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Backstepping Boundary Controllers and Observers for the Rayleigh Beam

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Abstract: We design an exponentially stabilizing feedback controller and observer for the Rayleigh beam using noncollocated measurement and actuation. Our strategy is to use a damping boundary feedback combined with a backstepping-like coordinate transformation to transform the system into an exponentially stable system. The same idea is used to design our observer. Simulation results are included to illustrate the performance of the closed-loop system.

1. INTRODUCTION

The dynamic behavior of many physical systems can be described by partial differential equations and boundary controller design for these systems is a topic of considerable interest. In this paper, we consider a boundary controller design for the Rayleigh beam.

The Rayleigh beam model is the beam model which adds the rotary inertia effects to the Euler-Bernoulli beam, and is the formal limit of the Timoshenko beam when neglecting the shear distortion [1]. This model can be found in some mechanical systems such as the rotor-bearing systems [2], [3]. Previous works on the control of the Rayleigh beam include [4]-[8] and most of them use the Riesz basis approach. In this paper, we use the idea of damping boundary feedback [9] combined with the backstepping approach [10]-[11] to design a stabilizing controller and observer using noncollocated measurement and actuation which is more implementable to several applications than collocated control.

Our paper is organized as follows. In Section 2, we present the Rayleigh beam model and a change of variable that reduces the beam model to a wave equation. In Section 3, we design a boundary controller using a backstepping-like integral transformation that transforms our reduced model into an exponentially stable system. In Section 4, we use the same idea to design an observer, which employs measurement only at the beam tip. In Section 5, simulation results are presented to illustrate the performance of the closed-loop system. Finally, conclusions are given in Section 6.

2. MODEL

The mathematical model of the Rayleigh beam is a second-order in time, fourth order in space PDE

$$\rho A w_{tt}(x,t) + w_{xxxx}(x,t) - \rho I w_{xxtt}(x,t) = 0 \quad x \in [0,1] \tag{1}$$

Here, we consider the clamped-end boundary conditions

$$w(0,t) = w_x(0,t) = 0 \tag{2}$$

where $w(x,t)$ denotes the tranverse displacement of the beam at the position x for time t, ρ is the density of the beam, A is the cross sectional area of the beam, and I is the mass moment of inertia of the beam's cross section. All parameters are dimensionless [1].

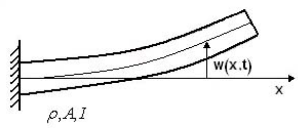

Fig 1: A Rayleigh beam clamped at $x = 0$.

We introduce a new variable

$$u(x,t) = \rho I w_{xx}(x,t) - \rho A w(x,t) \tag{3}$$

Then,

$$\begin{aligned} u_{tt}(x,t) &= \rho I w_{xxtt}(x,t) - \rho A w_{tt}(x,t) = w_{xxxx}(x,t) \\ &= \frac{1}{\rho I}\Big(u_{xx}(x,t) + \rho A w_{xx}(x,t)\Big) \\ &= \frac{1}{\rho I}\left(u_{xx}(x,t) + \frac{A}{I}\Big(u(x,t) + \rho A w(x,t)\Big)\right) \\ &= \frac{1}{\rho I}u_{xx}(x,t) + \frac{A}{\rho I^2}u(x,t) + \frac{A^2}{I^2}w(x,t) \end{aligned} \tag{4}$$

By solving an ODE in the spatial variable (3), we get

$$\begin{aligned} w(x,t) &= w(0,t)\cosh\sqrt{\frac{A}{I}}x + w_x(0,t)\sqrt{\frac{I}{A}}\sinh\sqrt{\frac{A}{I}}x \\ &\quad + \frac{1}{\rho\sqrt{AI}}\int_0^x \sinh\sqrt{\frac{A}{I}}(x-y)u(y,t)dy \\ &= \frac{1}{\rho\sqrt{AI}}\int_0^x \sinh\sqrt{\frac{A}{I}}(x-y)u(y,t)dy \end{aligned} \tag{5}$$

Substituting (5) into (4) yields

$$\begin{aligned} u_{tt}(x,t) &= \frac{1}{\rho I}u_{xx}(x,t) + \frac{A}{\rho I^2}u(x,t) \\ &\quad + \frac{A^{3/2}}{\rho I^{5/2}}\int_0^x \sinh\sqrt{\frac{A}{I}}(x-y)u(y,t)dy \end{aligned} \tag{6}$$

with the boundary condition

$$u(0,t) = \rho I w_{xx}(0,t) \tag{7}$$

Thus, we transform the beam model (1)-(2) into a wave equation (6)-(7). In the next section, we will stabilize (6)-(7) using a backstepping-like coordinate transformation, which transforms (6)-(7) into an exponentially stable system.

3. CONTROLLER DESIGN

We are going to stabilize (6)-(7) by employing a feedback control at the end $x = 1$ through two control inputs $w_x(1,t)$ and $w_{xxx}(1,t)$. It should be noted that our original model (1)-(2) is fourth-order in space. Therefore, it requires two boundary conditions at each end.
Consider the target system

$$v_{tt}(x,t) = \frac{1}{\rho I} v_{xx}(x,t) \tag{8}$$

with boundary conditions

$$v(0,t) = 0 \tag{9}$$

$$v_x(1,t) = -cv_t(1,t) \tag{10}$$

where $c > 0$ is a design parameter. It was proved in [9] that (8)-(10) is exponentially stable in the L_2 sense.
We look for a backstepping-like coordinate transformation

$$v(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy \tag{11}$$

which transforms (6)-(7) into (8)-(10).
From (11), we get

$$v(0,t) = u(0,t) \tag{12}$$

From (3) and (5), we get

$$\begin{aligned} w_x(x,t) = &\frac{1}{\rho\sqrt{AI}} \{\rho I w_{xx}(0,t) - u(0,t)\} \sinh\sqrt{\frac{A}{I}}x \\ &+\frac{1}{\rho I}\int_0^x \cosh\sqrt{\frac{A}{I}}(x-y)u(y,t)dy \end{aligned} \tag{13}$$

Setting $x = 1$ in (13), we can find an expression of $u(0,t)$ as follows.

$$\begin{aligned} u(0,t) = &\frac{-\rho\sqrt{AI}}{\sinh\sqrt{\frac{A}{I}}} \Bigg[w_x(1,t) - \sqrt{\frac{I}{A}} \sinh\sqrt{\frac{A}{I}} w_{xx}(0,t) \\ &-\frac{1}{\rho I}\int_0^1 \cosh\sqrt{\frac{A}{I}}(1-y)u(y,t)dy \Bigg] \end{aligned} \tag{14}$$

Thus, from (12) and (14) we choose our boundary feedback controller as

$$\begin{aligned} w_x(1,t) = &\sqrt{\frac{I}{A}} \sinh\sqrt{\frac{A}{I}} w_{xx}(0,t) \\ &+\frac{1}{\rho I}\int_0^1 \cosh\sqrt{\frac{A}{I}}(1-y)u(y,t)dy \end{aligned} \tag{15}$$

Then, we choose another boundary feedback controller in the form of the Neumann actuation.
By differentiating (11) with respect to x, we get

$$v_x(x,t) = u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy \tag{16}$$

Setting $x = 1$ in (16) and using (10) to find an expression of $u_x(1,t)$, we get

$$\begin{aligned} u_x(1,t) &= -cv_t(1,t) + k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)dy \\ &= k(1,1)u(1,t) - cu_t(1,t) + \int_0^1 k_x(1,y)u(y,t)dy \\ &\quad +c\int_0^1 k(1,y)u_t(y,t)dy \end{aligned} \tag{17}$$

Therefore, from (3) and (15) we have our boundary feedback controller as

$$\begin{aligned} w_{xxx}(1,t) = &\sqrt{\frac{A}{I}} \sinh\sqrt{\frac{A}{I}} w_{xx}(0,t) + \frac{k(1,1)}{\rho I}u(1,t) \\ &-\frac{c}{\rho I}u_t(1,t) + \frac{1}{\rho I}\int_0^1 \Big(k_x(1,y) \\ &+\frac{A}{I}\cosh\sqrt{\frac{A}{I}}(1-y)\Big)u(y,t)dy \\ &+\frac{c}{\rho I}\int_0^1 k(1,y)u_t(y,t)dy \end{aligned} \tag{18}$$

By differentiating (11) twice with respect to x and differentiating (11) twice again with respect to time, then substituting the result into (8), we get

$$\begin{aligned} 0 = &\int_0^x \Bigg\{ k_{xx}(x,y) - k_{yy}(x,y) - \frac{A}{I}k(x,y) \\ &+(\frac{A}{I})^{3/2}\sinh\sqrt{\frac{A}{I}}(x-y) \\ &-(\frac{A}{I})^{3/2}\int_y^x k(x,\xi)\sinh\sqrt{\frac{A}{I}}(\xi-y)d\xi \Bigg\} u(y,t)dy \\ &+\Big[\frac{A}{I} + 2\frac{d}{dx}k(x,x)\Big]u(x,t) + k(x,0)u_x(0,t) \end{aligned} \tag{19}$$

Because (19) holds for all $u(x,t)$, the kernel $k(x,y)$ must satisfy the following PDE

$$\begin{aligned} k_{xx}(x,y) = &k_{yy}(x,y) + \frac{A}{I}k(x,y) - (\frac{A}{I})^{3/2}\sinh\sqrt{\frac{A}{I}}(x-y) \\ &+(\frac{A}{I})^{3/2}\int_y^x k(x,\xi)\sinh\sqrt{\frac{A}{I}}(\xi-y)d\xi \end{aligned} \tag{20}$$

with boundary conditions

$$k(x,x) = -\frac{A}{2I}x, \quad k(x,0) = 0 \tag{21}$$

The well-posedness of the PDE (20)-(21) and the invertibility of the transformation (11) were proved in [12]. Hence, the closed-loop behavior of $u(x,t)$ is equivalent to the behavior of $v(x,t)$, which decays exponentially. From (5), we see that $w(x,t)$ also decays exponentially. Thus, we conclude that (1)-(2) with the controllers (15) and (18) is exponentially stable in the L_2 sense. However, the controllers (15) and (18) need the measurements of $u(x,t)$ and $u_t(x,t)$ along the whole beam. In the next section, we will show that such measurements are not necessary by using the same idea as in this section to design our observer.

4. OBSERVER DESIGN

We will design an observer to estimate the state $u(x,t)$ by employing measurement only at the end $x = 0$. Then, instead of (15) and (18), we use the boundary feedback controllers

$$\begin{aligned} w_x(1,t) &= \sqrt{\frac{I}{A}}\sinh\sqrt{\frac{A}{I}}w_{xx}(0,t) \\ &\quad +\frac{1}{\rho I}\int_0^1 \cosh\sqrt{\frac{A}{I}}(1-y)\hat{u}(y,t)dy \end{aligned} \tag{22}$$

$$\begin{aligned} w_{xxx}(1,t) &= \sqrt{\frac{A}{I}}\sinh\sqrt{\frac{A}{I}}w_{xx}(0,t) + \frac{k(1,1)}{\rho I}\hat{u}(1,t) \\ &\quad -\frac{c}{\rho I}\hat{u}_t(1,t) + \frac{1}{\rho I}\int_0^1 \Big(k_x(1,y) \\ &\quad +\frac{A}{I}\cosh\sqrt{\frac{A}{I}}(1-y)\Big)\hat{u}(y,t)dy \\ &\quad +\frac{c}{\rho I}\int_0^1 k(1,y)\hat{u}_t(y,t)dy \end{aligned} \tag{23}$$

where $\hat{u}(x,t)$ is generated by an observer to be designed. The observer equation is given by

$$\begin{aligned} \hat{u}_{tt}(x,t) &= \frac{1}{\rho I}\hat{u}_{xx}(x,t) + \frac{A}{\rho I^2}\hat{u}(x,t) \\ &\quad +\frac{A^{3/2}}{\rho I^{5/2}}\int_0^x \sinh\sqrt{\frac{A}{I}}(x-y)\hat{u}(y,t)dy \\ &\quad -\frac{1}{\rho I}\Big(p(x,0)p(0,0)-p_y(x,0)\Big)\Big(\rho I w_{xx}(0,t)-\hat{u}(0,t)\Big) \\ &\quad -\frac{1}{\rho I}p(x,0)\Big(\rho I w_{xxx}(0,t)-\hat{u}_x(0,t)\Big) \end{aligned} \tag{24}$$

$$\begin{aligned} \hat{u}(0,t) &= \rho I w_{xx}(0,t) + \frac{1}{p(0,0)-c_1}\Big(\rho I w_{xxx}(0,t)-\hat{u}_x(0,t)\Big) \\ &\quad -\frac{c_0}{\rho(0,0)-c_1}\Big(\rho I w_{xxt}(0,t)-\hat{u}_t(0,t)\Big) \end{aligned} \tag{25}$$

$$\hat{u}_x(1,t) = u_x(1,t) \tag{26}$$

The observer employs the shear measurement $w_{xxx}(0,t)$, the moment $w_{xx}(0,t)$ and its derivative $w_{xxt}(0,t)$. The input $u_x(1,t)$ in (26) is substituted from the controller (17) with $u(x,t)$ replaced by $\hat{u}(x,t)$. Unlike [11], eq. (35), we can avoid using the measurement at $x = 1$ in our observer. Denoting the observer error as $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, then substituting (24)-(26) from (6), (7), we obtain the observer error dynamics

$$\begin{aligned} \tilde{u}_{tt}(x,t) &= \frac{1}{\rho I}\tilde{u}_{xx}(x,t) + \frac{A}{\rho I^2}\tilde{u}(x,t) \\ &\quad +\frac{A^{3/2}}{\rho I^{5/2}}\int_0^x \sinh\sqrt{\frac{A}{I}}(x-y)\tilde{u}(y,t)dy \\ &\quad +\frac{1}{\rho I}\Big(p(x,0)p(0,0)-p_y(x,0)\Big)\tilde{u}(0,t) \\ &\quad +\frac{1}{\rho I}p(x,0)\tilde{u}_x(0,t) \end{aligned} \tag{27}$$

$$\tilde{u}_x(0,t) = -(p(0,0)-c_1)\tilde{u}(0,t) + c_0\tilde{u}_t(0,t) \tag{28}$$

$$\tilde{u}_x(1,t) = 0 \tag{29}$$

We look for a backstepping-like coordinate transformation

$$\tilde{u}(x,t) = \tilde{v}(x,t) - \int_0^x p(x,y)\tilde{v}(y,t)dy \tag{30}$$

which transforms (27)-(29) into an exponentially stable system in the L_2 sense

$$\tilde{v}_{tt}(x,t) = \frac{1}{\rho I}\tilde{v}_{xx}(x,t) \tag{31}$$

with boundary conditions

$$\tilde{v}_x(0,t) = c_0\tilde{v}_t(0,t) + c_1\tilde{v}(0,t) \tag{32}$$

$$\tilde{v}_x(1,t) = 0 \tag{33}$$

where $c_0, c_1 > 0$ are design parameters.

Using (27)-(33), we get

$$\begin{aligned} 0 &= \int_0^x \Bigg\{ p_{xx}(x,y) - p_{yy}(x,y) + \frac{A}{I}p(x,y) \\ &\quad -(\frac{A}{I})^{3/2}\sinh\sqrt{\frac{A}{I}}(x-y) \\ &\quad +(\frac{A}{I})^{3/2}\int_y^x p(\xi,y)\sinh\sqrt{\frac{A}{I}}(x-\xi)d\xi\Bigg\}\tilde{v}(y,t)dy \\ &\quad +\left[2\frac{d}{dx}p(x,x) - \frac{A}{I}\right]\tilde{v}(x,t) \end{aligned} \tag{34}$$

$$p_x(1,y) = 0, \quad p(1,1) = 0 \tag{35}$$

Because (34)-(35) should hold for all $\tilde{v}(x,t)$, the kernel $p(x,y)$ must satisfy the following PDE

$$p_{yy}(x,y) = p_{xx}(x,y) + \frac{A}{I}p(x,y) - (\frac{A}{I})^{3/2}\sinh\sqrt{\frac{A}{I}}(x-y) + (\frac{A}{I})^{3/2}\int_y^x p(\xi,y)\sinh\sqrt{\frac{A}{I}}(x-\xi)d\xi \quad (36)$$

with boundary conditions

$$p(x,x) = \frac{A}{2I}(x-1), \quad p_x(1,y) = 0 \quad (37)$$

By the change of variable $p(x,y) = \bar{p}(\bar{x},\bar{y})$, where $\bar{x} = 1-y$ and $\bar{y} = 1-x$, (36)-(37) are transformed into

$$\bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) = \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) + \frac{A}{I}\bar{p}(\bar{x},\bar{y}) - (\frac{A}{I})^{3/2}\sinh\sqrt{\frac{A}{I}}(\bar{x}-\bar{y}) + (\frac{A}{I})^{3/2}\int_{\bar{y}}^{\bar{x}} p(\bar{x},\xi)\sinh\sqrt{\frac{A}{I}}(\xi-\bar{y})d\xi \quad (38)$$

$$\bar{p}(\bar{x},\bar{x}) = -\frac{A}{2I}\bar{x}, \quad \bar{p}_{\bar{y}}(\bar{x},0) = 0 \quad (39)$$

It was proved in [12] that (38)-(39) is well-posed. Hence, the observer error dynamics (27)-(29) is exponentially stable in the L_2 sense.

5. SIMULATION RESULTS

First, we consider the gain kernels $k(1,y)$ and $k_x(1,y)$ used in the controller (18) and given by (20)-(21). By the method of successive approximations [12] we obtain the recursive relation for $k(x,y)$

$$k(x,y) = \lim_{n\to\infty} k_n(x,y) \quad (40)$$

$$k_0(x,y) = -\frac{A}{2I}y\cosh\sqrt{\frac{A}{I}}(x-y) \quad (41)$$

$$k_{n+1}(x,y) = k_0(x,y) + \frac{A}{I}\int_{\frac{x-y}{2}}^{\frac{x+y}{2}}\int_0^{\frac{x-y}{2}} k_n(\sigma+s,\sigma-s)dsd\sigma + (\frac{A}{I})^{3/2}\int_{\frac{x-y}{2}}^{\frac{x+y}{2}}\int_0^{\frac{x-y}{2}}\int_{\sigma-s}^{\sigma+s} k_n(\sigma+s,\xi) \times\sinh\sqrt{\frac{A}{I}}(\xi-\sigma+s)d\xi dsd\sigma \quad (42)$$

The same idea is used to determine the observer gain kernels $p(x,0)$ and $p_y(x,0)$ used in the observer (24) and given by (36)-(37). Thus, we obtain

$$\bar{p}(\bar{x},\bar{y}) = \lim_{n\to\infty}\bar{p}_n(\bar{x},\bar{y}) \quad (43)$$

$$\bar{p}_0(\bar{x},\bar{y}) = -\frac{A}{2I}\left(\bar{y}\cosh\sqrt{\frac{A}{I}}(\bar{x}-\bar{y}) + \sqrt{\frac{I}{A}}\sinh\sqrt{\frac{A}{I}}(\bar{x}-\bar{y})\right) \quad (44)$$

$$\begin{aligned}\bar{p}_{n+1}(\bar{x},\bar{y}) &= \bar{p}_0(\bar{x},\bar{y}) + \frac{A}{I}\int_{\frac{\bar{x}-\bar{y}}{2}}^{\frac{\bar{x}+\bar{y}}{2}}\int_0^{\frac{\bar{x}-\bar{y}}{2}}\bar{p}_n(\sigma+s,\sigma-s)dsd\sigma \\ &+ \frac{2A}{I}\int_0^{\frac{\bar{x}-\bar{y}}{2}}\int_0^{\sigma}\bar{p}_n(\sigma+s,\sigma-s)dsd\sigma \\ &+ (\frac{A}{I})^{3/2}\int_{\frac{\bar{x}-\bar{y}}{2}}^{\frac{\bar{x}+\bar{y}}{2}}\int_0^{\frac{\bar{x}-\bar{y}}{2}}\int_{\sigma-s}^{\sigma+s}\bar{p}_n(\sigma+s,\xi) \times\sinh\sqrt{\frac{A}{I}}(\xi-\sigma+s)d\xi dsd\sigma \\ &+ 2(\frac{A}{I})^{3/2}\int_0^{\frac{\bar{x}-\bar{y}}{2}}\int_0^{\sigma}\int_{\sigma-s}^{\sigma+s}\bar{p}_n(\sigma+s,\xi) \times\sinh\sqrt{\frac{A}{I}}(\xi-\sigma+s)d\xi dsd\sigma \end{aligned} \quad (45)$$

$$p(x,0) = \bar{p}(1,1-x), \quad p_y(x,0) = -\bar{p}_{\bar{x}}(1,1-x) \quad (46)$$

Using symbolic calculation in MATLAB with parameters $\rho = 100$, $A = I = 0.01$, we get the numerical values of the gain kernels $k(1,y)$, $k_x(1,y)$, $p(x,0)$ and $p_y(x,0)$ as shown in the Figures 2, 3, 4 and 5 respectively.

Next, we will simulate a dynamic behavior of the Rayleigh beam model (1)-(2) in both the uncontrolled case and the controlled case with and without the observer. However, we simulate our transformed model (6) instead and use the transformation (5) to convert back to (1)-(2). By discretizing (6) in space using the step size $h = 0.01$, our PDE system turns into a large system of linear ODEs with constant coefficients. Finally, we use Zakian I_{MN} recursions [14] (see also [15]) which is A-stable for $N-2 \le M \le N-1$. We choose $M = 3$, $N = 4$ and use the step size $t_s = 0.05$ sec. The parameters are chosen as $c = 0.5$, $c_0 = 1$ and $c_1 = 0.9$ with the initial conditions

$$w(x,0) = x^2(1-x)^2, \quad w_t(x,0) = 0$$

for the plant and

$$\hat{u}(x,0) = \frac{u(x,0)}{2}, \quad \hat{u}_t(x,0) = 0$$

for the observer.

Figure 6 shows the beam response for the uncontrolled case with the boundary conditions $w(1,t) = w_x(1,t) = 0$. It is obvious that the response is oscillatory.

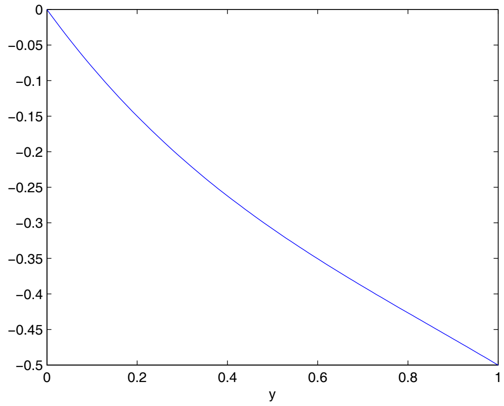

Fig 2: Gain kernel $k(1, y)$ of the controller (18).

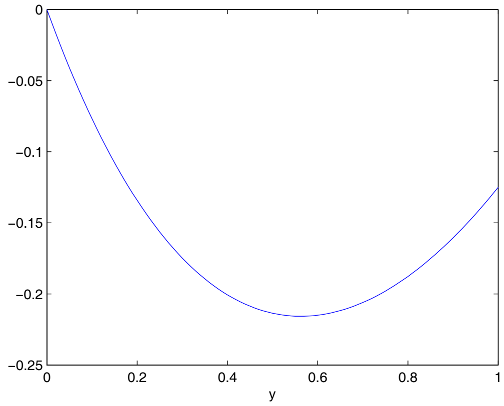

Fig 3: Gain kernel $k_x(1, y)$ of the controller (18).

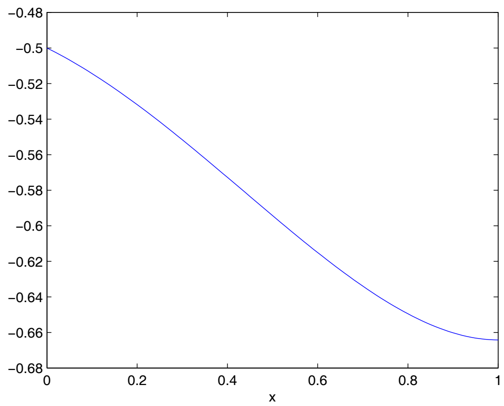

Fig 4: Gain kernel $p(x, 0)$ of the observer (24).

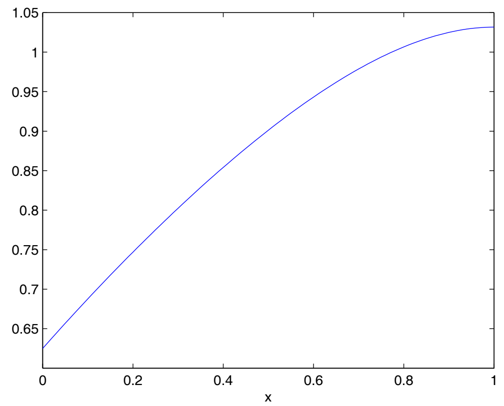

Fig 5: Gain kernel $p_y(x, 0)$ of the observer (24).

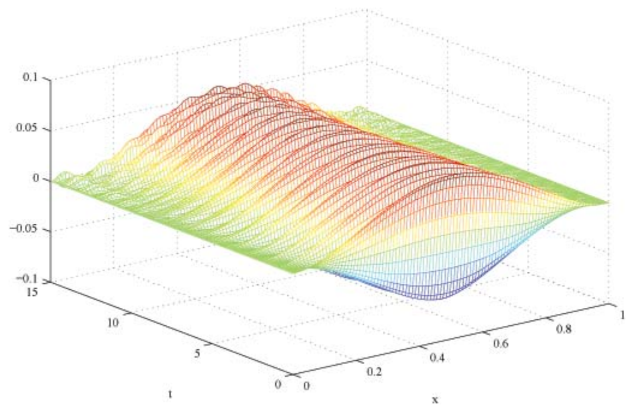

Fig 6: Beam response $w(x, t)$ (uncontrolled case, clamped at $x = 1$).

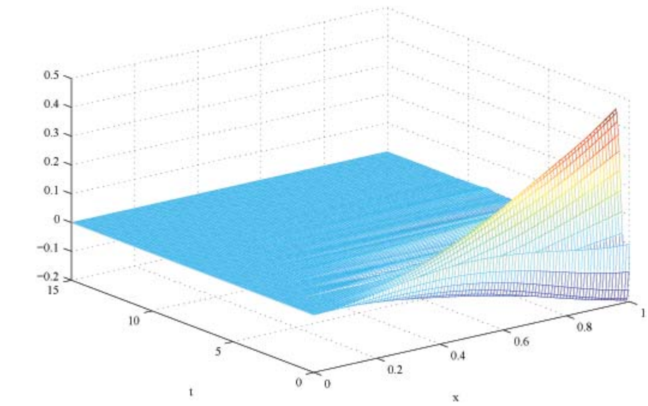

Fig 7: Beam response $w(x, t)$ (controlled case with controller (15),(18)).

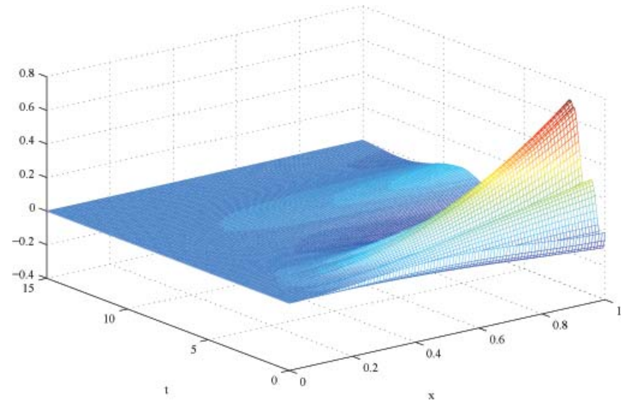

Fig 8: Beam response $w(x, t)$ (controlled case with observer).

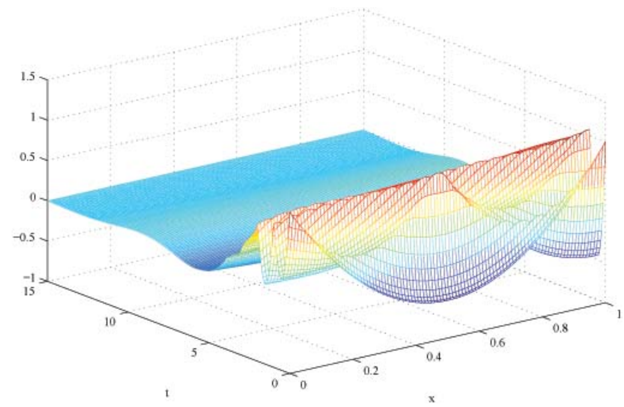

Fig 9: Observer error response $\tilde{u}(x, t)$ in (27)-(29).

Figure 7 shows the asymptotic stability of the beam response for the controlled-case without the observer, while Figure 8 shows the asymptotic stability of the beam response for controlled case with the observer (24)-(26). The response shown in Figure 8 has a slower rate of decay than the one in Figure 7.

Figure 9 shows the observer error response, which approaches zero as $t \to \infty$.

6. CONCLUSIONS

In this paper, we design a boundary feedback stabilizing controller for the Rayleigh beam. First, we transform a fourth-order in space beam equation into a second-order in space wave equation. To stabilize the transformed model, we use a backstepping-like integral transformation to transform it into an exponentially stable target system, and solve the kernel equation for the controller gains. The same idea is used to design an observer, which employs measurement only the the beam tip. Simulation results are presented, which show the stability of the closed-loop system.

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