair (V, K) matching the conditions outlined in the statement of *Proposition 1*.

Procedure - SOS-V-K(x)
Setup: Given the initial state $x(0) \in \mathbb{R}^n$. Set $l_1 = \varepsilon \sum x_i^m$ for some small $\varepsilon > 0$ and m the maximum degree of V. Each step of the iteration, indexed by k, consists of four substeps, three of which also involve iterations. These inner iterations will be indexed by j. The procedure starts by choosing a V_0, that is a Control Lyapunov Function (CLF) (Henrion and Garulli (2005)) for the linearized system around the desired set point and a control law $K_0(x)$ that asymptotically stabilizes (1). The algorithm is initialized with $V^{(k=0)} = V_0$, $K^{(k=0)} = K_0$.

(1) **Polynomials derivation** - Set $V = V^{(k-1)}$, $K = K^{(k-1)}$, and the inner iteration index $j = 1$. Solve the following SDP:

$$\left\{\hat{s}_6^k,\ \hat{s}_8^k,\ \hat{s}_9^k, \{\hat{s}_{11,i}^k,\ \hat{s}_{12,i}^k\}_{i=1}^m\right\} := \arg \max_{\substack{s_6,s_8,s_9,s_{11,i},s_{12,i} \in \Sigma[x],\\ i=1,\ldots,m}} \beta \quad (14)$$

s.t.

$$-\left((\beta - p)\,s_6 + (V-1)\right) \in \Sigma[x] \quad (15)$$

$$-\Big((1-V)\,s_8 + (V(f(x,K(x)) - V(x) + x'\,\Psi_x\,x + K(x)'\,\Psi_u\,K(x))\,s_9 + l_2\Big) \in \Sigma[x] \quad (16)$$

$$(\bar{u}_i - K_i) - (1-V)\,s_{11,i} \in \Sigma[x], i = 1,\ldots,m \quad (17)$$

$$(\bar{u}_i + K_i) - (1-V)\,s_{12,i} \in \Sigma[x], i = 1,\ldots,m \quad (18)$$

(2) **Controller synthesis** - Set $V = V^{(k-1)}$, solve the following SDP:

$$\left\{\hat{\beta}_k, \hat{s}_8^k,\ \hat{s}_{11,i}^k,\ \{\hat{s}_{11,i}^k,\ \hat{s}_{12,i}^k\}_{i=1}^m,\ \hat{K}\right\} := \arg \max_{s_8,s_{11,i},s_{12,i},s_{14} \in \Sigma[x],\ \ K \in \mathcal{R}[x]} \beta \quad (19)$$

s.t.

$$-\left((\beta - p)\,s_6 + (V-1)\right) \in \Sigma[x] \quad (20)$$

$$-\Big((1-V)\,s_8 + (V(f(x,K(x)) - V(x) + x'\,\Psi_x\,x + K^{k-1}(x)'\,\Psi_u\,K^{k-1}(x))\,\hat{s}_9 + l_2\Big) \in \Sigma[x] \quad (21)$$

$$(\bar{u}_i - K_i) - (1-V)\,s_{11,i} \in \Sigma[x], \quad i = 1,\ldots,m \quad (22)$$

$$(\bar{u}_i + K_i) - (1-V)\,s_{12,i} \in \Sigma[x], \quad i = 1,\ldots,m \quad (23)$$

$$(-\varepsilon + (x - x(0))^T (x - x(0)))\,s_{14} + (1-V) \in \Sigma[x] \quad (24)$$

(3) **Lyapunov Function Synthesis** - Solve the following SDP:

$$\left\{\hat{\alpha}_k,\ \hat{s}_6^k, \hat{s}_8^k,\ \{\hat{s}_{11,i}^k,\ \hat{s}_{12,i}^k\}_{i=1}^m,\ \hat{s}_{14}^k,\ \hat{V}_k\right\} := \arg \min_{\alpha_k,\ s_6,\ s_8,\ s_{11,i}, s_{12,i},\ s_{14},\ V\in\Sigma[x]} \alpha \quad (25)$$

s.t.

$$V - l_1 \in \Sigma[x] \quad (26)$$

$$-\left((\hat{\beta}_k - p)\, s_6 + (V - \alpha)\right) \in \Sigma[x] \quad (27)$$

$$-\Big((\alpha - V)\, \hat{s}_8 + (V(f(x, \hat{K}(x)) - V(x) + x'\, \Psi_x\, x + \hat{K}(x)'\, \Psi_u\, \hat{K}(x))\, s_9 + l_2\Big) \in \Sigma[x] \quad (28)$$

$$(\bar{u}_i - \hat{K}_i) - (\alpha - V)\, \hat{s}_{11,i} \in \Sigma[x], \quad i = 1, \ldots, m \quad (29)$$

$$(\bar{u}_i + \hat{K}_i) - (\alpha - V)\, \hat{s}_{12,i} \in \Sigma[x], \quad i = 1, \ldots, m \quad (30)$$

$$(-\varepsilon + (x - x(0))^T (x - x(0)))\, s_{14} + (\alpha - V) \in \Sigma[x] \quad (31)$$

redefine $\hat{V}_k := \hat{V}_k / \hat{\alpha}_k$,

(4) **Stopping Criterion** If

$$\left|\hat{\beta}_k - \hat{\beta}_{k-1}\right| \leq \varepsilon$$

stop else $k \leftarrow k + 1$ and goto Step 1.

5. A LOW-DEMANDING RECEDING HORIZON CONTROL ALGORITHM

This section is devoted to exploit the proposed procedure **SOS-V-K**(x) within a RHC framework. To this end, we will follow the algorithm proposed in (Wan and Kothare (2003a)) for the robust linear case.

We will first state the following problem:

RHC Problem - Given the non-linear system (1), determine, at each time instant, on the basis of the current state $x(t)$, a stabilizing control law $K(x((t))$ and a positive invariant region

$$\mathcal{E} := \{x \in \mathbb{R}^n | V(x) \leq 1\}$$

which minimizes a suitable upper bound V

$$J(K(x((t)), x(t)) \leq V(x(t))$$

to the cost index (4) under the prescribed constraints (2). □

All the arguments developed in the previous sections allows one to write down a computable RHC scheme,denoted as **WK-SOS**, which consists of the following algorithm:

Algorithm-WK-SOS

Off-line

0.1 Given an initial feasible state x_1, put $r = 1$

0.2 Generate a sequence of control laws $K_r(\cdot)$, invariant regions $\mathcal{E}_r$ by solving the SOS program **SOS-V-K**(x_r) with the additional constraint $\mathcal{E}_r \subset \mathcal{E}_{r-1}$ translated as an extra SOS condition in the above **Lyapunov function synthesis** phase (3)

$$-((\alpha - V)\, s_{16} + (V_{k-1} - 1)) \in \Sigma[x] \quad (32)$$

$s_{16} \in \Sigma[x]$, $\partial\,(V\, s_{16}) \geq \partial\,(V_{k-1})$

0.3 Store in a lookup table $K_r(\cdot)$, $\mathcal{E}_r(\cdot)$;

0.4 If $r < N$, choose a new state x_{r+1} s.t. $x_{r+1} \in \mathcal{E}_r$, Let $r = r + 1$ and go to step 0.2

On-line

1.1 Given an initial feasible state $x(0)$ s.t. $x(0) \in \mathcal{E}_1$, put $t = 0$;

1.2 Perform a bisection search over the sets $\mathcal{E}_r$ in the look-up table to find

$$\hat{r} := \arg\min_r \quad \text{s.t.} \quad x(t) \in \mathcal{E}_r$$

1.3 Feed the plant by the input

$$K_{\hat{r}}(x(t)) \quad (33)$$

Compute the future state

$$x(t + 1) = f(x(t)) + g(x(t))\, K_{\hat{r}}(x(t)) \quad (34)$$

1.4 $t \leftarrow t + 1$ and go to step 1.2

Next lemma ensures that the proposed MPC scheme admits a feasible solution at each time t and the SOS-based input strategy (33) is a stabilizing control law for (1) under (2).

Lemma 1. Given the system (1), let the off-line steps of proposed scheme have solution at time $t = 0$. Then, it has solution at each future time instant t, satisfies the input constraints and yields an asymptotically stable closed-loop system.

Proof - Existence of the sequence of $K_r, \mathcal{E}_r$ ensures that any initial state $x(0) \in \mathcal{E}_1$ can be steered to the origin without constraints violation. In particular, because of the additional constraint $\mathcal{E}_r \subseteq \mathcal{E}_{r-1}$, the regulated state trajectory emanating from the initial state satisfies

$$x(t{+}1) = \begin{cases} f(x(t)) + g(x(t))\, K_r(x(t)) & \text{if } \begin{cases} x(t) \in \mathcal{E}_r \\ x(t) \notin \mathcal{E}_{r+1} \\ r \neq N \end{cases} \\ f(x(t)) + g(x(t))\, K_N(x(t)) & \text{if } x(t) \in \mathcal{E}_N \end{cases} \quad (35)$$

Then, under both conditions $x(t) \in \mathcal{E}_r$ and $x(t) \notin \mathcal{E}_{r+1}$, $r = 1, \ldots, N - 1$, the control law $K_r(\cdot)$ is guaranteed to ultimately drive the state from $\mathcal{E}_r$ into the ellipsoid $\mathcal{E}_{r+1}$ because the Lyapunov difference $V(f(x, \hat{K}_r(x)) - V(x)$ is strictly negative (eq. (28)). Finally, the positive invariance of $\mathcal{E}_N$ and contractivity of K_N guarantee that the state remains within $\mathcal{E}_N$ and converges to the desired set point. □

6. ILLUSTRATIVE EXAMPLE

The aim of this section is to give a measure of the improvements achievable by exploiting the SOS programming framework. To this end, the RHC algorithm of (Wan and Kothare (2003a)) **Algorithm-WK** will be contrasted with the **Algorithm-WK-SOS**, described in Section 3. The simulations are instrumental to show especially the reduction of conservativeness in terms of achievable basins of attraction, when compared with its linear counterpart (**WK**). All the computations have been carried out on a PC Pentium 4-based using the YALMIP Toolbox (Löfberg (2004)).

A controlled Van Der Pol nonlinear oscillator (El Ghaoui (1994)) is taken into consideration

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) - (1 - x_1^2(t))\, x_2(t) + u(t) \end{cases} \quad (36)$$

It has an unstable limit cycle and a stable origin. The problem of finding its region of attraction have been extensively studied (see Chiang and Thorpe (1989) and references

therein), and more recently in (Papachristodoulou (2005)) where SOS programming and polynomial Lyapunov functions have been used to find a provable basin of attraction. The system (36) has been discretized using forward Euler differences with a sampling time $T_c = 0.1$ sec. It has been assumed: weighting matrices $R_x = diag\,([0.01\ 0.01])$, and $R_u = 1$, input saturation constraint $|u(t)| \leq 0.2,\ \ \forall t$.
As well-known, the region of attraction is enclosed by its limit cycle, therefore initially we have chosen the candidate function $p(x)$ as $p(x) = x_1^2 + x_2^2$. The other design knobs are here summarized:

- Candidate Lyapunov function degree: $\partial\,(V(x)) = 6$;
- Candidate stabilizing controller degree: $\partial\,(K(x)) = 4$.

The SOS free parameter polynomials have been chosen such that

$$\partial\,(s_6) = 2,\ \partial\,(s_8) = 4,\ \partial\,(s_9) = 0,\ \partial\,(s_{11}) = 2,$$
$$\partial\,(s_{12}) = 2,\ \partial\,(s_{14}) = 2,\ \partial\,(s_{16}) = 2,$$

in order to satisfy the degree solvability conditions (14). Finally in (32), the quantity ε has been chosen equal to 10^{-8}.
Fig. 1 reports the basins of attraction for the two control schemes. As expected, **WK-SOS** (continuous line) enjoys an enlarged region of feasible initial states w.r.t. **WK**(dashed). Moreover, when one computes the basins of attraction under the more stringent saturation constraint $|u(t)| \leq 0.15$ it results that no solutions exist for the **WK** algorithm whereas a restricted region (dotted in Fig. 1) is found for **WK-SOS**

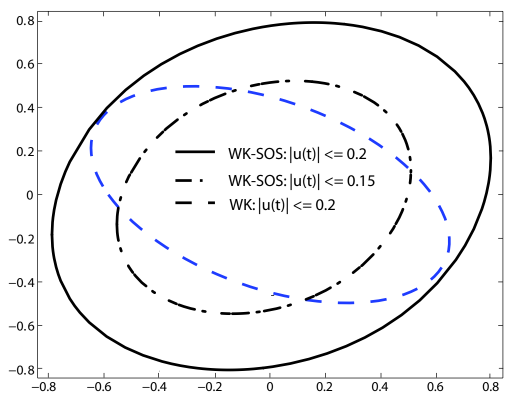

Fig. 1. State Attraction Region with input bound constraints - **WK-SOS**, $|u(t)| \leq 0.2$ (Continuous line), $|u(t)| \leq 0.15$ (Dash-dotted line); **WK**, $|u(t)| \leq 0.2$ (Dashed line)

In the following, only the input constraint $|u(t)| \leq 0.2$ will be considered. Four pairs $(K_i, \mathcal{E}_i)$ have been determined with the **SOS-V-B(x)** algorithm initialized each time with four states of components respectively $x_1^{set} = [0.4\ 0.3\ 0.15\ 0.1]$ and $x_2^{set} = [0\ 0\ 0\ 0]$, and the initial state has been set to $x(0) = [\,0.25\ 0.68\,]^T$.

Finally, Figs. 4-5 depict respectively: the cost, the regulated phase portraits (with four invariant regions $\mathcal{E}_i$) and time evolutions of the state and the input for the two schemes.

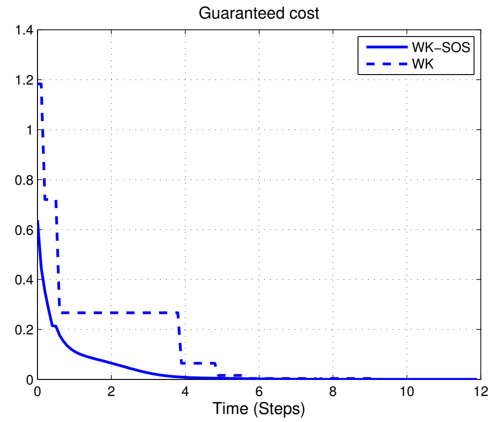

Fig. 2. Guaranteed cost - **WK-SOS** - Continuous line, **WK** - Dashed line

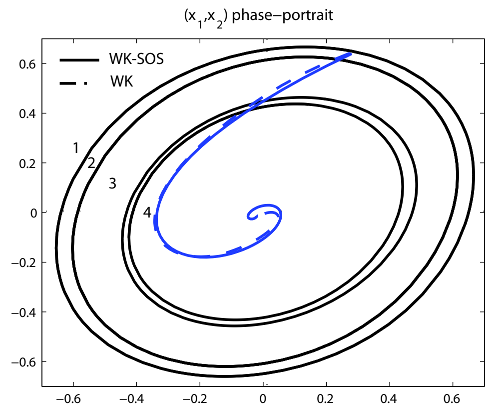

Fig. 3. Phase portrait with input bound constraints $|u(t)| \leq 0.2$.

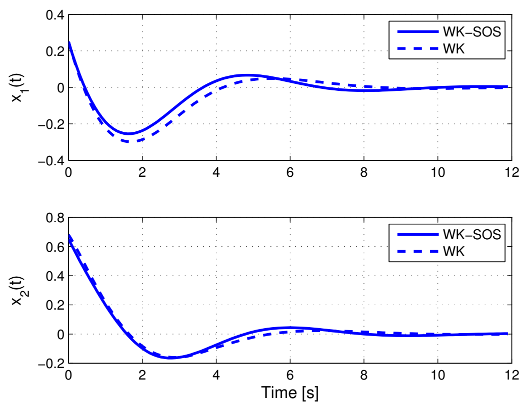

Fig. 4. State evolutions with input bound constraints $|u(t)| \leq 0.2$

7. CONCLUSIONS

In this paper, we have developed an off-line RHC algorithm for constrained polynomial nonlinear systems by means of SOS programming. The advantage of this algorithm is that it provides off-line a set of stabilizing polynomial control laws, corresponding to a nested set of positive invariant regions. Up to our knowledge this is a first attempt in

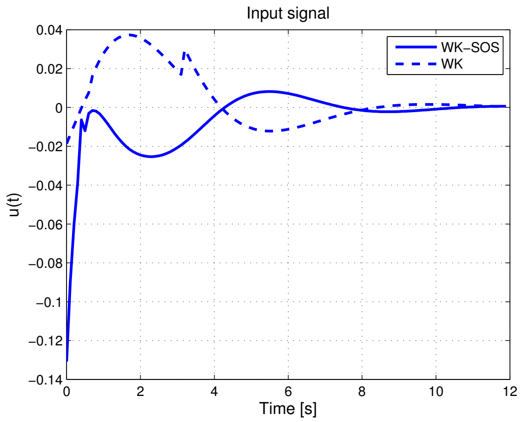

Fig. 5. Input signal with input bound constraints $|u(t)| \leq 0.2$

literature to formulate a RHC problem using SOS machinery. Numerical experiments have shown the benefits of the proposed RHC strategy w.r.t. linear embedding MPC schemes and makes SOS based MPC schemes potentially attractive.

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