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Continuous- and Discrete-Time Path-Following Design of Mixed H2 / H∞ State-Feedback Controllers

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Abstract: Among the few methods available to solve bilinear matrix inequalities (BMIs) occurring in control design, the path-following method, published some years ago for continuous-time systems, appears to be one of the best approaches, as far as linearization methods are concerned. However few details are given in the literature about its implementation. Here, this method is applied to the design of mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ controllers, for continuous-time systems as well as for discrete-time systems, with full details of the algorithm and some improvements over the one which has been published for this kind of application in the continuous-time case a few years ago. The results obtained in both cases with a numerical example are compared with those given by a direct BMI-solving program.

1. INTRODUCTION

The mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ control problem consists in the design of a controller which minimizes the $\mathcal{H}_2$ norm of a given closed-loop transfer function while satisfying an $\mathcal{H}_\infty$ norm constraint on the same or some other closed-loop transfer function.

This mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ control problem has been introduced in the early 1990's by Khargonekar and Rotea (1991) and by Kaminer, Khargonekar and Rotea (1993), who transformed the problem of optimal control with robust stability constraint of Bernstein and Haddad (1989) into a convex optimization approach. Zhou *et al.* (1990) and Doyle *et al.* (1994) have proposed a solution based on coupled Riccati equations, which may not be very easy to solve.

The formulation of this problem by means of linear matrix inequalities (LMIs) and bilinear matrix inequalities (BMIs) has been established later, for continuous-time systems and state-feedback first (Boyd *et al.* 1994), then extended to output feedback (Chilali & Gahinet 1996; Scherer & Gahinet 1997; Leibfritz 2001), and was later applied to discrete-time systems (Hindi, Hassibi & Boyd 1998; de Oliveira, Geromel & Bernussou 1999). A nice compact presentation of the continuous and the discrete cases was given more recently by Kanev *et al*. (2003). Du *et al*. (2005) have proposed an LMI approach to the design of mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ control for discrete-time systems, based on the introduction of additional slack variables, at the cost however of an increased formulation complexity, and applied it successfully to the design of disk drives.

The formulation of the mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ control problem involves BMIs. In order to solve these matrix inequalities without having recourse to a BMI-solving program, an elegant step-by-step method, implying linearization at its central step, the *Path-Following Method*, has been published some years ago (Hassibi, How & Boyd 1999). One of the applications of this paper dealt with the design of mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ state-feedback controllers for continuous-time systems. A significant advantage of this approach, to solve the mixed control problem in terms of LMIs, is that it does not impose the use of a single Lyapunov matrix for both the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ performances, which introduces usually conservatism in the design. The path-following algorithm for that design had however not been given in details in the paper of Hassibi, How and Boyd (1999), and it was also limited to continuous-time systems.

The purpose of this paper is to present an improved version of this algorithm and its extension to discrete-time systems, with full details and an additional feature, which influences the speed and accuracy of the convergence towards a solution by means of an automatic adaptation of the perturbation sizes in the linearization step.

The paper is organized as follows. Section 2 describes the $\mathcal{H}_2 / \mathcal{H}_\infty$ control problem formulation and our path-following algorithm in detail. The continuous case and the discrete case are treated simultaneously, to ease comparisons of the formulas and avoid repetitions. Section 3 applies this algorithm, in the continuous case, to the numerical example of Hassibi, How and Boyd (1999) and compares the results with those given by these authors. Section 4 illustrates the application of the discrete-time version of our algorithm to an academic example. The results of Sections 3 and 4 are also compared with those obtained with a direct BMI-solving software. Finally Section 5 will conclude this work with some comments.

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2. CONTINUOUS-TIME MIXED $\mathcal{H}_2$ / $\mathcal{H}_\infty$ CONTROLLER DESIGN

2.1 Problem Formulation

Consider the following linear system:

$$\begin{cases} \sigma x = Ax + B_w w + B_u u \\ z = C_z x + D_{zw} w + D_{zu} u \\ y = C_y x + D_{yw} w \end{cases} \tag{1}$$

where σ represents the time-derivative operator, $\sigma x(t) = dx(t)/dt$, for continuous-time systems, and the one sample-time forward-shift operator, $\sigma x(t) = x(t+1)$, for discrete-time systems, and where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $z(t) \in \mathbb{R}^{n_z}$ is the controlled outputs vector, $y(t) \in \mathbb{R}^{n_y}$ contains the measured outputs, $u(t) \in \mathbb{R}^{n_u}$ is the control signal and $w(t) \in \mathbb{R}^{n_w}$ is a vector of exogenous inputs, which may be reference, disturbance or noise signals. It will be assumed in the remaining of this paper that there exists no direct path from control input and measured output, $D_{yu} = 0$, as is the case in most practical situations.

In the following, only static state feedback control will be taken into account, which is obtained by setting, in (1), $C_y = I$, $D_{yw} = 0$, which yields $y(t) = x(t)$, and $u(t) = Kx(t)$. The closed loop has then the following state-space description:

$$\begin{cases} \sigma x = (A + B_u K)x + B_w w = \tilde{A}x + B_w w \\ z = (C_z + D_{zu}K)x + D_{zw} w = \tilde{C}x + D_{zw} w \end{cases} \tag{2}$$

where the closed-loop matrices $\tilde{A}$ and $\tilde{C}$ have been defined.

The aim is to compute the gain matrix K such that the $\mathcal{H}_2$ norm $\eta = \left\| L_2(\mathcal{T}(s) - D_{zw})R_2 \right\|_2$ of the closed-loop transfer function from an input to an output, determined respectively by the selection matrices R_2 and L_2, is minimized, while the closed-loop $\mathcal{H}_\infty$ norm on an input-output channel determined by R_∞ and L_∞ is less than some imposed level γ, thus satisfies the inequality $\left\| L_\infty \mathcal{T}(s) R_\infty \right\|_\infty < \gamma$

According to Kanev *et al*. (2003), from whom we borrow the use of the • symbol to denote matrix entries that follow from symmetry, and that of boldface letters to denote variable matrices (decision variables) appearing in the inequalities, the BMI optimization formulation is as follows, for the continuous-time and the discrete-time cases:

Continuous case :

minimize η^2

subject to

$$\boldsymbol{P}_1 \succ 0,\ \boldsymbol{P}_2 \succ 0,$$

$$\mathcal{H}_\infty : \begin{bmatrix} \tilde{\boldsymbol{A}}^T \boldsymbol{P}_1 + \boldsymbol{P}_1 \tilde{\boldsymbol{A}} & \boldsymbol{P}_1 B_w R_\infty & \tilde{\boldsymbol{C}}^T L_\infty^T \\ \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0, \quad \text{(a)}$$

$$\mathcal{H}_2 : \begin{cases} \begin{bmatrix} \tilde{\boldsymbol{A}}^T \boldsymbol{P}_2 + \boldsymbol{P}_2 \tilde{\boldsymbol{A}} & \boldsymbol{P}_2 B_w R_2 \\ \bullet & -I \end{bmatrix} \prec 0, & \text{(b)} \\ \begin{bmatrix} \boldsymbol{P}_2 & \tilde{\boldsymbol{C}}^T L_2^T \\ \bullet & \boldsymbol{Z} \end{bmatrix} \succ 0,\ \mathrm{Tr}(\boldsymbol{Z}) < \eta^2. & \text{(c)} \end{cases} \tag{3}$$

Discrete case :

minimize η^2

subject to

$$\boldsymbol{P}_1 \succ 0,\ \boldsymbol{P}_2 \succ 0,$$

$$\mathcal{H}_\infty : \begin{bmatrix} -\boldsymbol{P}_1 & \boldsymbol{P}_1 \tilde{\boldsymbol{A}} & \boldsymbol{P}_1 B_w R_\infty & 0 \\ \bullet & -\boldsymbol{P}_1 & 0 & \tilde{\boldsymbol{C}}^T L_\infty^T \\ \bullet & \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0, \quad \text{(a)}$$

$$\mathcal{H}_2 : \begin{cases} \begin{bmatrix} -\boldsymbol{P}_2 & \boldsymbol{P}_2 \tilde{\boldsymbol{A}} & \boldsymbol{P}_2 B_w R_2 \\ \bullet & -\boldsymbol{P}_2 & 0 \\ \bullet & \bullet & -I \end{bmatrix} \prec 0, & \text{(b)} \\ \begin{bmatrix} \boldsymbol{P}_2 & \tilde{\boldsymbol{C}}^T L_2^T \\ \bullet & \boldsymbol{Z} \end{bmatrix} \succ 0,\ \mathrm{Tr}(\boldsymbol{Z}) < \eta^2. & \text{(c)} \end{cases} \tag{4}$$

In both of these inequalities, the matrices $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{C}}$ are also typed in boldface since they contain the variable matrix $\boldsymbol{K}$.

2.2 Path-Following Algorithm Steps

The path-following algorithm used to solve the BMI (3) is basically the approach of Hassibi, How and Boyd (1999) for the continuous-time case and uses the same five steps. However it will be described here with full details and contains a few corrections of some of their formulas and an improvement in terms of speed and accuracy of convergence (Ostertag 2008). Furthermore this algorithm will be extended here to discrete-time systems, as already mentioned.

Step 1: Initialization

An initial, suboptimal value of $\boldsymbol{K}$ is computed according to the method of Khargonekar and Rotea (1991), with the assumption that a common Lyapunov matrix is searched for the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ problems ($\boldsymbol{P}_2 = \boldsymbol{P}_1$). This restrictive choice permits the bilinear matrix inequalities of (3) to be converted to equivalent LMIs. This is done in the continuous case by multiplying (3)(a) from right and left by the symmetric matrix $\mathrm{diag}(\boldsymbol{P}_1^{-1}, I, I)$ and its transpose, (3)(b) and (3)(c) by $\mathrm{diag}(\boldsymbol{P}_1^{-1}, I)$ and its transpose, and by applying the change of variables $\boldsymbol{Y} = \boldsymbol{P}_1^{-1}$ and $\boldsymbol{W} = \boldsymbol{K}\boldsymbol{Y} = \boldsymbol{K}\boldsymbol{P}_1^{-1}$, as is now common practice (Khargonekar & Rotea 1991; Boyd *et al.* 1994; El Ghaoui & Balakrishnan 1994). In the discrete case, the conversion consists in multiplying (4)(a) and (4)(b) respectively by $\mathrm{diag}(\boldsymbol{P}_1^{-1}, \boldsymbol{P}_1^{-1}, I, I)$, $\mathrm{diag}(\boldsymbol{P}_1^{-1}, \boldsymbol{P}_1^{-1}, I)$ and their transposes, (4)(c) being treated as (3)(c).

The BMIs (3) are thus replaced by the following set of LMIs in $\boldsymbol{Y}$, $\boldsymbol{W}$ and $\boldsymbol{Z}$:

Continuous case:
minimize η^2
subject to

$$\begin{bmatrix} \begin{pmatrix} \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{A}^T + \\ +B_u\boldsymbol{W} + \boldsymbol{W}^T B_u^T \end{pmatrix} & B_w R_\infty & (C_z\boldsymbol{Y} + D_{zu}\boldsymbol{W})^T L_\infty^T \\ \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{A}^T + B_u\boldsymbol{W} + \boldsymbol{W}^T B_u^T & B_w R_2 \\ \bullet & -I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \boldsymbol{Y} & (C_z\boldsymbol{Y} + D_{zu}\boldsymbol{W})^T L_2^T \\ \bullet & \boldsymbol{Z} \end{bmatrix} \succ 0, \ \mathrm{Tr}(\boldsymbol{Z}) < \eta^2, \ \boldsymbol{Y} \succ 0. \qquad (5)$$

Discrete case:
minimize η^2
subject to

$$\begin{bmatrix} -\boldsymbol{Y} & \boldsymbol{A}\boldsymbol{Y} + B_u\boldsymbol{W} & B_w R_\infty & 0 \\ \bullet & -\boldsymbol{Y} & 0 & (C_z\boldsymbol{Y} + D_{zu}\boldsymbol{W})^T L_\infty^T \\ \bullet & \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} -\boldsymbol{Y} & \boldsymbol{A}\boldsymbol{Y} + B_u\boldsymbol{W} & B_w R_2 \\ \bullet & -\boldsymbol{Y} & 0 \\ \bullet & \bullet & -I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \boldsymbol{Y} & (C_z\boldsymbol{Y} + D_{zu}\boldsymbol{W})^T L_2^T \\ \bullet & \boldsymbol{Z} \end{bmatrix} \succ 0, \ \mathrm{Tr}(\boldsymbol{Z}) < \eta^2, \ \boldsymbol{Y} \succ 0. \qquad (6)$$

These LMIs are solved for $\boldsymbol{Y}$ and $\boldsymbol{W}$, from which $P_1 = \boldsymbol{Y}^{-1}$ and $K = K_{\mathrm{ini}} = \boldsymbol{W}\boldsymbol{Y}^{-1} = \boldsymbol{W}P_1$ are obtained.

Step 2: Computation of η and P_2

Let $u = Kx$. The $\mathcal{H}_2$ norm η of the closed-loop system and the corresponding Lyapunov matrix P_2 are then determined by solving (3)(b) and (3)(c) in the continuous case, respectively (4)(b) and (4)(c) in the discrete case, where the matrices $\tilde{A}$ and $\tilde{C}$ have been recalculated for the present value of K according to their expressions (2) and are thus again constants, the decision variables being this time $\boldsymbol{P}_2$ and $\boldsymbol{Z}$.

Step 3: Linearization

The BMI (3) is then linearized to a first order approximation around K, P_1, η^2 and P_2 by means of perturbations $\boldsymbol{\delta K}$, $\boldsymbol{\delta P}_1$, $\boldsymbol{\delta\eta}^2$ and $\boldsymbol{\delta P}_2$. The first one induces in turn on two of the matrices of the closed-loop system (2) the following perturbations:

$$\boldsymbol{\delta\tilde{A}} = B_u\boldsymbol{\delta K}, \ \boldsymbol{\delta\tilde{C}} = D_{zu}\boldsymbol{\delta K},$$

the two other ones, B_w and D_{zw}, remaining unaffected. To keep these perturbations small, two additional LMIs expressing the constraints $\|\boldsymbol{\delta P}_1\| < \alpha\|P_1\|$ and $\|\boldsymbol{\delta P}_2\| < \alpha\|P_2\|$ are introduced. At the difference to (Hassibi, How & Boyd 1999), the "smallness" of the perturbations is determined by this parameter α, which, after an initial value of 0.2, will be adjusted during the iterative process, as will be seen below. The decision variables being now the four perturbations $\boldsymbol{\delta P}_1$, $\boldsymbol{\delta P}_2$, $\boldsymbol{\delta\tilde{A}}$ and $\boldsymbol{\delta\tilde{C}}$, the linearized problem is expressed by the following LMIs to be solved in this step:

Continuous case:
minimize $\delta\eta^2$
subject to

$$P_1 + \boldsymbol{\delta P}_1 \succ 0, \ P_2 + \boldsymbol{\delta P}_2 \succ 0,$$

$$\begin{bmatrix} \alpha P_1 & \boldsymbol{\delta P}_1 \\ \bullet & \alpha P_1 \end{bmatrix} \succ 0, \ \begin{bmatrix} \alpha P_2 & \boldsymbol{\delta P}_2 \\ \bullet & \alpha P_2 \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} \begin{pmatrix} (\tilde{A} + \boldsymbol{\delta\tilde{A}})^T P_1 + \\ P_1(\tilde{A} + \boldsymbol{\delta\tilde{A}}) + \\ \tilde{A}^T\boldsymbol{\delta P}_1 + \boldsymbol{\delta P}_1\tilde{A} \end{pmatrix} & (P_1 + \boldsymbol{\delta P}_1)B_w R_\infty & (\tilde{C} + \boldsymbol{\delta\tilde{C}})^T L_\infty^T \\ \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \begin{pmatrix} (\tilde{A} + \boldsymbol{\delta\tilde{A}})^T P_2 + P_2(\tilde{A} + \boldsymbol{\delta\tilde{A}}) + \\ \tilde{A}^T\boldsymbol{\delta P}_2 + \boldsymbol{\delta P}_2\tilde{A} \end{pmatrix} & (P_2 + \boldsymbol{\delta P}_2)B_w R_2 \\ \bullet & -I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} P_2 + \boldsymbol{\delta P}_2 & (\tilde{C} + \boldsymbol{\delta\tilde{C}})^T L_2^T \\ \bullet & \boldsymbol{Z} \end{bmatrix} \succ 0, \ \mathrm{Tr}(\boldsymbol{Z}) < \eta^2 + \delta\eta^2.$$

Discrete case :

minimize $\delta\eta^2$

subject to

$$P_1 + \boldsymbol{\delta P_1} \succ 0,\ P_2 + \boldsymbol{\delta P_2} \succ 0,$$

$$\begin{bmatrix} \alpha P_1 & \boldsymbol{\delta P_1} \\ \bullet & \alpha P_1 \end{bmatrix} \succ 0,\ \begin{bmatrix} \alpha P_2 & \boldsymbol{\delta P_2} \\ \bullet & \alpha P_2 \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} -(P_1 + \boldsymbol{\delta P_1}) & \begin{pmatrix} (P_1 + \boldsymbol{\delta P_1})\tilde{A} \\ +P_1 \boldsymbol{\delta \tilde{A}} \end{pmatrix} & \begin{pmatrix} (P_1 + \boldsymbol{\delta P_1}) \\ \cdot B_w R_\infty \end{pmatrix} & 0 \\ \bullet & -(P_1 + \boldsymbol{\delta P_1}) & 0 & (\tilde{C} + \boldsymbol{\delta \tilde{C}})^T L_\infty^T \\ \bullet & \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} -(P_2 + \boldsymbol{\delta P_2}) & (P_2 + \boldsymbol{\delta P_2})\tilde{A} + P_2 \boldsymbol{\delta \tilde{A}} & (P_2 + \boldsymbol{\delta P_2}) B_w R_2 \\ \bullet & -(P_2 + \boldsymbol{\delta P_2}) & 0 \\ \bullet & \bullet & -I \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} P_2 + \boldsymbol{\delta P_2} & (\tilde{C} + \boldsymbol{\delta \tilde{C}})^T L_2^T \\ \bullet & \boldsymbol{Z} \end{bmatrix} \succ 0,\ \mathrm{Tr}(\boldsymbol{Z}) < \eta^2 + \delta\eta^2.$$

Step 4: Update

Let $K := K + \delta K$, $\tilde{A} := \tilde{A} + B_u \delta K$, $\tilde{C} := \tilde{C} + D_{zu} \delta K$.

Step 5: Computation of a New P_1

Solve the SDP

Continuous case :

minimize t

subject to

$$\begin{bmatrix} \tilde{A}^T \boldsymbol{P} + \boldsymbol{P} \tilde{A} & \boldsymbol{P} B_w R_\infty & \tilde{C}^T L_\infty^T \\ \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0,$$

$$-t\,I \prec \boldsymbol{P} - (P_1 + \delta P_1) \prec t\,I,\ \ \boldsymbol{P} \succ 0. \quad (7)$$

Discrete case :

minimize t

subject to

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{P}\tilde{A} & \boldsymbol{P} B_w R_\infty & 0 \\ \bullet & -\boldsymbol{P} & 0 & \tilde{C}^T L_\infty^T \\ \bullet & \bullet & -I & R_\infty^T D_{zw}^T L_\infty^T \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \prec 0,$$

$$-t\,I \prec \boldsymbol{P} - (P_1 + \delta P_1) \prec t\,I,\ \ \boldsymbol{P} \succ 0. \quad (8)$$

- If this SDP is feasible and no numerical problems have occurred, the obtained Lyapunov matrix P proves a level γ in the $\mathcal{H}_\infty$ norm for the closed-loop system and is closest to the first-order perturbed P_1. Then, let $P_1 = P$ and go to step 2.
- If not, let $\alpha := \alpha/2$, restore values of K , $\tilde{A}$ and $\tilde{C}$ prior to step 4, and go to step 3. This gives the possibility to refine the convergence towards the minimum value of η in step 3, since the infeasibility of (7) or (8), or the numerical problems encountered during the resolution of these LMIs, indicate that the value of K obtained at step 4 may already be beyond its optimal value. This added feature improves significantly the ability of the algorithm to converge towards the final solution, as compared with the original one.

The iterative loop from steps 3 to 5 is stopped when the relative improvement in η at step 3 is inferior to a desired accuracy or when a preset number of iterations is reached.

Several verifications are then performed. First the closed-loop system is built with the last value of the feedback matrix K obtained at step 4 of the algorithm. The obtained value of η is then verified by applying the *norm(lti_sys, 2)* instruction of the MATLAB Control Systems Toolbox to the closed-loop subsystem with input and output as given by R_2 and L_2 . The corresponding value of γ is then also computed, by solving for the obtained value of K the inequalities (3)(a), respectively (4)(a), which are then LMIs, with the objective of minimizing γ^2 , or by calculating the LTI model norm of the closed-loop subsystem determined by R_∞ and L_∞ with the *norm(lti_sys, inf)* instruction.

All the LMIs of our algorithm are solved with either the SeDuMi-1.1 solver (Sturm 1999), or its improved version contained in the CVX toolbox (Grant, Boyd & Ye 2006), and have been programmed with the MATLAB-interface YALMIP (Löfberg 2004).

3. CONTINUOUS-TIME EXAMPLE

3.1 Example of Hassibi et al.

The numerical example used here is the example given by Hassibi, How and Boyd (1999) in Subsection 4.3 of their paper, reproduced here with our notations:

$$A = \begin{bmatrix} -1.40 & -0.49 & -1.93 \\ -1.73 & -1.69 & -1.25 \\ 0.99 & 2.08 & -2.49 \end{bmatrix},\ B_w = \begin{bmatrix} -0.16 & -1.29 \\ 0.81 & 0.96 \\ 0.41 & 0.65 \end{bmatrix},$$

$$B_u = \begin{bmatrix} 0.25 \\ 0.41 \\ 0.65 \end{bmatrix},\ C_z = \begin{bmatrix} -0.41 & 0.44 & 0.68 \\ -1.77 & 0.50 & -0.40 \end{bmatrix},$$

$$D_{zu} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T,\ D_{zw} = 0_{2\times 2},\ C_y = I_3,\ D_{yw} = 0_{3\times 2},$$

$$L_\infty = \begin{bmatrix} 1 & 0 \end{bmatrix},\ R_\infty = I_2,\ L_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ R_2 = I_2,\ \gamma = 2\ .$$

With our algorithm, the initial (suboptimal) value obtained for K at step 1 is

$$K_{\text{ini}} = \begin{bmatrix} 1.3434 & -0.2886 & 0.4851 \end{bmatrix},$$

resulting in $\eta = 1.0392$. If the limit of relative improvement in η in step 3 is set to 0.1%, the algorithm stops after 4 iterations and the $\mathcal{H}_2$ norm is reduced to $\eta = 0.7489$ with

$$K = \begin{bmatrix} 1.950 & 0.4011 & -0.2109 \end{bmatrix}.$$

These results differ significantly from the values obtained by Hassibi, How and Boyd (1999), probably due to the errors mentioned here in Section 2.

At its output our algorithm gave the following Lyapunov matrices:

$$P_1 = \begin{bmatrix} 0.6385 & 0.5027 & -0.0647 \\ 0.5027 & 0.6494 & -0.0200 \\ -0.0647 & -0.0200 & 0.9839 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 8.3292 & 9.7819 & -1.6480 \\ 9.7819 & 12.3536 & -2.9923 \\ -1.6480 & -2.9923 & 10.2801 \end{bmatrix}.$$

These two matrices are significantly different and differ also from the common Lyapunov matrix

$$P_1 = Y^{-1} = \begin{bmatrix} 0.5867 & 0.3977 & 0.0113 \\ 0.3977 & 0.5986 & 0.0147 \\ 0.0113 & 0.0147 & 0.8128 \end{bmatrix},$$

which is obtained as solution of (5) in Step 1 of the algorithm. This illustrates the suppression of the conservatism of the solution corresponding to a single Lyapunov matrix in the two constraints, which brings the reduction of η form the initial value of 1.0392 to the final value of 0.7489.

3.2 Comparison with a Direct BMI-Solving Program

The BMI (3) can also be solved directly by means of a BMI-solver such as the program PENBMI, from PENOPT. This algorithm is based on a combination of penalty barrier methods with the Augmented Lagrangian method (Kočvara & Stingl 2003). The result obtained for the same numerical example as previously is

$$K = \begin{bmatrix} 1.951 & 0.4015 & -0.2112 \end{bmatrix},$$

with a corresponding $\mathcal{H}_2$ norm of $\eta = 0.7489$. This result is very close to ours, well within the relative accuracy chosen to stop our algorithm.

4. DISCRETE-TIME EXAMPLE

4.1 Academic example

Assume that, for the discrete-time system

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D_{zu} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad D_{zw} = 0_{4\times 1},$$
$$C_y = I_2, \quad D_{yw} = 0_{2\times 1},$$
$$L_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad R_\infty = 1, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_2 = 1,$$

we want to calculate a state feedback controller which minimizes the $\mathcal{H}_2$ norm η of the closed loop from w to $z_2 = L_2 z$ while the $\mathcal{H}_\infty$ norm of the closed loop from w to $z_\infty = L_\infty z$ is less than some imposed level γ.

For this example, the optimal $\mathcal{H}_2$ controller that we have calculated (with no constraint on $\left\|\mathcal{T}_{wz_\infty}\right\|_\infty$) provides the closed-loop performance $\eta_{\min} = 2.3375$ and the optimal $\mathcal{H}_\infty$ controller (with no constraint on $\left\|\mathcal{T}_{wz_2}\right\|_2$) gives a closed-loop performance $\gamma_{\min} = 2$. We have thus applied our mixed-objective, discrete-time algorithm with an upper bound of $\gamma = 2.5$ and a desired accuracy for η of 0.1 %. After four iterations, the following results were obtained:

$$K = \begin{bmatrix} -2.398 & -0.6026 \end{bmatrix},$$

with $\eta = 2.4179$ and $\gamma = 2.3414$.

The two following Lyapunov matrices are produced at the output of the algorithm:

$$P_1 = \begin{bmatrix} 1.5776 & 0.9190 \\ 0.9190 & 0.7961 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 4.1138 & 1.4298 \\ 1.4298 & 0.9590 \end{bmatrix}$$

which had been initialized in step 1 with the following solution of (6):

$$P_1 = \begin{bmatrix} 2.3477 & 1.2117 \\ 1.2117 & 0.9836 \end{bmatrix}.$$

The same comment applies here as in the continuous case example.

4.2 Comparison with a Direct BMI-Solving Program

Again, the direct resolution of (4) by PENBMI yields for this example the following results:

$$K = \begin{bmatrix} -2.3972 & -0.6025 \end{bmatrix},$$

with $\eta = 2.3418$ and $\gamma = 2.4178$.

5. CONCLUDING REMARKS

In this work, we have presented the path-following method as an alternative to the solution of mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ state-feedback controller design, without having recourse to a BMI-solver code. The algorithm, described in details both for the continuous and the discrete case, is straightforward and can be implemented with pure LMI solving tools. The advantage over BMI solvers is that it gives a better overview during the convergence process, and that there exist very efficient LMI solvers, available freely. Two numerical examples, one coming from the original publication concerning the mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ state-feedback design in the continuous case and an academic example in the discrete case, have shown that the results given by our algorithm are as good as the ones yielded by a commercially available direct BMI-solving program, in both cases. A possible extension to our method to the design of full order dynamic output feedback is under investigation.

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